#### Analysis of Nonlinear Fractional Differential Equations Involving Atangana–Baleanu–Caputo Derivative

Kishor D. Kucche<sup>1</sup> kdkucche@gmail.com

Sagar T. Sutar  $^2$  sutar.sagar007@gmail.com

<sup>1</sup> Department of Mathematics, Shivaji University, Kolhapur-416 004, Maharashtra, India.
<sup>2</sup> Department of Mathematics, Vivekanand College (Autonomous), Kolhapur-416003, Maharashtra, India.

#### Abstract

In the present paper, we determine the estimations on Atangana–Baleanu–Caputo fractional derivative at extreme points. With the assistance of the estimations obtained, we derive the comparison results. Peano's type existence results established for non-linear fractional differential equations involving Atangana–Baleanu–Caputo fractional derivative. The acquired comparison results are then utilized to deal with the existence of local, extremal and global solution.

Key words: Fractional differential equations, Atangana–Baleanu–Caputo fractional derivative, Fractional differential inequalities, Comparison results, Local and global existence.
2010 Mathematics Subject Classification: 26A33, 34A12, 34A40, 35B50, 34A99

### 1 Introduction

The fundamental theory of fractional calculus has been exhibited principally with the two fractional derivative operators Riemann-Liouville and Caputo fractional derivatives. The Riemann-Liouville (RL) fractional derivative plays an important role mainly in the development of the theory of fractional derivatives, integrals and for application to develop various theories in pure mathematics. The RL fractional integral of the Caputo fractional derivative generates the initial conditions in the form of classical derivatives. As a result of which we have a clear physical interpretation in the modeling of real-world phenomena in the form of fractional differential equations (FDEs). Basic calculus and the theoretical development FDEs involving these two fractional derivative operators have been excellently presented in the monographs [1, 2, 3, 4, 5].

Kai et al. [6] researched the existence and uniqueness of solutions, structural stability and the dependence of the solution on the order of the differential equation and on the initial condition. Daftardar-Gejji et al. [7] broadened these investigations for the system of fractional differential equations. Lakshmikantham and Vatsala [8, 9] built up the theory of fractional differential and integral inequalities and employed it to explore the existence of extremal and global solutions of nonlinear FDEs. Basic development relating to investigations on the theory of nonlinear FDEs can be found in [10, 11, 12, 13] and ongoing advancements can be found in [14, 15] and references therein. Researchers working in the field of applied mathematics have been developing the theory of fractional calculus in various directions by defining different types of arbitrary order derivatives and integrals. It is very well known that the traditional fractional derivative operators provide better mathematical modeling of many real-world phenomena than the classical integer derivatives. In spite of this reality, numerous researchers accept that that worthy exactness may not be accomplished in the modeling of physical phenomena involving memory effect in the whole of the time duration due to the presence of a singular kernel in the definition of traditional fractional derivatives.

To eliminate the singular kernel, a non-singular fractional derivative operator with exponential kernel is proposed in [16], which is well known as Caputo–Fabrizio (CF) fractional derivative. For the development of the theory relating to FDEs involving CF derivative and it's real-world application one can refer[17, 18, 19, 20] and the references given therein. Motivated by the investigations of [16], Atangana and Baleanu in [21] proposed new fractional derivative having Mittag-Leffler (ML) function as its kernel, which is well known as Atangana–Baleanu–Caputo (ABC) fractional derivative. The basic calculus of ABCfractional derivative can be found in [22, 23, 24, 25, 26]. Non-locality of ABC-fractional derivative with singular ML kernel effectively permits taking care of the nonlocal dynamics, computational purposes and capturing the various features of realistic systems more suitably. Mathematical modeling via ABC-FDEs of the various outbreak, such as dengue fever, the free motion of a coupled oscillator, a tumor-immune surveillance mechanism etc. and efficient numerical method to tackle this has been explored in [27, 28, 29, 30, 31, 32].

Jarad et al. [33] provided sufficient conditions for the existence and uniqueness of the solution of nonlinear ABC-FDEs. Authors derived the Gronwall inequality in the frame of Atangana-Baleanu fractional integral and through it investigated Ulam-Hyers stability of nonlinear ABC-FDEs. Baleanu et al.[34] considered the existence and uniqueness of solution nonlinear ABC-FDEs and structured a numerical procedure dependent on the fractional Euler and predictor-corrector technique. Syam et al.[35] determined existence and uniqueness results for the linear and nonlinear ABC-FDEs and exhibited a numerical technique dependent on the Cheby-shev collocation technique. Afshari et al. [36] demonstrated the existence results for ABC-FDEs utilizing the fixed point theorems for contractive mappings such as  $\alpha$ - $\gamma$ -Geraghty type,  $\alpha$ -type  $\mathcal{F}$ -contraction in  $\mathcal{F}$ -complete metric space. Ravichandran et al. [38, 39, 40, 41] examined the existence and uniqueness of solution for ABC-fractional differential and integrodifferential equations. Shah et al. [42] analyzed the qualitative theory of existence and stability theory of Ulams type for evolution ABC-FDEs.

Even if, many researchers have investigated nonlinear ABC-FDEs, it ought to be seriously analyzed for the qualitative properties. In this view, motivated by the applications and the interesting literature on ABC-FDEs, on the line of [8, 9, 25], we investigates estimation on ABC-fractional derivatives at extreme points, comparison results, local and global existence of a solution and extremal solution for nonlinear ABC-FDEs of the form

$${}^{ABC}{}_0\mathcal{D}^{\alpha}_{\tau}\omega(\tau) = f\left(\tau,\omega(\tau)\right), \tau \in J = [0,T], \ T > 0, \tag{1.1}$$

$$\omega(0) = \omega_0, \tag{1.2}$$

where  $0 < \alpha < 1$ ,  ${}^{ABC}{}_0 \mathcal{D}^{\alpha}_{\tau}$  is the ABC- fractional derivative operator,  $\omega$ ,  ${}^{ABC}{}_0 \mathcal{D}^{\alpha}_{\tau} \omega \in C(J)$ and  $f \in C(J \times \mathbb{R}, \mathbb{R})$  is continuous non-linear function. The local existence of solution to ABC-FDEs (1.1)–(1.2) is based on Peano's theorem and the comparison results which we have derived in the present paper. The novelty of the present paper is that we have obtained comparison results, local and global existence of a solution, and extremal solution without demanding the monotonicity and Holder continuity assumption on the nonlinear function associated with ABC-FDEs (1.1)-(1.2). Further, we provided alternative proofs to few results of [25] pertain to fractional integral inequalities.

This paper is organized as follows. In section 2, we recall basic definitions and results related with ABC-fractional derivative. Section 3 deals with estimation on ABC-fractional derivative at extreme points. In section 4, we derive comparison results for ABC-FDEs involving initial and boundary conditions. Section 5 deals with local existence and extremal of solution of ABC-FDEs (1.1)-(1.2). In section 5, the global existence of solution is proved for ABC-FDEs (1.1)-(1.2).

## 2 Preliminaries

In this section, we recall some definitions and basic results about ABC-fractional derivative operator and generalized Mittag–Leffler function.

**Definition 2.1** [35] Let  $p \in [1, \infty)$  and  $\Omega$  be an open subset of  $\mathbb{R}$  the Sobolev space  $H^p(\Omega)$ 

is defined as

$$H^{p}(\Omega) = \left\{ f \in L^{2}(\Omega) : D^{\beta}f \in L^{2}(\Omega), \text{for all } |\beta| \leq p \right\}.$$

**Definition 2.2** [21] Let  $\omega \in H^1(0,T)$  and  $\alpha \in [0,1]$ , the left Atangana–Baleanu–Caputo fractional derivative of  $\omega$  of order  $\alpha$  is defined by

$${}^{ABC}{}_0\mathcal{D}^{\alpha}_{\tau}\omega(\tau) = \frac{B(\alpha)}{1-\alpha} \int_0^{\tau} \mathbb{E}_{\alpha} \left[ -\frac{\alpha}{1-\alpha} (\tau-\sigma)^{\alpha} \right] \omega'(\sigma) d\sigma,$$

where  $B(\alpha) > 0$  is a normalization function satisfying B(0) = B(1) = 1 and  $\mathbb{E}_{\alpha}$  is one parameter Mittag-Leffler function [3, 5] defined by

$$\mathbb{E}_{\alpha}(z) = \sum_{n=0}^{n=\infty} \frac{z^n}{\Gamma(n\alpha+1)}.$$

The associated fractional integral is defined by

$${}^{AB}{}_0I^{\alpha}_{\tau}\omega(\tau) = \frac{1-\alpha}{B(\alpha)}\omega(\tau) + \frac{\alpha}{B(\alpha)} {}_0I^{\alpha}_{\tau}\omega(\tau).$$

where

$${}_0I^{\alpha}_{\tau}\omega(\tau) = \frac{1}{\Gamma(\alpha)} \int_0^{\tau} (\tau - \sigma)^{\alpha - 1} \omega(\sigma) d\sigma,$$

is the Riemann-Liouville fractional integral [3, 5] of  $\omega$  of order  $\alpha$ .

**Lemma 2.1** [22] If  $0 < \alpha < 1$ , then  ${}^{AB}{}_0I^{\alpha}_{\tau} \left({}^{ABC}{}_0\mathcal{D}^{\alpha}_{\tau}\omega(\tau)\right) = \omega(\tau) - \omega(0)$ .

**Definition 2.3** [43, 44, 45] The generalized Mittag-Leffler function  $\mathbb{E}^{\gamma}_{\alpha,\beta}(z)$  for the complex numbers  $\alpha, \beta, \gamma$  with  $Re(\alpha) > 0$  is defined as

$$\mathbb{E}_{\alpha,\beta}^{\gamma}(z) = \sum_{k=0}^{\infty} \frac{(\gamma)_k}{\Gamma(\alpha k + \beta)} \frac{z^k}{k!},$$

where  $(\gamma)_k$  is the Pochhammer symbol given by

$$(\gamma)_0 = 1, \ (\gamma)_k = \gamma(\gamma+1)\cdots(\gamma+k-1), \ k = 1, 2, \cdots$$

Note that,

$$\mathbb{E}^1_{\alpha,\beta}(z) = \mathbb{E}_{\alpha,\beta}(z) \text{ and } \mathbb{E}^1_{\alpha,1}(z) = \mathbb{E}_{\alpha}(z).$$

**Lemma 2.2** [22] Let  $0 < \alpha < 1$  and  $\beta, \sigma, \lambda \in \mathbb{C}$  ( $Re(\beta) > 0$ ). Then

$${}^{ABC}{}_{0}\mathcal{D}^{\alpha}_{\tau}\left[\tau^{\beta-1} \mathbb{E}^{\sigma}_{\alpha,\beta}\left(\lambda \,\tau^{\alpha}\right)\right] = \frac{B(\alpha)}{1-\alpha} \,\tau^{\beta-1} \mathbb{E}^{1+\sigma}_{\alpha,\beta}(\lambda \,\tau^{\alpha}).$$

## 3 Estimates on ABC fractional derivatives at extreme points

**Theorem 3.1** If m is any differentiable function defined on J such that  ${}^{ABC}{}_0\mathcal{D}^{\alpha}_{\tau}m \in C(J)$ and there exists  $\tau_0 \in (0,T]$  with  $m(\tau_0) = 0$ ,  $m(\tau) \leq 0$ ,  $\tau \in [0,\tau_0)$ , then  ${}^{ABC}{}_0\mathcal{D}^{\alpha}_{\tau}m(\tau_0) \geq 0$ .

*Proof:* Using integration by parts we write

$${}^{ABC}{}_{0}\mathcal{D}^{\alpha}_{\tau}m(\tau) = \frac{B(\alpha)}{1-\alpha} \int_{0}^{\tau} \mathbb{E}_{\alpha} \left( -\frac{\alpha}{1-\alpha} (\tau-\sigma)^{\alpha} \right) m'(\sigma) d\sigma$$
  
$$= \frac{B(\alpha)}{1-\alpha} \left\{ \left[ \mathbb{E}_{\alpha} \left( -\frac{\alpha}{1-\alpha} (\tau-\sigma)^{\alpha} \right) m(\sigma) \right]_{\sigma=0}^{\sigma=\tau} - \int_{0}^{\tau} \left( \frac{d}{d\sigma} \mathbb{E}_{\alpha} \left( -\frac{\alpha}{1-\alpha} (\tau-\sigma)^{\alpha} \right) \right) m(\sigma) d\sigma \right\}$$
  
$$= \frac{B(\alpha)}{1-\alpha} \left\{ m(\tau) - \mathbb{E}_{\alpha} \left( -\frac{\alpha}{1-\alpha} \tau^{\alpha} \right) m(0) - \int_{0}^{\tau} \left( \frac{d}{d\sigma} \mathbb{E}_{\alpha} \left( -\frac{\alpha}{1-\alpha} (\tau-\sigma)^{\alpha} \right) \right) m(\sigma) d\sigma \right\}.$$

Since  $m(\tau_0) = 0$ , we have

$${}^{ABC}{}_{0}\mathcal{D}^{\alpha}_{\tau}m(\tau_{0}) = -\frac{B(\alpha)}{1-\alpha} \left\{ \mathbb{E}_{\alpha} \left( -\frac{\alpha}{1-\alpha}\tau_{0}^{\alpha} \right) m(0) + \int_{0}^{\tau_{0}} \left( \frac{d}{d\sigma} \mathbb{E}_{\alpha} \left[ -\frac{\alpha}{1-\alpha}(\tau_{0}-\sigma)^{\alpha} \right] \right) m(\sigma) d\sigma \right\}.$$

$$(3.1)$$

From [25], we have

$$\mathbb{E}_{\alpha}(-\tau^{\alpha}) = \int_{0}^{\infty} e^{-r\tau} K_{\alpha}(r) dr, \ 0 < \alpha < 1, \text{ for all } \tau > 0,$$

where

$$K_{\alpha}(r) = \frac{1}{\pi} \frac{r^{\alpha - 1} \sin(\alpha \pi)}{r^{2\alpha} + 2r^{\alpha} \cos(\alpha \pi) + 1} > 0.$$

Since  $K_{\alpha}(r)$ ,  $e^{-r\left(\frac{\alpha}{1-\alpha}\right)^{\frac{1}{\alpha}}\tau_0} > 0$ , for r > 0 and  $0 < \alpha < 1$ , we have

$$\mathbb{E}_{\alpha}\left(-\frac{\alpha}{1-\alpha}\tau_{0}^{\alpha}\right) = \mathbb{E}_{\alpha}\left(-\left(\left[\frac{\alpha}{1-\alpha}\right]^{\frac{1}{\alpha}}\tau_{0}\right)^{\alpha}\right) = \int_{0}^{\infty}e^{-r\left(\frac{\alpha}{1-\alpha}\right)^{\frac{1}{\alpha}}\tau_{0}}K_{\alpha}(r)\,dr > 0. \quad (3.2)$$

Since,  $B(\alpha) > 0$  and  $m(0) \le 0$ , from (3.2) it follows that

$$-\frac{B(\alpha)}{1-\alpha}\mathbb{E}_{\alpha}\left(-\frac{\alpha}{1-\alpha}\tau_{0}^{\alpha}\right)m(0) \ge 0.$$
(3.3)

In view of inequality (3.3), Eq. (3.1) reduces to

$$^{ABC}{}_{0}\mathcal{D}^{\alpha}_{\tau}m(\tau_{0}) \geq -\frac{B(\alpha)}{1-\alpha} \int_{0}^{\tau_{0}} \left(\frac{d}{d\sigma}\mathbb{E}_{\alpha}\left(-\frac{\alpha}{1-\alpha}(\tau_{0}-\sigma)^{\alpha}\right)\right) m(\sigma)d\sigma.$$

Again from [25], we have

$$\frac{d}{d\tau}\mathbb{E}_{\alpha}\left(-\frac{\alpha}{1-\alpha}(\tau_0-\tau)^{\alpha}\right) \ge 0,$$

Therefore,  $m(\tau) \leq 0, \ \tau \in [0, \tau_0)$  gives

$$\frac{d}{d\tau} \mathbb{E}_{\alpha} \left[ -\frac{\alpha}{1-\alpha} (\tau_0 - \tau)^{\alpha} \right] (-m(\tau)) \ge 0, \ \tau \in [0, \tau_0).$$
(3.4)

Using the inequality (3.4) and the fact  $B(\alpha) > 0$ , it follows that

$${}^{ABC}{}_{0}\mathcal{D}^{\alpha}_{\tau}m(\tau_{0}) \geq \frac{B(\alpha)}{1-\alpha} \int_{0}^{\tau_{0}} \left(\frac{d}{d\sigma}\mathbb{E}_{\alpha}\left(-\frac{\alpha}{1-\alpha}(\tau_{0}-\sigma)^{\alpha}\right)\right) (-m(\sigma)) \, d\sigma \geq 0.$$

This completes the proof.

The dual of the Theorem 3.1 is also hold.

**Theorem 3.2** If *m* is any differentiable function defined on *J* such that  ${}^{ABC}{}_0\mathcal{D}^{\alpha}_{\tau}m \in C(J)$ and there exists  $\tau_0 \in (0,T]$  with  $m(\tau_0) = 0$ ,  $m(\tau) \ge 0$ ,  $\tau \in [0,\tau_0)$ , then  ${}^{ABC}{}_0\mathcal{D}^{\alpha}_{\tau}m(\tau_0) \le 0$ .

*Proof:* One can observe that if  $m(\tau)$  satisfies the assumptions of Theorem 3.2, then  $(-m)(\tau)$  satisfies the conditions of Theorem 3.1. Hence by applying Theorem 3.1 with m replaced by (-m) we obtain,  ${}^{ABC}{}_0\mathcal{D}^{\alpha}_{\tau}(-m)(\tau_0) \geq 0$ . This gives

$${}^{ABC}{}_0 \mathcal{D}^{\alpha}_{\tau} m(\tau_0) \le 0.$$

In the following Theorem, we give an alternative proof of Lemma 2.1 [25] utilizing the outcome that we have gotten in Theorem 3.1.

**Theorem 3.3** Let a differentiable function f defined on J attain its maximum at a point  $\tau_0 \in J$  and  ${}^{ABC}{}_0 \mathcal{D}^{\alpha}_{\tau} f \in C(J)$ . Then the inequality  ${}^{ABC}{}_0 \mathcal{D}^{\alpha}_{\tau} f(\tau_0) \geq 0$  holds true.

*Proof:* Let  $f_{\max} = f(\tau_0) = \max_{\tau \in J} f(\tau)$ . Define  $m(\tau) = f(\tau) - f_{\max}, \tau \in J$ . Then m is differentiable function defined on J such that,  ${}^{ABC}{}_0 \mathcal{D}^{\alpha}_{\tau} m \in C(J)$  and

$$m(\tau_0) = 0, \ m(\tau) = f(\tau) - f_{\max} < 0, \ \text{for all } \tau \in J \setminus \{\tau_0\}$$

Therefore,  $m(\tau_0) = 0$  and  $m(\tau) < 0$ , for all  $\tau \in [0, \tau_0)$ . Since *m* satisfies all the conditions of Theorem 3.1, by applying it we obtain  ${}^{ABC}{}_0\mathcal{D}^{\alpha}_{\tau}m(\tau_0) \ge 0$ . Since,

$${}^{ABC}{}_0\mathcal{D}^{\alpha}_{\tau}m(\tau) = \; {}^{ABC}{}_0\mathcal{D}^{\alpha}_{\tau}\left(f(\tau) - f_{\max}\right) = \; {}^{ABC}{}_0\mathcal{D}^{\alpha}_{\tau}f(\tau), \; \tau \in J,$$

we have

$${}^{ABC}{}_0\mathcal{D}^{\alpha}_{\tau}f(\tau_0) = {}^{ABC}{}_0\mathcal{D}^{\alpha}_{\tau}m(\tau_0) \ge 0$$

In the next Theorem, we provide an alternating proof of Lemma 2.2 [25].

**Theorem 3.4** Let a differentiable function f defined on J attain its minimum at a point  $\tau_0 \in J$  and  ${}^{ABC}{}_0 \mathcal{D}^{\alpha}_{\tau} f \in C(J)$ . Then the inequality  ${}^{ABC}{}_0 \mathcal{D}^{\alpha}_{\tau} f(\tau_0) \leq 0$  holds true.

*Proof:* Let  $f_{\min} = f(\tau_0) = \min_{\tau \in J} f(\tau)$  and define  $m(\tau) = f(\tau) - f_{\min}$ ,  $\tau \in J$ . Then, one can complete the remaining proof by applying Theorem 3.2 and following the similar types of steps as in the proof of Theorem 3.3.

#### 4 Comparison Results

**Theorem 4.1** Let  $f \in C(J \times \mathbb{R}, \mathbb{R})$ . Let v, w be any differentiable functions on J such that  ${}^{ABC}{}_0\mathcal{D}^{\alpha}_{\tau} v, \; {}^{ABC}{}_0\mathcal{D}^{\alpha}_{\tau} w \in C(J), \; satisfying$ 

(i) 
$${}^{ABC}{}_0 \mathcal{D}^{\alpha}_{\tau} v(\tau) \leq f(\tau, v(\tau)), \ \tau \in J,$$

(ii) 
$${}^{ABC}{}_0 \mathcal{D}^{\alpha}_{\tau} w(\tau) \ge f(\tau, w(\tau)), \ \tau \in J_{\tau}$$

one of the above inequalities being strict.

Then v(0) < w(0), implies

$$v(\tau) < w(\tau), \ \tau \in J.$$

*Proof:* Suppose that the conclusion of the theorem does not holds. Then by continuity of v, w there exits  $\tau_0 \in J$  such that

$$v(\tau_0) = w(\tau_0)$$
 and  $v(\tau) < w(\tau)$  for all  $\tau \in [0, \tau_0)$ .

Define  $m(\tau) = v(\tau) - w(\tau)$ ,  $\tau \in J$ . Then the function  $m(\tau)$  is differentiable on J with  ${}^{ABC}{}_0 \mathcal{D}^{\alpha}_{\tau} m \in C(J)$  and  $\tau_0 \in J$  is such that

$$m(\tau_0) = 0$$
 and  $m(\tau) < 0$  for all  $\tau \in [0, \tau_0)$ .

Since *m* satisfies all assumptions of Theorem 3.1, we get  ${}^{ABC}{}_0 \mathcal{D}^{\alpha}_{\tau} m(\tau_0) \geq 0$ . This gives

$${}^{ABC}{}_0 \mathcal{D}^{\alpha}_{\tau} v(\tau_0) \geq {}^{ABC}{}_0 \mathcal{D}^{\alpha}_{\tau} w(\tau_0)$$

Suppose that the inequality (i) is strict, then we get

$$f(\tau_0, v(\tau_0)) > {}^{ABC}{}_0 \mathcal{D}^{\alpha}_{\tau} v(\tau_0) \geq {}^{ABC}{}_0 \mathcal{D}^{\alpha}_{\tau} w(\tau_0) \geq f(\tau_0, w(\tau_0)).$$

This is contradiction with  $v(\tau_0) = w(\tau_0)$ . Therefore, we must have

$$v(\tau) < w(\tau)$$
, for all  $\tau \in J$ .

This completes the proof of theorem.

**Theorem 4.2** Assume that the conditions of Theorem 4.1 holds (ii). Suppose that

$$f(\tau,\omega) - f(\tau,\eta) \le L(\omega-\eta), \text{ for all } \omega,\eta \in \mathbb{R} \text{ with } \omega \ge \eta \text{ and } 0 < L < \frac{B(\alpha)}{1-\alpha}$$

Then  $v(0) \leq w(0)$  implies

$$v(\tau) \le w(\tau), \text{ for all } \tau \in J.$$

*Proof:* For  $\epsilon > 0$ , we define

$$w_{\epsilon}(\tau) = w(\tau) + \epsilon \mathbb{E}_{\alpha}(\tau^{\alpha}), \ \tau \in J.$$
(4.1)

By choice of w and Lemma 2.2, the function  $w_{\epsilon}$  is differentiable on J such that  ${}^{ABC}{}_{0}\mathcal{D}^{\alpha}_{\tau}w_{\epsilon} \in C(J)$  and

$$w_{\epsilon}(0) = w_0 + \epsilon > w(0).$$

Since  $w_{\epsilon}(\tau) \geq w(\tau), \ \tau \in J$ , by using Lipschitz condition on f, we have

$$f(\tau, w_{\epsilon}(\tau)) - f(\tau, w(\tau)) \le L(w_{\epsilon}(\tau) - w(\tau)) = L\epsilon \mathbb{E}_{\alpha}(\tau^{\alpha}).$$

Using the condition on L, we have

$$f(\tau, w(\tau)) \ge f(\tau, w_{\epsilon}(\tau)) - L\epsilon \mathbb{E}_{\alpha}(\tau^{\alpha}) > f(\tau, w_{\epsilon}(\tau)) - \frac{B(\alpha)}{1 - \alpha} \epsilon \mathbb{E}_{\alpha}(\tau^{\alpha}), \ \tau \in J.$$
(4.2)

By Lemma 2.2, we find

$${}^{ABC}{}_{0}\mathcal{D}^{\alpha}_{\tau}\left(\mathbb{E}_{\alpha}(\tau^{\alpha})\right) = {}^{ABC}{}_{0}\mathcal{D}^{\alpha}_{\tau}\left(\mathbb{E}^{1}_{\alpha,1}(\tau^{\alpha})\right) = \frac{B(\alpha)}{1-\alpha}\mathbb{E}^{2}_{\alpha,1}(\tau^{\alpha}), \ \tau \in J.$$
(4.3)

Since  $(2)_0 = 1$  and  $\frac{(2)_k}{k!} = k + 1 > 1$ ,  $k = 1, 2, \dots$ , we have

$$\mathbb{E}_{\alpha,1}^2(\tau^{\alpha}) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+1)} \frac{(2)_k}{k!} \ge \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+1)} = \mathbb{E}_{\alpha}(\tau^{\alpha}), \ \tau \in J.$$
(4.4)

Using the inequality (4.4), Eq.(4.3) reduces to

$${}^{ABC}{}_{0}\mathcal{D}^{\alpha}_{\tau}\left(\mathbb{E}_{\alpha}(\tau^{\alpha})\right) \geq \frac{B(\alpha)}{1-\alpha}\mathbb{E}_{\alpha}(\tau^{\alpha}), \ \tau \in J.$$

$$(4.5)$$

Utilizing the inequalities (ii), (4.2) and (4.5), for any  $\tau \in J$ , we have

Therefore,

$${}^{ABC}{}_{0}\mathcal{D}^{\alpha}_{\tau}w_{\epsilon}(\tau) > f(\tau, w_{\epsilon}(\tau)), \ \tau \in J.$$

Since  $v(0) < w_{\epsilon}(0)$ , by application of Theorem 3.1 with  $w(\tau) = w_{\epsilon}(\tau)$ , for each  $\epsilon > 0$  we have

$$v(\tau) < w_{\epsilon}(\tau), \ \tau \in J.$$

Taking limit as  $\epsilon \to 0$ , in the above inequality and utilizing Eq. (4.1), we obtain

$$v(\tau) \le w(\tau), \ \tau \in J.$$

**Corollary 4.3** If m is any differentiable function defined on J such that  ${}^{ABC}{}_0\mathcal{D}^{\alpha}_{\tau}m \in C(J)$ ,  $\tau \in J$  and

$${}^{ABC}{}_0 \mathcal{D}^{\alpha}_{\tau} m(\tau) \leq \frac{B(\alpha)}{1-\alpha} m(\tau), \ \tau \in J,$$
$$m(0) = m_0,$$

then  $m(\tau) \leq m_0 \mathbb{E}_{\alpha}(\tau^{\alpha}), \ \tau \in J.$ 

Proof: Define

$$\lambda(\tau) = m_0 \mathbb{E}_{\alpha}(\tau^{\alpha}), \ \tau \in J.$$

Then  $\lambda(0) = m_0$ . Further using the inequality (4.5), we have

$${}^{ABC}{}_0\mathcal{D}^{\alpha}_{\tau}\lambda(\tau) \ge \frac{B(\alpha)}{1-\alpha}\lambda(\tau), \ \tau \in J.$$

Applying Theorem 4.2, with v = m and  $w = \lambda$ , we get

$$m(\tau) \le \lambda(\tau) = m_0 \mathbb{E}_{\alpha}(\tau^{\alpha}), \tau \in J.$$

The following theorem is the comparison result for periodic boundary value problems involving ABC-fractional derivative. The proof of the same one can finish watching the comparable kind of steps of Theorem 2.6 [46].

**Theorem 4.4** Let v, w be any differentiable functions on J such that  ${}^{ABC}{}_{0}\mathcal{D}^{\alpha}_{\tau} v, {}^{ABC}{}_{0}\mathcal{D}^{\alpha}_{\tau} w \in$  $C(J), \tau \in J, and f \in C(J \times \mathbb{R}, \mathbb{R})$  satisfying

(i) 
$${}^{ABC}{}_0\mathcal{D}^{\alpha}_{\tau}v(\tau) \leq f(\tau, v(\tau)), \ \tau \in J, \ v(0) \leq v(T)$$

(ii) 
$${}^{ABC}{}_0\mathcal{D}^{\alpha}_{\tau}w(\tau) \ge f(\tau, w(\tau)), \ \tau \in J, \ w(0) \ge w(T)$$

If the function  $f(\tau, \omega)$  is non increasing in  $\omega$  for each  $\tau$  then

$$v(\tau) \le w(\tau), \ \tau \in J.$$

In the next Theorem, we will give an alternating proof of Lemma 2.3 [25] without utilizing Cauchy-Schwartz inequality.

**Theorem 4.5** If m is any differentiable function defined on J such that  ${}^{ABC}{}_0\mathcal{D}^{\alpha}_{\tau}m \in C(J)$ then  ${}^{ABC}{}_0\mathcal{D}^{\alpha}_{\tau}m(0) = 0.$ 

*Proof:* Let *m* is any differentiable function defined on *J* such that  ${}^{ABC}{}_0 \mathcal{D}^{\alpha}_{\tau} m \in C(J), \ \tau \in J$ . Since m' exists and is continuous on J, we have  $m, m' \in C(J)$ , this implies  $m, m' \in L^1(J)$ , therefore sup  $|m'(\tau)| \leq M$ . Using the definition of ABC-fractional derivative operator,  $\tau{\in}[0{,}T]$ 

$$\begin{split} |^{ABC}{}_{0}\mathcal{D}^{\alpha}_{\tau}m(\tau)| &\leq \frac{B(\alpha)}{1-\alpha} \int_{0}^{\tau} \left| \mathbb{E}_{\alpha} \left( -\frac{\alpha}{1-\alpha} (\tau-\sigma)^{\alpha} \right) \right| |m'(\sigma)| d\sigma \\ &\leq \frac{MB(\alpha)}{1-\alpha} \int_{0}^{\tau} \mathbb{E}_{\alpha} \left( \frac{\alpha}{1-\alpha} (\tau-\sigma)^{\alpha} \right) d\sigma = \frac{MB(\alpha)}{1-\alpha} \mathbb{E}_{\alpha,2} \left( \frac{\alpha}{1-\alpha} \tau^{\alpha} \right) \tau \\ &\text{is gives, } {}^{ABC}{}_{0}\mathcal{D}^{\alpha}_{\tau}m(0) = 0. \end{split}$$

This gives,  ${}^{ABC}{}_{0}\mathcal{D}^{\alpha}_{\tau}m(0) = 0.$ 

**Corollary 4.6** If  $\omega$ ,  ${}^{ABC}{}_0 \mathcal{D}^{\alpha}_{\tau} \omega$ ,  $u \in C(J)$  and  ${}^{ABC}{}_0 \mathcal{D}^{\alpha}_{\tau} \omega(\tau) = u(\tau)$ , then u(0) = 0. *Proof:* Proof follows from the Theorem 4.5.

**Remark 4.7** From the Corollary 4.6 it follows that the ABC-FDEs (1.1)–(1.2) and its equivalent fractional integral equation is consistent only if  $f(0, \omega(0)) = 0$ , where  $\omega$  is differentiable function with  ${}^{ABC}{}_0 \mathcal{D}^{\alpha}_{\tau} \omega \in C(J)$ .

### 5 Existence of Local and Extremal Solution

In this section, we determine the results about the existence of local and extremal solutions of the ABC-FDEs (1.1)-(1.2) through the equivalent fractional integral equation given in the following Lemma.

**Lemma 5.1** [38, 35] The equivalent fractional integral equation to the the ABC-FDEs (1.1)-(1.2) is given by

$$\omega(\tau) = \omega_0 + \frac{1-\alpha}{B(\alpha)} f(\tau, \omega(\tau)) + \frac{\alpha}{B(\alpha)\Gamma(\alpha)} \int_0^\tau (\tau - \sigma)^{\alpha - 1} f(\sigma, \omega(\sigma)) d\sigma.$$

Let  $\epsilon > 0$  be arbitrary. Consider the ABC-FDEs of the form,

$${}^{ABC}{}_{0}\mathcal{D}^{\alpha}_{\tau}\omega_{\epsilon}(\tau) = f\left(\tau,\omega_{\epsilon}(\tau)\right), \tau \in J, \tag{5.1}$$

$$\omega_{\epsilon}(\tau)|_{\tau=0} = \omega_{\epsilon}(0), \tag{5.2}$$

where  $\omega_{\epsilon}$ ,  ${}^{ABC}{}_{0}\mathcal{D}^{\alpha}_{\tau}\omega_{\epsilon} \in C(J)$  and  $f \in C(J \times \mathbb{R}, \mathbb{R})$ . Then by Lemma 5.1, the equivalent fractional integral equation of the ABC-FDEs (5.1)-(5.2) is given by

$$\omega_{\epsilon}(\tau) = \omega_{\epsilon}(0) + \frac{1-\alpha}{B(\alpha)}f(\tau,\omega_{\epsilon}(\tau)) + \frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_{0}^{\tau}(\tau-\sigma)^{\alpha-1}f(\sigma,\omega_{\epsilon}(\sigma))d\sigma, \ \tau \in J.$$

**Theorem 5.2** If the function  $f \in C(R_0, \mathbb{R})$ ,  $R_0 = \{(\tau, \omega) : \tau \in J, |\omega - \omega_0| \le b\}$  is such that

$$|f(\tau,\omega)| \leq M$$
, for all  $(\tau,\omega) \in R_0$ 

and satisfies the Lipschitz type condition,

$$|f(\tau_1,\omega) - f(\tau_2,\eta)| \le L_1 |\tau_1 - \tau_2| + L_2 |\omega - \eta|, \ \tau_1, \tau_2 \in J, \ \omega, \eta \in \mathbb{R},$$

where  $L_1 > 0$  and  $0 < L_2 < \frac{B(\alpha)}{(1-\alpha)}$ , then the family  $\{\omega_{\epsilon}\}$  of solution of the ABC-FDEs (5.1)–(5.2) is equicontinious on J.

*Proof:* Let  $\epsilon > 0$  be arbitrary. Let  $\omega_{\epsilon}(\tau)$  be the solution of the ABC-FDEs (5.1)–(5.2). Let  $\tau_1, \tau_2 \in J$  with  $0 < \tau_1 \leq \tau_2 < T$ . Then by hypotheses, we have

$$\begin{aligned} |\omega_{\epsilon}(\tau_{1}) - \omega_{\epsilon}(\tau_{2})| \\ &= \left| \left( \omega_{\epsilon}(0) + \frac{1 - \alpha}{B(\alpha)} f(\tau_{1}, \omega_{\epsilon}(\tau_{1})) + \frac{\alpha}{B(\alpha)\Gamma(\alpha)} \int_{0}^{\tau_{1}} (\tau_{1} - \sigma)^{\alpha - 1} f(\sigma, \omega_{\epsilon}(\sigma)) d\sigma \right) \right. \\ &\left. - \left( \omega_{\epsilon}(0) + \frac{1 - \alpha}{B(\alpha)} f(\tau_{2}, \omega_{\epsilon}(\tau_{2})) + \frac{\alpha}{B(\alpha)\Gamma(\alpha)} \int_{0}^{\tau_{2}} (\tau_{2} - \sigma)^{\alpha - 1} f(\sigma, \omega_{\epsilon}(\sigma)) d\sigma \right) \right| \\ &\leq \frac{1 - \alpha}{B(\alpha)} |f(\tau_{1}, \omega_{\epsilon}(\tau_{1})) - f(\tau_{2}, \omega_{\epsilon}(\tau_{2}))| \end{aligned}$$

$$+ \frac{\alpha}{B(\alpha)\Gamma(\alpha)} \left( \int_{0}^{\tau_{1}} ((\tau_{1} - \sigma)^{\alpha - 1} - (\tau_{2} - \sigma)^{\alpha - 1}) |f(\sigma, \omega_{\epsilon}(\sigma))| d\sigma + \int_{\tau_{1}}^{\tau_{2}} (\tau_{2} - \sigma)^{\alpha - 1} |f(\sigma, \omega_{\epsilon}(\sigma))| d\sigma \right) \\ \leq \frac{1 - \alpha}{B(\alpha)} (L_{1}|\tau_{1} - \tau_{2}| + L_{2}|\omega_{\epsilon}(\tau_{1}) - \omega_{\epsilon}(\tau_{2})|) + \frac{M\alpha}{B(\alpha)\Gamma(\alpha)} \int_{0}^{\tau_{1}} ((\tau_{1} - \sigma)^{\alpha - 1} - (\tau_{2} - \sigma)^{\alpha - 1}) d\sigma \\ + \frac{M\alpha}{B(\alpha)\Gamma(\alpha)} \int_{\tau_{1}}^{\tau_{2}} (\tau_{2} - \sigma)^{\alpha - 1} d\sigma$$

This gives,

$$\left|\omega_{\epsilon}(\tau_{1})-\omega_{\epsilon}(\tau_{2})\right| \leq \frac{1}{\left(1-\frac{1-\alpha}{B(\alpha)}L_{2}\right)} \left[\frac{(1-\alpha)L_{1}}{B(\alpha)}(\tau_{2}-\tau_{1})+\frac{M}{B(\alpha)\Gamma(\alpha)}\left(2(\tau_{2}-\tau_{1})^{\alpha}+\tau_{1}^{\alpha}-\tau_{2}^{\alpha}\right)\right]$$

Note that for any  $0 < \alpha < 1$ ,  $(\tau_2 - \tau_1)^{\alpha} \leq (\tau_2 - \tau_1)$  and  $\tau_1^{\alpha} - \tau_2^{\alpha} \leq 0$ . Therefore, we have

$$|\omega_{\epsilon}(\tau_1) - \omega_{\epsilon}(\tau_2)| \leq \frac{\Gamma(\alpha)(1-\alpha)L_1 + 2M}{\Gamma(\alpha) \{B(\alpha) - (1-\alpha)L_2\}} (\tau_2 - \tau_1)$$

One can check that for any  $\tilde{\epsilon} > 0$  there exists  $\tilde{\delta} = \tilde{\epsilon} \frac{\Gamma(\alpha) \{B(\alpha) - (1-\alpha)L_2\}}{\Gamma(\alpha)(1-\alpha)L_1 + 2M}$  such that if  $|\tau_1 - \tau_2| < \tilde{\delta}$ , then 6

$$|\omega_{\epsilon}(\tau_1) - \omega_{\epsilon}(\tau_2)| < \tilde{\epsilon}.$$

This proves that the family of solution  $\{\omega_{\epsilon}\}$  of the ABC-FDEs (5.1)–(5.2) is equicontinious on J. 

**Theorem 5.3** Assume that the conditions of Theorem 5.2 hold. If  $M(1 - \alpha) < bB(\alpha)$ , then the ABC-FDEs (1.1)–(1.2) has at least one solution on  $J' = [0,\beta]$ , where  $\beta = \min\left\{T, \left[\frac{\Gamma(\alpha)(bB(\alpha) - M(1-\alpha))}{M}\right]^{\frac{1}{\alpha}}\right\}.$ 

*Proof:* Fix  $\delta > 0$ . Let  $\omega_0 \in C[-\delta, 0]$  be any real valued function satisfying the conditions

$$\omega_0(0) = \omega_0, \ |\omega_0(\tau) - \omega_0| \le b.$$

For any  $\epsilon$ ,  $0 < \epsilon < \delta$ , define  $\beta_1 = \min \{\beta, \epsilon\}$ . Consider the ABC-fractional delay differential equations

$${}^{ABC}{}_{0}\mathcal{D}^{\alpha}_{\tau}\omega_{\epsilon}(\tau) = f\left(\tau, \omega_{\epsilon}(\tau-\epsilon)\right), \tau \in [0,\beta_{1}], \tag{5.3}$$

$$\omega_{\epsilon}(\tau) = \omega_0(\tau), \tau \in [-\delta, 0].$$
(5.4)

Then by Lemma 5.1, equivalent fractional integral equ. of the ABC-FDEs (5.3)-(5.4) is

$$\omega_{\epsilon}(\tau) = \begin{cases} \omega_{0}(\tau), \ \tau \in [-\delta, 0], \\ \omega_{0} + \frac{1 - \alpha}{B(\alpha)} f(\tau, \omega_{\epsilon}(\tau - \epsilon)) \\ + \frac{\alpha}{B(\alpha)\Gamma(\alpha)} \int_{0}^{\tau} (\tau - \sigma)^{\alpha - 1} f(\sigma, \omega_{\epsilon}(\sigma - \epsilon)) d\sigma, \ \tau \in [0, \beta_{1}]. \end{cases}$$
(5.5)

In the view of Corollary 4.6, the ABC-FDEs (5.3)–(5.4) is consistent only if

$$f(0,\omega_{\epsilon}(-\epsilon)) = f(0,\omega_0(-\epsilon)) = 0$$
(5.6)

One can observe that  $\omega_{\epsilon}$  is continuous on  $[-\delta, \beta_1]$  expect possibly at  $\tau = 0$ . Using continuity of f and Eq.(5.6), we have

$$\lim_{\tau \to 0^+} \omega_{\epsilon}(\tau) = \lim_{\tau \to 0^+} \left( \omega_0 + \frac{1 - \alpha}{B(\alpha)} f(\tau, \omega_{\epsilon}(\tau - \epsilon)) + \frac{\alpha}{B(\alpha)\Gamma(\alpha)} \int_0^{\tau} (\tau - \sigma)^{\alpha - 1} f(\sigma, \omega_{\epsilon}(\sigma - \epsilon)) d\sigma \right)$$
$$= \omega_0 + f(0, \omega_0(-\epsilon)) = \omega_0$$

Hence the function  $\omega_{\epsilon}(\tau): [-\delta, \beta_1] \to \mathbb{R}$  is continuous. Note that,

$$|\omega_{\epsilon}(\tau) - \omega_0| = |\omega_0(\tau) - \omega_0| \le b, \ \tau \in [-\delta, 0].$$
(5.7)

Also for any  $\tau \in [0, \beta_1]$ ,

$$\begin{aligned} |\omega_{\epsilon}(\tau) - \omega_{0}| &\leq \frac{1 - \alpha}{B(\alpha)} |f(\tau, \omega_{\epsilon}(\tau - \epsilon))| + \frac{\alpha}{B(\alpha)\Gamma(\alpha)} \int_{0}^{\tau} (\tau - \sigma)^{\alpha - 1} |f(\sigma, \omega_{\epsilon}(\sigma - \epsilon))| d\sigma \\ &\leq \frac{(1 - \alpha)M}{B(\alpha)} + \frac{M\alpha}{B(\alpha)\Gamma(\alpha)} \int_{0}^{\tau} (t - \sigma)^{\alpha - 1} d\sigma \\ &= \frac{M}{B(\alpha)} \left(1 - \alpha + \frac{\tau^{\alpha}}{\Gamma(\alpha)}\right) \leq \frac{M}{B(\alpha)} \left(1 - \alpha + \frac{\beta_{1}^{\alpha}}{\Gamma(\alpha)}\right). \end{aligned}$$
(5.8)

Since 
$$\beta_1 \leq \beta \leq \left[\frac{\Gamma(\alpha)(bB(\alpha) - M(1 - \alpha))}{M}\right]^{\frac{1}{\alpha}}$$
, we have  
$$\frac{\beta_1^{\alpha}}{\Gamma(\alpha)} \leq \frac{M}{B(\alpha)} \left(\frac{bB(\alpha)}{M} - 1 + \alpha\right)$$
(5.9)

From (5.8) and (5.9), we have

$$|\omega_{\epsilon}(\tau) - \omega_0| \le b, \tau \in [0, \beta_1] \tag{5.10}$$

From (5.7) and (5.10), we have

$$|\omega_{\epsilon}(\tau) - \omega_0| \le b, \tau \in [-\delta, \beta_1].$$
(5.11)

If  $\beta_1 < \beta$ , we consider fractional integral Eq.(5.5) on the interval  $[-\delta, \beta_2]$ , where  $\beta_2 = \min\{\beta, 2\epsilon\}$ , such that

$$|\omega_{\epsilon}(\tau) - \omega_0| \le b, \ \tau \in [-\delta, \beta_2].$$

Continuing in this way,  $\omega_{\epsilon}(\tau)$  can be extended to  $[-\delta,\beta]$ , such that

$$|\omega_{\epsilon}(\tau) - \omega_0| \le b, \ \tau \in [-\delta, \beta].$$

This gives

$$|\omega_{\epsilon}(\tau)| \le |\omega_{\epsilon}(\tau) - \omega_{0}| + |\omega_{0}| \le b + |\omega_{0}|, \ \tau \in [-\delta, \beta].$$

Therefore

$$\|\omega_{\epsilon}\| \le b + |\omega_0|, \ \tau \in [-\delta, \beta],$$

and hence  $\{\omega_{\epsilon}\}$  is uniformly bounded family of function defined on  $[-\delta,\beta]$ . Since, the hypothesis of Lemma 5.1 are satisfied, the family  $\{\omega_{\epsilon}\}$  is equicontinious. Applying Ascoli-Arzela's theorem there exists a decreasing sequence  $\{\epsilon_n\}$  with  $\epsilon_n > 0$  for all n and  $\epsilon_n \to 0$ such that  $\omega_{\epsilon_n} \to \omega$ ,  $\omega \in C([0,\beta],\mathbb{R})$  uniformly on  $[0,\beta]$ . Since  $f \in C(R_0,\mathbb{R})$ , we have  $f(\tau, \omega_{\epsilon_n}(\tau - \epsilon_n)) \to f(\tau, \omega)$ . By replacing  $\epsilon$  with  $\epsilon_n$  in equation (5.5) and then taking limit as  $n \to \infty$ , we obtain

$$\omega(\tau) = \omega_0 + \frac{1-\alpha}{B(\alpha)}f(\tau,\omega(\tau)) + \frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_0^\tau (\tau-\sigma)^{\alpha-1}f(s,\omega(s))d\sigma, \ \tau \in [0,\beta],$$

which gives the required solution of the ABC-FDEs (1.1)-(1.2).

**Theorem 5.4** Assume that the conditions of Theorem 5.2 hold. If  $(2M + b)(1 - \alpha) < bB(\alpha)$ , then the ABC-FDEs (1.1)–(1.2) has extremal solution on  $J'' = [0, \beta_0]$ , where  $\beta_0 = \min\left\{T, \left[\frac{(bB(\alpha) - (2M + b)(1 - \alpha))\Gamma(\alpha)}{(2M + b)}\right]^{\frac{1}{\alpha}}\right\}.$ 

*Proof:* We give the proof only for the existence of maximal solution of the ABC-FDEs (1.1)-(1.2), as the proof of existence of minimal solution one can complete on similar lines. For  $0 < \epsilon \leq \frac{b}{2}$ , consider the ABC-FDEs

$${}^{ABC}{}_{0}\mathcal{D}^{\alpha}_{\tau}\omega(\tau) = f(\tau,\omega(\tau)) + \epsilon := f_{\epsilon}(\tau,\omega(\tau)), \tau \in J, \tag{5.12}$$

$$\omega(0) = \omega_0 + \epsilon = \omega(0, \epsilon). \tag{5.13}$$

Define  $R_{\epsilon} = \left\{ (\tau, \omega) : \tau \in J, |\omega - \omega(0, \epsilon)| \leq \frac{b}{2} \right\}$ . Clearly  $R_{\epsilon} \subset R_0$ . Consider the function  $f_{\epsilon} : R_{\epsilon} \to \mathbb{R}$  defined by

$$f_{\epsilon}(\tau,\omega(\tau)) = f(\tau,\omega(\tau)) + \epsilon.$$

Then,  $f_{\epsilon}$  satisfies the Lipschitz type condition with same Lipschitz constants  $L_1, L_2$  as defined in Theorem 5.2. Further,

$$|f_{\epsilon}(\tau,\omega(\tau))| \le M + \frac{b}{2}, \ (\tau,\omega) \in R_{\epsilon}.$$

Since  $f_{\epsilon}$  satisfies all assumptions of Theorem 5.3, by applying it, the ABC-FDEs (5.12)–(5.13) has at least one solution  $\omega(\tau, \epsilon)$  on J''.

Let  $0 < \epsilon_2 < \epsilon_1 \leq \epsilon$ , then we have

$$\begin{split} &\omega(0,\epsilon_2) = \omega_0 + \epsilon_2 < \omega_0 + \epsilon_1 = \omega(0,\epsilon_1); \\ & {}^{ABC}{}_0\mathcal{D}^{\alpha}_{\tau}\omega(\tau,\epsilon_2) = f\left(\tau,\omega(\tau,\epsilon_2)\right) + \epsilon_2, \ \tau \in J''; \\ & {}^{ABC}{}_0\mathcal{D}^{\alpha}_{\tau}\omega(\tau,\epsilon_1) > f\left(\tau,\omega(\tau,\epsilon_1)\right) + \epsilon_2, \ \tau \in J''. \end{split}$$

Note that  $\omega(\tau, \epsilon_1)$  and  $\omega(\tau, \epsilon_2)$  respectively are lower and upper solutions of ABC-FDEs, with  $\omega(0, \epsilon_2) < \omega(0, \epsilon_1)$ . Therefore by applying Theorem 4.1 we have,

$$\omega(\tau,\epsilon_2) < \omega(\tau,\epsilon_1), \ \tau \in J''.$$

Next we show that the family  $\{\omega(\tau, \epsilon)\}$  of solutions of the ABC-FDEs (5.12)–(5.13) is uniformally bounded on J''. Proceeding as in the proof of Theorem 5.3, for any  $\tau \in J''$ , we have

$$|\omega(\tau,\epsilon) - \omega(0,\epsilon)| \le \frac{2M+b}{2B(\alpha)} \left(1 - \alpha + \frac{\beta_0^{\alpha}}{\Gamma(\alpha)}\right), \tau \in J''.$$

Since  $\beta_0 \leq \left[\frac{(bB(\alpha) - (2M + b)(1 - \alpha))\Gamma(\alpha)}{(2M + b)}\right]^{\frac{1}{\alpha}}$ , above inequality reduces to

$$|\omega(\tau,\epsilon) - \omega(0,\epsilon)| \le \frac{2M+b}{2B(\alpha)} \left(1 - \alpha + \frac{bB(\alpha)}{2M+b} - 1 + \alpha\right) \le \frac{b}{2} < b, \ \tau \in J''.$$

Since  $f_{\epsilon}$  satisfies assumptions of Theorem 5.2, the family  $\{\omega(\tau, \epsilon)\}$  is equicontinious on J. Applying Ascoli-Arzela's theorem there exists a decreasing sequence  $\{\epsilon_n\}$  with  $\epsilon_n > 0$  for all n and  $\epsilon_n \to 0$  such that  $\omega(\tau, \epsilon_n) \to \eta(\tau)$  uniformly on J'', where  $\eta \in C([0, \beta_0], \mathbb{R})$  uniformly on  $[0, \beta]$ . By uniform continuity of  $f_{\epsilon}$ , we have

$$f_{\epsilon_n}(\tau,\omega(\tau,\epsilon_n)) \to f(\tau,\eta(\tau)), \ \tau \in J''.$$

By replacing  $\epsilon$  with  $\epsilon_n$  in equation (5.12) and then taking limit as  $n \to \infty$ , we obtain

$$\begin{aligned} {}^{ABC}{}_0\mathcal{D}^{\alpha}_{\tau}\eta(\tau) &= f\left(\tau,\eta(\tau)\right), \tau \in J'', \\ \eta(0) &= \omega_0. \end{aligned}$$

This proves that  $\eta(\tau)$  is a solution of ABC-FDEs(1.1)–(1.2).

It remains to prove that  $\eta(\tau)$  is the maximal solution of the ABC-FDEs (1.1)–(1.2). Let  $\omega(\tau)$  be any solution of (1.1)–(1.2) on J''. Then for any  $\epsilon > 0$ ,

$$\begin{split} \omega_0 &= \omega(0) < \omega_0 + \epsilon = \omega(0, \epsilon); \\ {}^{ABC}{}_0 \mathcal{D}^{\alpha}_{\tau} \omega(\tau) > f\left(\tau, \omega(\tau, \epsilon_2)\right) + \epsilon, \ \tau \in J''; \\ {}^{ABC}{}_0 \mathcal{D}^{\alpha}_{\tau} \omega(\tau, \epsilon) &= f\left(\tau, \omega(\tau, \epsilon)\right) + \epsilon, \ \tau \in J''. \end{split}$$

Note that  $\omega(\tau)$  and  $\omega(\tau, \epsilon)$  respectively are lower and upper solutions of ABC-FDEs, with  $\omega_0 < \omega_0 + \epsilon$ . Therefore by applying Theorem 4.1 we have,

$$\omega(\tau) < \omega(\tau, \epsilon), \ \tau \in J''.$$

Taking limit as  $\epsilon \to 0$ , we obtain

$$\omega(\tau) \le \eta(\tau), \ \tau \in J''.$$

This proves that  $\eta(\tau)$  is the maximal solution of the ABC-FDEs (1.1)–(1.2) on J".

# 6 Existence of Global Solution

**Theorem 6.1** Assume that m is any differentiable function defined on J such that  ${}^{ABC}{}_0\mathcal{D}^{\alpha}_{\tau}m \in C(J)$  and  $g \in C(J \times \mathbb{R}_+, \mathbb{R}_+)$  such that

$${}^{ABC}{}_0\mathcal{D}^{\alpha}_{\tau}m(\tau) \le g(\tau, m(\tau)), \ \tau \in J$$
(6.1)

and let  $\eta(\tau)$  is the maximal solution of the ABC-FDEs

$$\begin{split} {}^{ABC}{}_0\mathcal{D}^{\alpha}_{\tau}u(\tau) &= g\left(\tau, u(\tau)\right), \tau \in J, \\ u(0) &= u_0. \end{split}$$

Then  $m(0) \le u(0)$ , implies  $m(\tau) \le \eta(\tau), \ \tau \in J$ .

*Proof:* Let  $\epsilon > 0$  be arbitrary. Let  $u(\tau, \epsilon)$  be a solution of the ABC-FDEs of the form

$${}^{ABC}{}_0\mathcal{D}^{\alpha}_{\tau}u(\tau) = g\left(\tau, u(\tau)\right) + \epsilon, \tau \in J,$$
$$u(0) = u_0 + \epsilon.$$

Therefore,

$${}^{ABC}{}_{0}\mathcal{D}^{\alpha}_{\tau}u(\tau,\epsilon) > g\left(\tau,u(\tau,\epsilon)\right), \tau \in J,$$
$$u(0) = u_{0} + \epsilon \tag{6.2}$$

Therefore m is lower solution and  $u(\tau, \epsilon)$  is upper solution of the ABC-FDE

$${}^{ABC}{}_{0}\mathcal{D}^{\alpha}_{\tau}\omega(\tau) = g\left(\tau,\omega(\tau)\right), \tau \in J$$

Further,

$$m(0) \le u(0) < u(0) + \epsilon = u(0, \epsilon).$$

By applying Theorem 4.1, we obtain

$$m(\tau) \le u(\tau,\epsilon), \ \tau \in J, \ \epsilon > 0.$$

Taking  $\epsilon \to 0$ , and following similar approach as in the proof of Theorem 5.4, we obtain

$$m(\tau) \le \eta(\tau), \ \tau \in J.$$

**Theorem 6.2** Assume that  $f \in C([0,\infty) \times \mathbb{R}, \mathbb{R})$  and  $g \in C([0,\infty) \times \mathbb{R}_+, \mathbb{R}_+)$  are such that  $|f(\tau,\omega)| \leq g(\tau, |\omega|)$ . Further, suppose that there exists local solution  $\omega(\tau, \omega_0)$  of the ABC-FDEs

$${}^{ABC}{}_{0}\mathcal{D}^{\alpha}_{\tau}\omega(\tau) = f(\tau,\omega(\tau)), \tau \in [0,\infty), \tag{6.3}$$

$$\omega(0) = \omega_0,\tag{6.4}$$

and maximal solution  $\eta(\tau)$  of

$$^{ABC}{}_0\mathcal{D}^{\alpha}_{\tau}u(\tau) = g\left(\tau, u(\tau)\right), \tau \in [0, \infty),$$
  
 $u(0) = u_0 \ge 0.$ 

Then the largest interval of existence of solution  $\omega(\tau, \omega_0)$  of the (6.3)–(6.4) such that  $|\omega_0| < u_0$  is  $[0, \infty)$ .

Proof: By assumption there exists a local solution  $\omega(\tau, \omega_0)$  of the (6.3)–(6.4) on the interval  $[0, \beta)$ , where  $\beta < \infty$  with  $|\omega_0| < u_0$ . Suppose on contrary  $\beta$  can not be increased further. Define  $m(\tau) = |\omega(\tau, \omega_0)|, \ \tau \in [0, \beta)$ . As  $|f|' \leq |f'|$ , for any  $\tau \in [0, \beta)$ , we find

$${}^{ABC}{}_{0}\mathcal{D}^{\alpha}_{\tau}m(\tau) = \frac{B(\alpha)}{1-\alpha} \int_{0}^{\tau} \mathbb{E}_{\alpha} \left(-\frac{\alpha}{1-\alpha}(\tau-\sigma)^{\alpha}\right) m'(\sigma) d\sigma$$
$$= \frac{B(\alpha)}{1-\alpha} \int_{0}^{\tau} \mathbb{E}_{\alpha} \left(-\frac{\alpha}{1-\alpha}(\tau-\sigma)^{\alpha}\right) |\omega(\sigma,\omega_{0}|' d\sigma)$$
$$\leq \frac{B(\alpha)}{1-\alpha} \int_{0}^{\tau} \mathbb{E}_{\alpha} \left(-\frac{\alpha}{1-\alpha}(\tau-\sigma)^{\alpha}\right) |\omega'(\sigma,\omega_{0}| d\sigma)$$
$$= \left|\frac{B(\alpha)}{1-\alpha} \int_{0}^{\tau} \mathbb{E}_{\alpha} \left(-\frac{\alpha}{1-\alpha}(\tau-\sigma)^{\alpha}\right) \omega'(\sigma,\omega_{0}) d\sigma\right|$$
$$= |{}^{ABC}{}_{0}\mathcal{D}^{\alpha}_{\tau}\omega(\tau,\omega_{0})|$$

Therefore,

$${}^{ABC}{}_0\mathcal{D}^{\alpha}_{\tau}m(\tau) \le |{}^{ABC}{}_0\mathcal{D}^{\alpha}_{\tau}\omega(\tau,\omega_0)| = |f(\tau,\omega(\tau,\omega_0))| \le g(\tau,|\omega(\tau,\omega_0)|) = g(\tau,m(\tau)), \ \tau \in [0,\beta).$$

This implies  $m(\tau)$  is lower solution of

$${}^{ABC}{}_0\mathcal{D}^{\alpha}_{\tau}u(\tau) = g\left(\tau, u(\tau)\right), \tau \in [0, \beta).$$

$$(6.5)$$

Further by assumption  $\eta(\tau)$  is the maximal solution of the problem (6.5). Since  $m(0) = |\omega_0| \le u_0$ , by applying Theorem 4.1, we obtain

$$m(\tau) = |\omega(\tau, \omega_0)| \le \eta(\tau), \ \tau \in [0, \beta).$$

By assumption  $\eta(\tau)$  exists on  $[0, \infty)$ . Therefore by continuity of g on  $[0, \beta] \times \mathbb{R}_+$ , there exists M > 0 such that,

$$|g(\tau,\eta(\tau))| \le M, \ \tau \in [0,\beta).$$

Let  $0 \le \tau_1 \le \tau_2 < \beta$ . Then following similar steps as in the proof of Theorem 4.1, we have

$$|\omega(\tau_1,\omega_0) - \omega(\tau_2,\omega_0)| \le \frac{\Gamma(\alpha)(1-\alpha)L_1 + 2M}{\Gamma(\alpha)\left[B(\alpha) - (1-\alpha)L_2\right]}(\tau_2 - \tau_1).$$

From above inequality it follows that,

$$\lim_{\tau_1, \tau_2 \to \beta^-} \omega(\tau_1, \omega_0) = \lim_{\tau_2, \tau_1 \to \beta^-} \omega(\tau_2, \omega_0),$$

for any  $\tau_1, \tau_2$  with  $0 \le \tau_1 \le \tau_2 < \beta$ . This implies that  $\lim_{\tau \to \beta^-} \omega(\tau, \omega_0)$  exists. Let

$$\omega(\beta,\omega_0) = \lim_{\tau \to \beta^-} \omega(\tau,\omega_0).$$

Then by assumption the ABC-FDEs

$$ABC_0 \mathcal{D}^{\alpha}_{\tau} \omega(\tau) = f(\tau, \omega(\tau)), \tau \ge \beta,$$
$$\omega(\beta) = \omega_{\beta},$$

has a local solution. This implies that  $\omega(\tau, \omega_0)$  can be continued beyond  $\beta$  which is a contradiction our assumption. Hence every solution  $\omega(\tau, \omega_0)$  of the ABC-FDEs (6.3)–(6.4) exists on  $[0, \infty)$ .

## Conclusion

The comparison results, local, extremal, and global existence of a solution, derived without demanding the monotonicity and Holder continuity assumption on the nonlinear function involving ABC-FDEs. The estimations on ABC-fractional derivative and the comparison results obtained to ABC-FDEs one can use to research the existence, uniqueness and qualitative properties of solutions for various classes of ABC-FDEs.

### References

- K. S. Miller, B. Ross, An introduction to the fractional calculus and differential equations, John Wiley, New York, 1993.
- [2] I. Podlubny, Fractional differential equations, Academic Press, San Diego, 1999.
- [3] A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, Theory and applications of fractional differential equations, North-Holland Mathematics Studies, vol. 204, Elsevier Science B.V., Amsterdam, 2006.
- [4] V. Lakshmikantham, S. Leela, Theory of fractional dynamic systems, Cambridge Scientific Publisher LTD. Cambridge, 2009.
- [5] K. Diethelm, The analysis of fractional differential equations, Lecture Notes in Mathematics, *Springer-verlag Berlin Heidelberg*, 2010.
- [6] K. Diethelm, N.J. Ford, Analysis of fractional differential equations, J. Math. Anal. Appl., 265 (2002), 229-248.

- [7] V Daftardar-Gejji, A Babakhani, Analysis of a system of fractional differential equations, Journal of Mathematical Analysis and Applications, 293 (2004) 511522.
- [8] V. Lakshimikantham, A. S. Vatsala, Theory of fractional differential inequalities and applications, *Communications in Applied Analysis*, 11 (2007), 395-402.
- [9] V. Lakshimikantham, A. S. Vatsala, Basic Theory of fractional differential equations, *Nonlinear Analysis*, 69 (2008), 2677-2682.
- [10] V. Lakshimikantham, Theory of fractional functional differential equations, Nonlinear Analysis, 69 (2008), 3337-3343.
- [11] V. Lakshimikantham, A. S. Vatsala, General uniqueness and monotone iterative technique for fractional differential equations *Applied Mathematics Letters*, 21 (2008), 828-834.
- [12] R. P. Agarwal, Yong Zhou, Yunyun Heb, Existence of fractional neutral functional differential equations, *Computers and Mathematics with Applications*, 59 (2010), 1095-1100.
- [13] J. Wang, L. Lv, Y. Zhou, Ulam stability and data dependence for fractional differential equations with Caputo derivative, *Electronic Journal of Qualitative Theory of Differential Equations*, 63 (2011), 1–10.
- [14] N. D. Cong, H. T. Tuan, H. Trinh, On asymptotic properties of solutions to fractional differential equations, *Journal of Mathematical Analysis and Applications*, (2)484, (2020).
- [15] G. Iskenderoglu, D. Kaya, Symmetry analysis of initial and boundary value problems for fractional differential equations in Caputo sense, *Chaos, Solitons and Fractals* 134 (2020), 109684.
- [16] M. Caputo, M. Fabrizio, A New Definition of Fractional Derivative Without Singular Kernel, Progress in Fractional Differentiation and Applications, 1(2015),73-85.
- [17] D. Baleuno, A. Jajarmi, H. Mohammadi, S. Rezapour, A new study of mathematical modelling of human liver with Caputo–Fabrizio fractional derivative, *Chaos, Solitons and Fractals*,134 (2020) 109705. Aydogan et al. Boundary Value Problems (2018) 2018:90
- [18] M. S. Aydogan, D. Baleanu, A. Mousalou and S. Rezapour, On high order fractional integro-differential equations including the CaputoFabrizio derivative, Aydogan et al. Boundary Value Problems (2018) 2018:90 https://doi.org/10.1186/s13661-018-1008-9.
- [19] J. Losada, Juan J. Nieto, Properties of a New Fractional Derivative without Singular Kernel, Progr. Fract. Differ. Appl. (2) 1 (2015), 87–92.
- [20] A. Atangana, D.Baleanu, Caputo-Fabrizio derivative applied to groundwater flow within confined aquifer, J. Eng. Mech., (5) 143 (2017), : D4016005.
- [21] A. Atangana, D. Baleanu, New fractional derivatives with nonlocal and non-singular kernel: theory and application to heat transfer model, *Therm. Sci.* 2016 20(2), 763-69.

- [22] T. Abdeljawad, D. Baleanu, Integration by parts and its applications of a new nonlocal fractional derivative with Mittag-Leffler nonsingular kernel, J Nonlinear Sci Appl (3)10(2017), 1098-1107.
- [23] D. Baleuno, A. Fernandez, On some new properties of fractional derivatives with Mittag-Leffler kernel, Commun Nonlinear Sci Numer Simulat, 59 (2018), 444-462.
- [24] A. Fernandez, A complex analysis approach to Atangana-Baleanu fractional calculus, http://arxiv.org/abs/1905.06834v1.
- [25] Mohammed Al-Refai, Comparison principles for differential equations involving Caputo fractional derivative with Mittag-Leffler Non-Singular kernel, *Electronic Journal of Differential Equations*, 36 (2018),1–10.
- [26] A. Atangana, I. Koca; Chaos in a simple nonlinear system with Atangana-Baleanu derivatives with fractional order, *Chaos, Solitons and Fractals*, 89 (2016), 447-454.
- [27] A. Jajarmi, B. Ghanbari, D. Baleuno, A new and efficient numerical method for the fractional modeling and optimal control of diabetes and tuberculosis co-existence, *Chaos*, 29, 093111 (2019).
- [28] A. Jajarmi, S. Arshad, D. Baleuno, A new fractional modeling and control strategy for the outbreak of dengue fever, *Physica A*, 535 (2019), 122524.
- [29] D. Baleuno, A. Jajarmi, S.S.Sajjadi, D. Mozyrska, A new fractional model and optimal control of a tumor-immune surveillance with non-singular derivative operator, *Chaos*, 29, 083127 (2019).
- [30] A. Jajarmi, D. Baleuno, S.S.Sajjadi, J. H. Asad, A new features of the fractional Euler-Lagrange equation for a coupled oscillator using a nonsingular operator approach, *Front. phys.* doi:10.3389/fPhy.2019.00196.
- [31] S. Ucar, E. Ucar, N. Ozdemir, Z. Hammouch, Mathematical analysis and numerical simulation for a smoking model with Atangana–Baleuno derivative, *Chaos, Solitons* and Fractals, 118 (2019), 300–306.
- [32] M. S. Abdo, K. Shah, H. A. Wahash, S. K. Panchal, On comprehensive model of the novel coronavirus (COVID-19) under Mittag-Leffler derivative, *Chaos, Solitons and Fractals*, 135 (2020), 109867.
- [33] Fahd Jarad, Thabet Abdeljawad, Zakia Hammouch, On a class of ordinary differential equations in the frame of AtanganaBaleanu fractional derivative, *Chaos, Solitons and Fractals* 117 (2018) 16-20.
- [34] D. Baleuno, A. Jajarmi, M. Hajipour, On the nonlinear dynamical systems within the generalized fractinal derivative with Mittag–Leffler kernel, *Nonlinear Dyn* 94 (2018), 397–414.
- [35] M.I. Syam, Mohammed Al-Refai, Fractional differential equations with Atangana-Baleanu fractional derivative: Analysis and applications, *Chaos, Solitons and Fractals*, X 2 (2019) 100013.

- [36] Hojjat Afshari, Dumitru Baleanu, Applications of some fixed point theorems for fractional differential equations with Mittag-Leffler kernel, Advances in Difference Equations, (2020) 2020:140.
- [37] K. Logeswari, C. Ravichandran, A new exploration on existence of fractional neutral integrodifferential equations in the concept of Atangana-Baleanu derivative, *Physica A* 544 (2020), 123454.
- [38] C. Ravichandran, K. Logeswari, Fahd Jarad, New results on existence in the framework of AtanganaBaleanu derivative for fractional integro-differential equations, *Chaos, Solitons and Fractals*, 125 (2019), 194–200.
- [39] C. Ravichandran, K. Logeswari, S. K. Panda, K. S. Nisar, On new approach of fractional derivative by Mittag-Leffler kernel to neutral integro-differential systems with impulsive conditions, Chaos, *Chaos, Solitons and Fractals*, 139 (2020) 110012.
- [40] C. Ravichandran, K. Logeswari, S. K. Panda, K. S. Nisar, On new approach of fractional derivative by Mittag–Leffler kernel to neutral integro–differential systems with impulsive conditions, *Chaos, Solitons and Fractals*,139 (2020), 110012.
- [41] N. Valliammal, C. Ravichandran, K. S. Nisar, Solutions to fractional neutral delay differential nonlocal systems, *Chaos, Solitons and Fractals*, 138 (2020) 109912
- M.Sher, [42] K. Т. Abdeljawad, Study of Shah, evolution problem under Mittag-Leffler type fractional order derivative, Alexandria Eng. (2020),htpp://doi.org/10.1016/j.aej.2020.06.050
- [43] T. R. Prabhakar, A singular integral equation with a generalized Mittag-Leffler function in the kernel, Yokohama. Math. J., 19, 7–15.(1971).
- [44] A. A. Kilbas, M. Saigo, K. Saxena, Generalized Mittag-Leffler function and generalized fractional calculus operators, *Integral Transforms Spec. Funct.*, 15 (2004) 31-49.
- [45] A. Erdelyi, W. Magnus, F. Oberhettinger, and F. G. Tricomi,(1953), Higher Transcendental Functions, Vol. I. McGraw-Hill, NewYork-Toronto-London.
- [46] J. D. Ramirez, A. S. Vatsala, Generalized monotone iterative technique for Caputo fractional differential equation with perodic boundry conditions via Initial value problem, *International Journal of Differential Equations*, doi:10.1155/2012/842813.