unit : I.

M.C.Q.

- Newton's law of motion : Low of Inertia.

Every partical continuous to move in a straight state of uniform motion in a straight line & remains at rest, unless acted upon by an a external force.

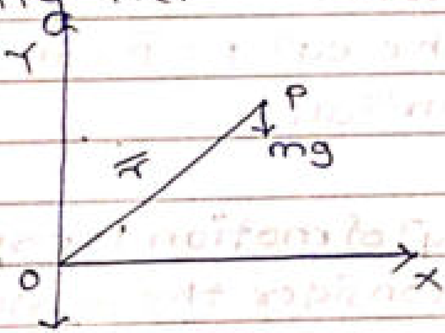
- Linear momentum :

consider a partical of mass m and position vector $\bar{r}$ linear momentum ($\bar{p}$) of a partical is a product of mass and it's velocity i.e.

$$\bar{p} = m \cdot \bar{v}$$

$$= m \cdot \frac{d\bar{r}}{dt}$$

$$\bar{p} = m \cdot \dot{\bar{r}}$$



- Second law of motion :

The time rate of change of linear momentum of a partical is proportional to force acting on it & is in the direction of these force

$\bar{F} \propto \frac{d\bar{p}}{dt}$

$$\bar{F} \propto \frac{d\bar{p}}{dt}$$

$\bar{F} = \dot{\bar{p}}$

$$\bar{F} = k \cdot \dot{\bar{p}}$$

Where k is the constant of proportionality. Whose value depends on unit choosen. In general we take $k = 1$ by a special choice of unit of force.

$$\therefore \bar{F} = \dot{\bar{p}}$$

$$= \frac{d}{dt}(mv)$$

$$= m\frac{d\bar{v}}{dt}$$

$$\bar{F} = m\bar{a}$$

where $\bar{a}$ is acceleration

- Third law of motion:
The forces of action & reaction betn two interacting bodies are equally in magnitude & opposite in direction and are co-linear

Note:
The eqn $\bar{F} = \dot{\bar{P}} = m\bar{a}$ is eqn of motion of a single partical

- Eqn of motion for system of particles
consider the system of n particales of masses $m_1, m_2, \ldots, m_n$ and position vectors $\bar{r}_1, \bar{r}_2, \ldots, \bar{r}_n$
Any partical of the system will experience two types of forces.
1) External force - $\bar{F}_i^{(e)}$; $i = 1, 2, \ldots, n$
2) Internal force - $\bar{F}_{ji}^{(int)}$; $i, j = 1, 2, \ldots, n$
Therefore, the total force acting on i^{th} partical is

$$\bar{F}_i = \bar{F}_i^{(e)} + \sum_j \bar{F}_{ji}^{(int)}$$

Where. $\sum_j \bar{F}_{ji}^{(int)}$ is called total internal force acting on i^{th} partical due to intension of all other $(n-1)$ particales in the system.
$\therefore$ By Newton's second law we have

$$\bar{F}_i = \dot{\bar{P}}_i$$

$\therefore$ We have

F_{ji} → jth portical acting force on ith portical.

$$\bar{F}_i^{(e)} + \sum_j \bar{F}_{ji}^{(int)} = \dot{\bar{p}}_i$$

Where, $\bar{p}_i$ is linear momentum of ith portical.
The eq^n of motion of the system is obtained by summing the eq^n

$$\therefore \sum_i \bar{F}_i^{(e)} + \sum_i \sum_j \bar{F}_{ji}^{(int)} = \sum_i \dot{\bar{p}}_i$$

i.e. $\sum_i \bar{F}_i^{(e)} + \sum_{i,j} \bar{F}_{ji}^{(int)} = \sum_i \dot{\bar{p}}_i$ --- $(\because \sum_i \sum_j = \sum_{i,j})$ --- (1)

Now $\bar{F}_{ii}^{(int)} = 0$ i.e. force on portical by itself is zero. and by using Newton's third law we have,

$$\bar{F}_{ij}^{(int)} = -\bar{F}_{ji}^{(int)}$$

$\therefore$ Eq^n (1) becomes

$$\sum_i F_i^{(e)} = \sum_i \dot{\bar{p}}_i \qquad \Rightarrow \dot{\bar{p}} = \bar{F}^{(e)}$$

$$\Rightarrow \dot{\bar{p}} = \bar{F}$$

i.e. $\bar{F}^{(e)} = \dot{\bar{p}}$ ----- (2) $\qquad \bar{F} = \bar{F}^{(e)}$.

Where $\bar{F}^{(e)}$ is total external force and $\bar{p}$ total linear momentum of system

Eq^n (2) is called eq^n of motion of system of portical.

I.M.P

- Theorem

State and prove conservation theo. for linear momentum of system of particles

statement- If the sum of external forces acting on the particle in the system is zero then total linear momentum of that system is conserved / const.

proof: From eq^n (2) i.e. eq^n of motion of system of particle we have

If $\bar{F}^{(e)} = 0$

$$\Rightarrow \dot{\bar{p}} = 0$$

$$\Rightarrow \frac{d\bar{p}}{dt} = 0$$

$$\Rightarrow \bar{p} = \text{constant}$$

i.e. linear momentum is constant

- **Angular momentum:**
If $\bar{p} = m\bar{v}$ is a linear momentum of a particle of mass m & position vector $\bar{r}$ then the angular momentum of the particle is $\bar{L} = \bar{r} \times \bar{p}$

- **Torque:**
Torque is time rate of change of angular momentum
i.e. $\bar{N} = \frac{d\bar{L}}{dt}$

$$= \frac{d}{dt}(\bar{r} \times \bar{p})$$

$\bar{N}_i = \frac{d_i}{d}$ $\quad \bar{L} = \bar{r} \times \bar{p}$

$$= \bar{r} \times m\bar{v}$$

$$= \bar{r} \times m\dot{\bar{r}}$$

$$\frac{d\bar{L}}{dt} = \frac{d}{dt}(\bar{r} \times m\dot{\bar{r}})$$

$$= \bar{r} \times m\ddot{\bar{r}} + \dot{\bar{r}} \times m\dot{\bar{r}} \quad (\because \dot{\bar{r}} \times m\dot{\bar{r}} = 0$$
$$\bar{r} \times \bar{r} = 0)$$

$$\bar{N} = \frac{d\bar{L}}{dt} = \bar{r} \times m\bar{a}$$

$$\bar{N} = \bar{r} \times \bar{F} \quad (\because \bar{F} = m\bar{a})$$

- **Theo.**
Angular momentum of system of particles
consider a system of particles of masses $m_1, m_2, \ldots, m_n$ and position vectors $\bar{r}_1, \bar{r}_2, \ldots, \bar{r}_n$ respectively.
The angular momentum of ith particle is
$$\bar{L}_i = \bar{r}_i \times \bar{p}_i$$

$F = \frac{dP}{dt} = m\dot{v} = ma = m\ddot{r}$ $\quad \bar{r}\times\bar{P}$

$N = \frac{dL}{dt}$ $\quad \bar{r}\times\bar{F}$

$\therefore$ Total angular momentum of system is

$$\bar{L} = \sum_{i=1}^{n} \bar{L}_i$$

$$= \sum_i \bar{r}_i \times \bar{p}_i$$

$\therefore$ Total Torque of the system is

$$\bar{N} = \frac{d\bar{L}}{dt}$$

$$= \frac{d}{dt}\left(\sum_i \bar{r}_i \times \bar{p}_i\right)$$

$$= \sum_i \bar{r}_i \times \dot{\bar{p}}_i + \underbrace{\sum_i \dot{\bar{r}}_i \times \bar{p}_i}_{=0} \quad \text{---- (1)}$$

We know that,

$$\bar{p}_i = m\bar{v}_i$$

$$\bar{p}_i = m\dot{\bar{r}}_i$$

$$\therefore \quad \dot{\bar{r}}_i \times m\dot{\bar{r}}_i = 0$$

$\therefore$ Eq^n (1) becomes

$$\bar{N} = \sum_i \bar{r}_i \times \dot{\bar{p}}_i \quad \text{----- (2)}$$

Now,

$$\dot{\bar{p}}_i = \bar{F}_i$$

But,

$$\bar{F}_i = \bar{F}_i^{(e)} + \sum_j \bar{F}_{ji}^{(int)}$$

$$\therefore \quad \bar{r}_i \times \dot{\bar{p}}_i = \bar{r}_i \times \left[\bar{F}_i^{(e)} + \sum_j \bar{F}_{ji}^{(int)}\right]$$

$$= \bar{r}_i \times \bar{F}_i^{(e)} + \bar{r}_i \times \sum_j \bar{F}_{ji}^{(int)}$$

$$= \bar{N}_i^{(e)} + \bar{r}_i \times \sum_j \bar{F}_{ji}^{(int)} \quad \text{---- (3)}$$

Where,

$\bar{N}_i^{(e)} = \bar{r}_i \times \bar{F}_i^{(e)}$ is external torque of ith particle.

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$\therefore$ By (2) and (3) we have

$$\bar{N} = \sum_i \bar{N}_i^{(e)} + \sum_i \bar{r}_i \times \sum_j \bar{F}_{ji}^{(int)} \quad ----(4)$$

consider the 2nd term from the above eqn

$$\sum_i \bar{r}_i \times \sum_j \bar{F}_{ji}^{(int)} = \bar{r}_1 \times \bar{F}_{11}^{(int)} + \bar{r}_1 \times \bar{F}_{21}^{(int)} + \cdots + \bar{r}_1 \times \bar{F}_{n1}^{(int)}$$
$$+ \bar{r}_2 \times \bar{F}_{12}^{(int)} + \bar{r}_2 \times F_{22}^{(int)} + \cdots + \bar{r}_2 \times \bar{F}_{n2}^{(int)}$$
$$+$$
$$\vdots$$
$$+ \bar{r}_n \times \bar{F}_{1n}^{(int)} + \bar{r}_n \times \bar{F}_{2n}^{(int)} + \cdots + \bar{r}_n \times \bar{F}_{nn}^{(int)}$$

$$= 0 - \bar{r}_1 \times \bar{F}_{12}^{(int)} - \bar{r}_1 \times \bar{F}_{13}^{(int)} - \cdots (-\bar{r}_1 \times \bar{F}_{1n}^{(int)})$$
$$+ \bar{r}_2 \times \bar{F}_{12}^{(int)} + 0 + \bar{r}_2 \times \bar{F}_{32}^{(int)} + \cdots + \bar{r}_2 \times \bar{F}_{n2}^{(int)}$$
$$+$$
$$\vdots$$
$$+ \bar{r}_n \times \bar{F}_{1n}^{(int)} + \bar{r}_n \times \bar{F}_{2n}^{(int)} + \cdots + \underbrace{\cancel{\bar{r}_n \times \bar{F}_{nn}^{(int)}}}_{0}$$

$$= (\bar{r}_2 - \bar{r}_1) \times \bar{F}_{12}^{(int)} + (\bar{r}_3 - \bar{r}_1) \times \bar{F}_{13}^{(int)}$$
$$+ (\bar{r}_4 - \bar{r}_1) \times \bar{F}_{14}^{(int)} + \cdots$$

$$= |\bar{r}_i - \bar{r}_j| \times \bar{F}_{ji}^{(int)}; \quad i, j = 1, 2, \ldots, n.$$

$$\sum_i \bar{r}_i \times \sum_j \bar{F}_{ji}^{(int)} = \bar{r}_{ij} \times \bar{F}_{ji}^{(int)} \quad ; \text{where } \bar{r}_{ij} = |\bar{r}_i - \bar{r}_j| \quad ------(5)$$

Interchanging i and j on R.H.S of eqn (5) we have,

$$\sum_i \bar{r}_i \times \sum_j \bar{F}_{ji}^{(int)} = \bar{r}_{ji} \times \bar{F}_{ij}^{(int)}$$
$$= -\bar{r}_{ij} \times \bar{F}_{ji}^{(int)} \quad ----(6)$$

Adding eqn (5) & (6) we have

$$2\left(\sum_i \bar{r}_i \times \sum_j \bar{F}_{ji}^{(int)}\right) = 0$$

$$\sum_i \bar{r}_i \times \sum_j \bar{F}_{ji}^{(int)} = 0$$

$\therefore$ from eqn (4) we have,

$$\bar{N} = \sum_i \bar{N}_i^{(e)}$$

i.e. $\bar{N} = \bar{N}^{(e)}$ ------- (7)

Thus, total torque on the system is the sum of the external torque actiong on particles in the system.

I.M.P. • Theo.

State and prove conservation theo. for angular momentum of system of particles.

Statement:

If the total external torque acting on system of particles is zero then total angular momentum of system of particles is conserved or constant.

proof:

from eqn (7).

If $\bar{N}^{(e)} = 0$

$\Rightarrow \bar{N} = 0$

$\Rightarrow \frac{d\bar{L}}{dt} = 0$

$\Rightarrow \bar{L} = \text{const.}$

i.e. angular momentum is constant.

Hence, the proof

VIMP

• constraints and constrained motion:

Defin: Some times the motion of physical system is not free & it is limited by putting some restrictions on position co-ordinate of the particle involved in the system.

The motion under such restriction is called as constrained motion. and restriction are called as constraints.

The mathematical relation betn position co-ordinate due to constraints are called as constraints relation

- Examples of constrained motion:

i) In the case of rigid body distance betn any two points is always const.

ii) Motion of simple pendulum.

iii) Motion of particle on a plane curve $y=f(x)$

iv) The motion of particle placed on sphere is restricted so that it can move either on the surface of the sphere or outside the sphere

IMP

- classification of constraints:

A) classification based on nature of the constraint relation.

a) Holonomic constraint:

If the constraint relation of the system is described by the eqn of the form $f_j(q_j, t) = 0$ Where, q_j are position-co-ordinates and t is time, then the constraints are called as holonomic constraints.

eg //

i) rigid body.

ii) simple pendulum

b) Non-holonomic constraint:

The constraints which can be expressed in the form of inequality are called as non-holonomic constraints

e.g.:

i) Gas molecule moving in the closed container.

B) classification based on explicit involvement of time in a relation:-

a) Scleronomic constraints: (scleronomus constraints)

If the constraints relation do not involve time explicitly then they are called as scleronomic constraints.

e.g.

i) simple pendulum

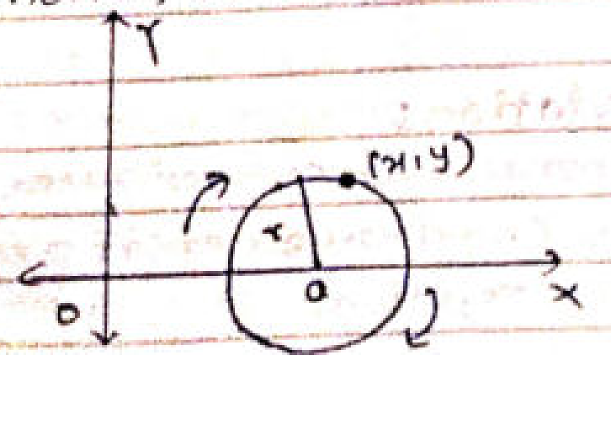
ii) particle moving on parabola. $y^2 = 4ax$ i.e. $y^2 = x$

b) Rheonomic constraints:

If constraints contains time explicitely are known as rheonomic constraints.

e.g.

i) consider a bead is sliding on a moving wire. if the centre of this wire is on a x-axis then the eqn of constraints will be $(x-a(t))^2 + y^2 = r^2$ where order $P \cdot (x,y)$ are position co-ordinates of a bead, r is radius of wire. (in this case, the centre 'a' of a circle is changing with time due to motion of circular wire & hence $a = a(t)$)



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- Degrees of freedom :- [DOF]

Defin:- The least possible No. of independent co-ordinate required to specify the motion of the system completely by taking in a count the contraints is called degree of freedom

e.g.

i) For system of n particles free from constraints moving independent of each other then the system has $3n$ DOF

ii) If the system contains n particles in the space & there are k No. of holonomic constraints then the system has $3n-k$ DOF

- Generalized co-ordinates :-

Defin- The independent co-ordinate required to describe the position of system completely are called generalized co-ordinate & it is denoted by q_j ; $j = 1, 2, ---, n$

Note: i) The generalized co-ordinates need not be the position co-ordinates, but they can be angles

e.g. simple pendulum, charges, momentum etc

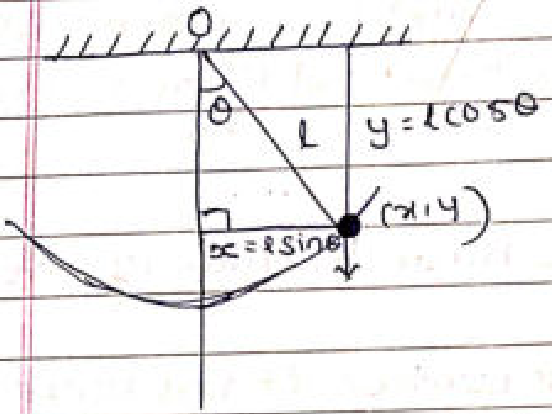
ii) No. of DOF - No. of constraints = generalized co-ordinateds

- Transformation relations:

The relation betn generalized co-ordinates and position co-ordinates (and vice versa) are called transformation relations.

example. simple pendulum

consider a particle of mass m attached to a fixed support (pt. O), by a light inextensible string of length l. The motion is in a plane.



If order pair (x, y) is position of these particle, then x & y are not free.
Throughout the motion, the bob is at distance l from O
i.e. $x^2 + y^2 = l^2$ for any time.

In this case the angle θ made by string with fixed vertical line passing through O can be treated as generalized co-ordinate because it is sufficient to describe the position of bob at any time t

Therefore, DOF equal to <u>one</u>.

from fig. transformation relations are.

$\underset{\text{p.c.}}{x} = l \sin \underset{\text{G.C.}}{\theta}$, $\underset{\text{p.s}}{y} = l \cos \underset{\text{G.C.}}{\theta}$ i.e. $l = \sqrt{x^2+y^2}$, $\theta = \tan^{-1}\left(\frac{x}{y}\right)$

- Work:

consider a particle of mass m and position vector $\bar{r}$.

m

$\bar{r}$

Suppose that the particle is acted upon a force $\bar{F}$ and it is displaced through an infinetisimal distance $d\bar{r}$, then the work done by $\bar{F}$ is

$dW = \bar{F}.d\bar{r}$

~~dW = F dr~~ $dW = \bar{F}.d\bar{r}$

$W = \int_{r_1}^{r_2} \bar{F}.d\bar{r}$

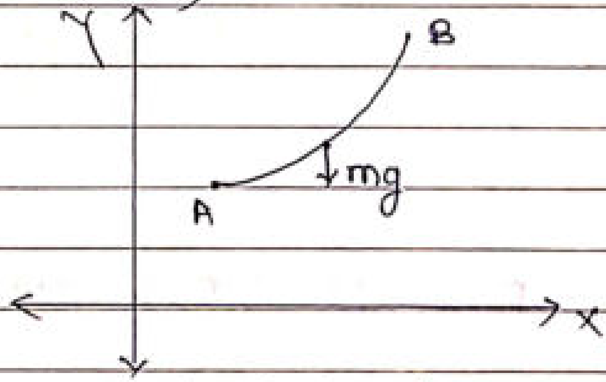
If the particle is displaced through finite distance from position r_1 to r_2 the work done by $\bar{F}$ is $W = \int_{r_1}^{r_2} \bar{F}.d\bar{r}$

- conservative force:

Defiⁿ- If the work done by force $\bar{F}$ in moving a particle from one position to another depends only on initial and final pt.s and is independent from the path ~~travels~~ traversed traversed then the $\bar{F}$ is called conservative force.

Note: From above defiⁿ we have following one more criteria.
The force $\bar{F}$ is conservative if the work done by $\bar{F}$ along any closed path is zero i.e. Work
$\oint \bar{F} d\bar{r} = 0$

Result: Show that gravitational force is conservative.
proof: consider a particle of mass m and position vector $\bar{r} = (x, y)$



The component of the force along x-axis is zero i.e. $F_x = 0$
And along Y-axis it is $F_y = -mg$
$\therefore \bar{F} = (0, -mg)$
The work done by $\bar{F}$ in moving this particle from pt. A to pt. B is

$$W = \int_A^B \bar{F} d\bar{r}$$

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$$W = \int_A^B (0, -mg)(dx, dy)$$

$\bar{r} = (x, y)$.
$d\bar{r} = (dx, dy)$

$$W = \int_{y_1}^{y_2} -mg\, dy$$

$m.\ \bar{r}(x, y)$.
$W = \int_A^B (0, -mg)(dx, dy)$

$$W = -mg \int_{y_1}^{y_2} dy$$

$W = \int_{y_1}^{y_2} -mg\, dy$

$$W = -mg(y_2 - y_1) \qquad = -mg(y_2 - y_1).$$

Thus, work W is independent of the path and depend only on extreme pt.s.
Implise that gravitational force is conservative.
Hence, the proof.

Que. Show that if $\bar{F}$ is conservative force then $\bar{F} = -\nabla V$ for some scalar funⁿ V.

$\int \nabla \times \bar{F}\, ds = 0$

$\nabla \times \bar{F} = 0$

→ $\bar{F}$ is conservative force iff

$$\oint_C \bar{F}\, d\bar{r} = 0 \quad \text{---} (1)$$

If $\bar{F} = \oint \bar{F}\, d\bar{r} = 0$

∴ By stoke theo. We have $\quad \nabla = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)$

$\Rightarrow \bar{F} = -\nabla V$

$$\oint_C \bar{F}\, d\bar{r} = \int_S \nabla \times \bar{F} \cdot ds$$

$\text{grd}\,\phi = \nabla\phi = \left(\frac{\partial\phi}{\partial x}\hat{i}, \frac{\partial\phi}{\partial y}\hat{j}, \frac{\partial\phi}{\partial z}\hat{k}\right)$

from eqⁿ (1) We have

$$\int \nabla \times \bar{F}\, ds = 0$$

ϕ - scalar

$$\Rightarrow \nabla \times \bar{F} = 0$$

$\nabla\phi$ = vector

Thus $\bar{F}$ is conservative iff

curl $\bar{F} = \nabla \times \bar{F} = 0$

How ever $\nabla \times \bar{F} = 0$ iff $\bar{F}$ is gradient of some scalar funⁿ.

i.e. $\bar{F} = \nabla V$ (∵ for some scalar funⁿ V)

The -ve sign is for convension.

- Virtual displacement:
  A virtual (infinitesimal) displacement of a system refers to a change in the configuration of a system as a result of arbitrary infinitesimal in the co-ordinates consistant with forces and constraints imposed on the system at a given instant t

Note: Actual displacement occurs in a time interval dt but, virtual displacement occurs at a fixed time instant t

This change is denoted by δ and $\delta t = 0$ because time is fixed

- Virtual Work:
  The work done by force $\bar{F}$ is causing virtual displacement is called virtual work

- principle of virtual work: stable.
  If the system is in equilibrium i.e (the total force on each particle vanishes, $\bar{F_i} = 0$) then the virtual work of $\bar{F_i}$ during the virtual displacement $\delta \bar{r_i}$ also vanishes i.e. $\bar{F_i} \cdot \delta \bar{r_i} = 0$

$$\therefore \sum_i \bar{F_i} \, \delta \bar{r_i} = 0$$

Note: The principle of ~~r~~ virtual work is applicable in statics (i.e. for system in equilibrium) the analogous principle in dynamics was proposed by D'Alembert

- D'Alembert's principle:
  Eq^n of motion of particle is $\bar{F_i} = \dot{\bar{p_i}}$, where

$\bar{p}_i$ is the linear momentum of the ith particle.

This can be written as $\bar{F}_i - \dot{\bar{P}}_i = 0$

Hence, $\sum_i (\bar{F}_i - \dot{\bar{P}}_i) = 0$

Implying a system of particle is equilibrium.

This eq^n state that the dynamical system appears to be in equilibrium under the action of applied force $\bar{F}_i$ and equal and opposite effective force $\dot{\bar{P}}_i$

In this way dynamics reduces to statics.

Thus, $\sum_i (\bar{F}_i - \dot{\bar{P}}_i) = 0 \Leftrightarrow$ system is in equilibrium

Hence, virtual work done by the force is zero

$\Rightarrow \sum_i (\bar{F}_i - \dot{\bar{P}}_i) \delta \bar{r}_i = 0$

These is known as the mathematical form of D'Alembert's principle which states that "A system of particles moves in such way that, the total virtual work done by applied forces and reverse effective forces is zero".

i.e. $\sum_i (\bar{F}_i - \dot{\bar{P}}_i) \delta \bar{r}_i = 0$

Note: All the laws in the mechanics can be derived by De'Alembert's principle. Hence it is called as fundamental principle of mechanics.

- Generalized velocity:

From the transformation relation we have

$\bar{r}_i = \bar{r}_i (q_1, q_2, \cdots, q_n, t)$

$= \bar{r}_i (q_j, t)$; $i, j = 1, 2, \cdots, n$ ------(1)

Where, $q_1, q_2, \cdots, q_n$ are generalized coordinates

diff. eq^n (1) w.r.t. t. we have

$$\frac{d\bar{r}_i}{dt} = \left[\frac{\partial \bar{r}_i}{\partial q_1} \cdot \frac{dq_1}{dt} + \frac{\partial \bar{r}_i}{\partial q_2} \cdot \frac{dq_2}{dt} + \frac{\partial \bar{r}_i}{\partial q_3} \cdot \frac{dq_3}{dt} + \cdots + \frac{\partial \bar{r}_i}{\partial q_n} \cdot \frac{dq_n}{dt}\right]$$

$$\frac{\partial \bar{r}_i}{\partial q_i} \cdot \frac{\partial q_i}{dt} \qquad + \frac{\partial \bar{r}_i}{\partial t} \cdot \frac{dt}{dt}$$

$q_k, \dot{q}_k, Q_k = \sum \bar{F}_i \frac{\partial \bar{r}_i}{\partial q_k}$

$$\dot{\bar{r}}_i = \sum_{k=1}^{n} \frac{\partial \bar{r}_i}{\partial q_k} \cdot \frac{dq_k}{dt} + \frac{\partial \bar{r}_i}{\partial t} = \sum_{k=1}^{n} \frac{\partial \bar{r}_i}{\partial q_k} \dot{q}_k + \frac{\partial \bar{r}_i}{\partial t}$$

These expression is called velocity of ith particle and the term $\dot{q}_k$ are called generalized velocity.

- δ-variation of $\bar{r}_i$:

From eqn (1) we have

$$\delta \bar{r}_i = \sum_{k=1}^{n} \frac{\partial \bar{r}_i}{\partial q_k} \delta q_k + \frac{\partial \bar{r}_i}{\partial t} \cdot \delta t$$

$$\delta \bar{r}_i = \sum_{k=1}^{n} \frac{\partial \bar{r}_i}{\partial q_k} \cdot \delta q_k \qquad \because \delta t = 0$$

- Generalized force:

If $\bar{F}_i$ is a force acting on ith particle whose position vector is $\bar{r}_i$ then the virtual work done by all these $\bar{F}_i$ is

$$\delta W = \sum_i \bar{F}_i \, \delta \bar{r}_i$$

$$= \sum_i \bar{F}_i \left[\sum_{k=1}^{n} \frac{\partial \bar{r}_i}{\partial q_k} \cdot \delta q_k \right]$$

$$= \sum_i \sum_k \bar{F}_i \frac{\partial \bar{r}_i}{\partial q_k} \cdot \delta q_k$$

Interchanging order of summation.

$$= \sum_k \left[\sum_i \bar{F}_i \frac{\partial \bar{r}_i}{\partial q_k} \right] \delta q_k$$

$$= \sum_k Q_k \, \delta q_k$$

Where $Q_k = \sum_i \bar{F}_i \frac{\partial \bar{r}_i}{\partial q_k}$ are called as components of generalized forces.

9.

IMP

- Lagrange's eq^n of motion:

Que. Derived Lagrange's eq^n of motion from De'Alembert's principle.

2. IMP

$\Rightarrow$ consider a system of particles of masses $m_1, m_2, \ldots m_n$ and position vectors $r_1, r_2, \ldots, r_n$.

If $q_1, q_2, \ldots, q_n$ are generalized coordinates

$$\bar{r}_i = \bar{r}_i(q_1, q_2, \cdots q_n, t)$$

$$= \bar{r}_i(q_j, t) \qquad ; i, j = 1, 2, \cdots, n.$$

$$\delta\bar{r}_i = \sum_j \frac{\partial \bar{r}_i}{\partial q_j} \delta q_j$$

from De'Alembert's eq^n,

$$\sum_i (\bar{F}_i - \dot{\bar{P}}_i)\delta\bar{r}_i = 0$$

$$\therefore \sum_i \bar{F}_i \delta\bar{r}_i = \sum_i \dot{\bar{P}}_i \delta\bar{r}_i \qquad \frac{d}{dt}(m\dot{\bar{r}}) = m\ddot{\bar{r}}$$

$$\therefore \sum_i \bar{F}_i \left(\sum_j \frac{\partial \bar{r}_i}{\partial q_j} \delta q_j\right) = \sum_i m_i\ddot{\bar{r}}_i \left(\sum_j \frac{\partial \bar{r}_i}{\partial q_j} \delta q_j\right)$$

Where $\dot{\bar{P}}_i = m_i\ddot{\bar{r}}_i$

$$\therefore \sum_j \left(\sum_i \bar{F}_i \frac{\partial \bar{r}_i}{\partial q_j}\right) \delta q_j = \sum_i \sum_j m_i\ddot{\bar{r}}_i \frac{\partial \bar{r}_i}{\partial q_j} \delta q_j$$

$$\therefore \sum_j Q_j \cdot \delta q_j = \sum_i \sum_j m_i\ddot{\bar{r}}_i \frac{\partial \bar{r}_i}{\partial q_j} \cdot \delta q_j \quad \text{-----(1)}$$

$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} = Q_j$ Where $Q_j = \sum_i \bar{F}_i \cdot \frac{\partial \bar{r}_i}{\partial q_j}$ are component of generalized force.

consider,

$$\star \quad \frac{d}{dt}\left(\dot{\bar{r}}_i \frac{\partial \bar{r}_i}{\partial q_j}\right) = \ddot{\bar{r}}_i \frac{\partial \bar{r}_i}{\partial q_j} + \dot{\bar{r}}_i \frac{d}{dt}\left(\frac{\partial \bar{r}_i}{\partial q_j}\right)$$

$$\therefore \ddot{\bar{r}}_i \frac{\partial \bar{r}_i}{\partial q_j} = \frac{d}{dt}\left(\dot{\bar{r}}_i \frac{\partial \bar{r}_i}{\partial q_j}\right) - \dot{\bar{r}}_i \frac{d}{dt}\left(\frac{\partial \bar{r}_i}{\partial q_j}\right)$$

from eq^n(1) we have

$$\rightarrow \sum_k \frac{\partial \bar{r}_i}{\partial q_k} \cdot \dot{q}_k = \frac{\partial \bar{r}_i}{\partial q_1}\dot{q}_1 + \frac{\partial \bar{r}_i}{\partial q_2}\cdot\dot{q}_2 + \cdots \frac{\partial \bar{r}_i}{\partial q_j}\dot{q}_j$$ 31/07/2018

$$\sum_j Q_j \delta q_j = \sum_{i,j} m_i \left[\frac{d}{dt}\left(\dot{\bar{r}}_i \frac{\partial \bar{r}_i}{\partial q_j}\right) - \dot{\bar{r}}_i \frac{d}{dt}\left(\frac{\partial \bar{r}_i}{\partial q_j}\right)\right]\delta q_j \quad ----(2)$$

☆ Now,

$$\dot{\bar{r}}_i = \sum_k \frac{\partial \bar{r}_i}{\partial q_k}\dot{q}_k + \frac{\partial \bar{r}_i}{\partial t} \quad ----(3)$$

$r_i(q_j, t)$ Differentiating eq^n (3) by w.r.t. $\dot{q}_j$

$$\rightarrow \frac{d\dot{\bar{r}}_i}{d\dot{q}_j} = \frac{\partial \bar{r}_i}{\partial q_j} \quad ----(4)$$

Diff. eq^n (3) w.r.t. q_j

$$\frac{d\dot{\bar{r}}_i}{dq_j} = \frac{\partial \bar{r}_i}{\partial q_1}\dot{q}_1 + \frac{\partial \bar{r}_i}{\partial q_2}\dot{q}_2 + \cdots + \frac{\partial \bar{r}_i}{\partial q_n}\dot{q}_n + \frac{\partial \bar{r}_i}{\partial t}$$

$$= \frac{\partial^2 \bar{r}_i}{\partial q_1 \partial q_j}\dot{q}_1 + \frac{\partial^2 \bar{r}_i}{\partial q_2 \partial q_j}\dot{q}_2 + \cdots + \frac{\partial^2 \bar{r}_i}{\partial q_n \partial q_j}\dot{q}_n + \frac{\partial \bar{r}_i}{\partial t \partial q_j}$$

$$\frac{d\dot{\bar{r}}_i}{dq_j} = \sum_k \frac{\partial^2 \bar{r}_i}{\partial q_k \partial q_j}\dot{q}_k + \frac{\partial^2 \bar{r}_i}{\partial t \partial q_j} \quad ---(5)$$

☆ consider.

?

$$\frac{d}{dt}\left(\frac{\partial \bar{r}_i}{\partial q_j}\right) = \sum_k \frac{\partial^2 \bar{r}_i}{\partial q_k \partial q_j}\dot{q}_k + \frac{\partial^2 \bar{r}_i}{\partial q_j \partial t} \quad ----(6)$$

from eq^n (6) & (5) We get

$$\frac{d}{dt}\left(\frac{\partial \bar{r}_i}{\partial q_j}\right) = \frac{\partial \dot{\bar{r}}_i}{\partial q_j} \quad ----(7)$$

using eq^n 4 & 7 in eq^n 2 we get

$$\sum_j Q_j \delta q_j = \sum_{i,j} m_i \left[\frac{d}{dt}\left(\dot{\bar{r}}_i \frac{\partial \dot{\bar{r}}_i}{\partial \dot{q}_j}\right) - \dot{\bar{r}}_i \frac{\partial \dot{\bar{r}}_i}{\partial q_j}\right]\delta q_j \quad -⑧$$

since,

~~∴ above eqⁿ becomes~~

$$\cancel{\sum_j Q_j \delta q_j = \sum_{i,j} m_i \left[\frac{d}{dt}\left(\frac{1}{2}\partial\right.\right.}$$

$$\star \quad \frac{1}{2}\frac{\partial}{\partial q_j}(\dot{\vec{r}}_i \cdot \dot{\vec{r}}_i) = \frac{1}{2}\frac{\partial}{\partial q_j}(\dot{\vec{r}}_i^{\,2}) = \frac{1}{2} 2\dot{\vec{r}}_i \frac{\partial \dot{\vec{r}}_i}{\partial q_j}$$

$$= \dot{\vec{r}}_i \frac{\partial \dot{\vec{r}}_i}{\partial q_j}$$

$$\star \quad \& \; \frac{1}{2}\frac{\partial}{\partial \dot{q}_j}(\dot{\vec{r}}_i \dot{\vec{r}}_i) = \dot{\vec{r}}_i \frac{\partial \dot{\vec{r}}_i}{\partial \dot{q}_j}$$

$$\sum_j Q_j \delta q_j = \sum_{i,j} m_i \left[\frac{d}{dt}\left(\frac{1}{2}\frac{\partial}{\partial \dot{q}_j}(\dot{\vec{r}}_i \cdot \dot{\vec{r}}_i)\right) - \frac{1}{2}\frac{\partial}{\partial q_j}(\dot{\vec{r}}_i \cdot \dot{\vec{r}}_i)\right]\delta q_j$$

$$= \sum_j \left\{\frac{d}{dt}\left(\frac{\partial}{\partial \dot{q}_j}\left[\sum_i \frac{1}{2} m_i \dot{\vec{r}}_i \dot{\vec{r}}_i\right]\right) - \frac{\partial}{\partial q_j}\left[\sum_i \frac{1}{2} m_i \dot{\vec{r}}_i \dot{\vec{r}}_i\right]\right\}\partial q_j$$

$$= \sum_j \left[\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j}\right]\delta q_j \quad \text{——— (9)}$$

Where,

$$T = \sum_i \frac{1}{2} m_i \dot{\vec{r}}_i^{\,2} = \sum_i \frac{1}{2} m_i v_i^2$$

is total kinetic energy of this system.

∴ Eqⁿ (9) can be written as,

$$\sum_j \left[Q_j - \left(\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j}\right)\right]\delta q_j = 0$$

If the system is holonomic then q_j are L.I and hence δq_j are also linearly independent.

Therefore, we have,

$$Q_j = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} \quad \text{——— (10)}$$

This are called as Lagranges eq^n for <u>holonomic</u> system.
$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} = Q_j$$

<u>case 1)</u> System is conservative:-

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In this case $\bar{F}_i = -\nabla_i V$

$$= -\frac{\partial V}{\partial \bar{r}_i}$$

Where, V is potential energy of the system.

Therefore, $\cancel{Q_j} = Q_j = \sum_i \bar{F}_i \frac{\partial \bar{r}_i}{\partial q_j}$

$$= \sum_i \frac{-\partial V}{\partial \bar{r}_i} \frac{\partial \bar{r}_i}{\partial q_j}$$

$$= \cancel{\sum_i} \; -\frac{\partial V}{\partial q_j}$$

$$= \frac{-\partial V}{\partial q_j}$$

$z = f(x, y)$

$\frac{\partial z}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} - \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial t}$

$\therefore$ from eq^n(10) we have,

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} = \frac{-\partial V}{\partial q_j}$$

$$\therefore \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} + \frac{\partial V}{\partial q_j} = 0$$

q_j

q_j

$$\therefore \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial}{\partial q_j}(\partial T - \partial V) = 0$$

$$\therefore \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial}{\partial q_j}(T - V) = 0 \quad \text{———(11)}$$

For conservative system, the potential energy will not depend on generalized velocity.

$\therefore$ we have,

conservative → $V = V(q_j)$

non-conservative → $V(q_j, \dot{q}_j, t)$

$$\frac{\partial V}{\partial \dot{q}_j} = 0$$

∴ We have,

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j} - \frac{\partial V}{\partial \dot{q}_j}\right) - \frac{\partial}{\partial q_j}(T - V) = 0$$

i.e. $$\frac{d}{dt}\left(\frac{\partial}{\partial \dot{q}_j}(T - V)\right) - \frac{\partial}{\partial q_j}(T - V) = 0 \quad \text{-----} (12)$$

Define $L = T - V$, a lagrangian of given conservative system

Where $L = L(q_j, \dot{q}_j, t)$

∴ eqn (12) becomes,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0$$

These are lagranges eqn of motion for conservative holonomic system.

VIMP

case II) System is non-conservative :-

For non-conservative system the lagranges eqn are given by eqn (10)

In this case the potential energy is depends on generalized velocity also i.e. $V = V(q_j, \dot{q}_j, t)$

In some practical cases (for eg a charged partical moving in electromagnetic field) the component of generalized forces can be ~~exproes~~ expressed as

$$Q_j = \frac{-\partial V}{\partial q_j} + \frac{d}{dt}\left(\frac{\partial V}{\partial \dot{q}_j}\right) \quad \text{———} (*)$$

∴ From eqn (10) We have

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} = \frac{-\partial V}{\partial q_j} + \frac{d}{dt}\left(\frac{\partial V}{\partial \dot{q}_j}\right)$$

$$\text{i.e. } \frac{d}{dt}\left(\frac{\partial}{\partial \dot q_j}(T-V)\right) - \frac{\partial}{\partial q_j}(T-V) = 0$$

$$\text{i.e. } \frac{d}{dt}\left(\frac{\partial L}{\partial \dot q_j}\right) - \frac{\partial L}{\partial q_j} = 0 \quad \text{---(13)}$$

Note: Note that, for a non-conservative system the eqn of motion are given by (10) & not (13) ingeneral. The eqn (13) hold only when Q_j has the form (*)

Case III) System is partially conservative & partially non-conservative:-

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consider the system with conservative forces $\bar F_i$ and non-conservative forces $\bar F_i^{(d)}$

$$\therefore \bar F_i = -\nabla_i V = -\frac{\partial V}{\partial \bar r_i}$$

Now.

$$Q_j = \sum_i \left[\bar F_i + \bar F_i^{(d)}\right]\frac{\partial \bar r_i}{\partial q_j}$$

$$= \sum_i \bar F_i \frac{\partial \bar r_i}{\partial q_j} + \sum_i \bar F_i^{(d)} \frac{\partial \bar r_i}{\partial q_j}$$

$$= \sum_i \left(\frac{-\partial V}{\partial \bar r_i}\right)\frac{\partial \bar r_i}{\partial q_j} + \cancel{\sum_i \left(\frac{-\partial V}{\partial \bar r_i}\right)\frac{\partial \bar r_i}{\partial q_j}} \; Q_j^{(d)}$$

$$\text{where, } Q_j^{(d)} = \sum_i \bar F_i^{(d)} \frac{\partial \bar r_i}{\partial q_j}$$

$$\therefore Q_j = \frac{-\partial V}{\partial q_j} + Q_j^{(d)}$$

$\therefore$ from eqn (10) we have

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot q_j}\right) - \frac{\partial T}{\partial q_j} = \frac{-\partial V}{\partial q_j} + Q_j^{(d)}$$

$K.E = T = \frac{1}{2} m \dot{r}_i^2$; $P.E = V = mgh$.

$$\therefore \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial}{\partial q_j}(T-V) = Q_j^{(d)}$$

$$\therefore \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = Q_j^{(d)} \quad \text{------} (14)$$

$\bar{r}_i = \bar{r}_i(q_j, t)$

If the forces are fractional forces then it is observed that,

$$F_i^{(d)} = -\lambda_i \dot{\bar{r}}_i \quad ; \lambda_i \text{ is const.}$$

$$\dot{\bar{r}}_i = \sum_k \frac{\partial \bar{r}_i}{\partial q_k} \dot{q}_k + \frac{\partial \bar{r}_i}{\partial t}$$

$$\therefore Q_j^{(d)} = \sum_i -\lambda_i \dot{\bar{r}}_i \cdot \frac{\partial \bar{r}_i}{\partial q_j}$$

$$\frac{\partial \dot{\bar{r}}_i}{\partial \dot{q}_j} = \frac{\partial \bar{r}_i}{\partial q_k}$$

$$= -\sum_i \lambda_i \dot{\bar{r}}_i \frac{\partial \dot{\bar{r}}_i}{\partial \dot{q}_j} \qquad \left(\frac{\partial \bar{r}_i}{\partial q_j} = \frac{\partial \dot{\bar{r}}_i}{\partial \dot{q}_j}\right)$$

$$= -\sum_i \lambda_i \frac{\partial}{\partial \dot{q}_j}\left(\frac{1}{2} \dot{\bar{r}}_i \dot{\bar{r}}_i\right) \quad \text{for M.C.Q.}$$

$$= \frac{-\partial R}{\partial \dot{q}_j} \quad ; \text{where } R = \frac{1}{2}\sum_i \lambda_i \dot{\bar{r}}_i^2$$

is called Rayleigh's dissipation funⁿ.

$\therefore$ eqⁿ (14) becomes,

$$T = \frac{1}{2}\sum_i m_i \dot{\bar{r}}_i^2$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = \frac{-\partial R}{\partial \dot{q}_j}$$

Also, the given system is ~~non~~-conservative.

$$\therefore \text{We have, } \frac{\partial V}{\partial \dot{q}_j} = 0$$

$\therefore$ above eqⁿ becomes,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = \frac{-\partial R}{\partial \dot{q}_j}$$

Que: Use De' Alembert's principal to find the eqⁿ of motion of a simple pendulum.

→ consider a particle of mass m and position vector $\bar{r}$ is (x, y) attach to a consid fixed

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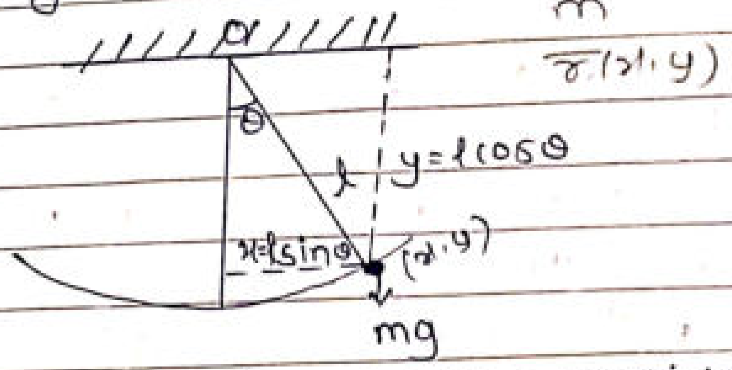
support by a rigid string of a length l.



The only force acting on the particle is its wt. mg in the downward direction.

If F_x and F_y are components of force along x and y axis then F_x will be zero and $F_y = -mg$

$\therefore\ \bar{F} = (0, -mg)$

Now $\bar{p} = m\bar{v} = m\dot{\bar{r}}$

$\dot{\bar{p}} = m\ddot{\bar{r}}$

$= (m\ddot{x}, m\ddot{y}) \quad ;\because \bar{r} = \bar{r}(x, y)$

Now, by De' Alemberts eq^n,

$$(\bar{F} - \dot{\bar{p}})\delta\bar{r} = 0$$

$$(-m\ddot{x}, -mg - m\ddot{y})(\delta x, \delta y) = 0$$

$$(-m\ddot{x}\delta x, -(mg + m\ddot{y})\delta y = 0$$

$$m\ddot{x}\delta x + (mg + m\ddot{y})\delta y = 0 \quad \text{———} (1)$$

Now x and y are not independent but related by $x^2 + y^2 = l^2$

Applying δ on both side.

$$2x\delta x + 2y\delta y = 0$$

$$x\delta x + y\delta y = 0$$

$$\delta x = \frac{-y}{x}\delta y$$

$\therefore$ Eq^n(1) becomes

$$m\ddot{x}\left(\frac{-y}{x}\right)\delta y + (mg + m\ddot{y})\delta y = 0$$

13. Simple pendulum $\Rightarrow$ $\ddot{\theta} = \frac{-g}{l}\sin\theta.$

$$\ddot{x}\cdot\left(\frac{-y}{x}\right)\delta y + (g+\ddot{y})\delta y = 0$$

$$-\ddot{x}y\,\delta y + (xg + x\ddot{y})\,\delta y = 0$$

$$\therefore\ (-\ddot{x}y + xg + x\ddot{y})\,\delta y = 0$$

Since, $\delta y \neq 0$ as y is not constant

$$\therefore\ -\ddot{x}y + xg + x\ddot{y} = 0 \quad ----(2)$$

Since, x and y are not independent, we can transform this eqn (2) & involving single variable say θ. $x = l\sin\theta$.

For this, we use transformation relation

$x = l\sin\theta$ and $y = l\cos\theta$

$\dot{x} = l\cos\theta\,\dot{\theta}$ & $\dot{y} = -l\sin\theta.\dot{\theta}$

$\ddot{x} = -l\sin\theta\cdot\dot{\theta}^2 + l\cos\theta\,\ddot{\theta}$ and $\ddot{y} = -l\cos\theta\,\dot{\theta}^2 - l\sin\theta\,\ddot{\theta}$

Substituting the values of $x, y, \ddot{x}, \ddot{y}$ in eqn (2)

We get,

$$-[-l\sin\theta\,\dot{\theta}^2 + l\cos\theta.\ddot{\theta}]\,l\cos\theta + l\sin\theta\, g + l\sin\theta\,[-l\cos\theta\dot{\theta}^2 - l\sin\theta\ddot{\theta}] = 0$$

$\Rightarrow \cancel{l^2\sin\theta\,\dot{\theta}^2 - l^2}$

$$\Rightarrow l^2\sin\theta\,\dot{\theta}^2\cos\theta - l^2\cos^2\theta\,\ddot{\theta} + l\sin\theta\, g - l^2\sin\theta\cos\theta.\dot{\theta}^2 - l^2\sin^2\theta\,\ddot{\theta} = 0$$

$$\Rightarrow -l^2\ddot{\theta}(\cos^2\theta + \sin^2\theta) + l\sin\theta\, g = 0$$

$$\Rightarrow -l^2\ddot{\theta} + l\sin\theta\, g = 0$$

$$\Rightarrow \ddot{\theta} = \frac{g}{l}\sin\theta$$

$(F - \bar{F})\delta r_i = 0$

$(0, -mg) - (-m\ddot{x}, m\ddot{y})$.

As θ decreases we have

$(\delta x, \delta y)$

$$\ddot{\theta} = \frac{-g}{l}\sin\theta$$

$(-m\ddot{x}, -mg + m\ddot{y})\,dx\,dy$

Which is required eqn of motion of simple pendulum.

2/08/2018

1 show that the lagranges eq^n $\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} = Q_j$

can also written in the form $\frac{\partial \dot{T}}{\partial \dot{q}_j} - 2\frac{\partial T}{\partial q_j} = Q_j$

→ The lagranges eq^n of motion is,

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} = Q_j \quad \text{——(1)}$$

The kinetic ~~eng~~ energy of the system is,

$$T = T(q_j, \dot{q}_j, t)$$

$$\therefore \dot{T} = \frac{dT}{dt} = \sum_k \left[\frac{\partial T}{\partial q_k} \cdot \frac{dq_k}{dt} + \frac{\partial T}{\partial \dot{q}_k} \frac{d\dot{q}_k}{dt}\right] + \frac{\partial T}{\partial t} \cdot \frac{dt}{dt}$$

$$\dot{T} = \sum_k \left[\frac{\partial T}{\partial q_k} \cdot \dot{q}_k + \frac{\partial T}{\partial \dot{q}_k} \ddot{q}_k\right] + \frac{\partial T}{\partial t} \quad \text{——(2)}$$

Diff. eq^n (2) w.r.t. $\dot{q}_j$

$$\frac{\partial \dot{T}}{\partial \dot{q}_j} = \sum_k \frac{\partial^2 T}{\partial \dot{q}_j \partial q_k} \cdot \dot{q}_k + \frac{\partial T}{\partial q_j} + \sum_k \frac{\partial^2 T}{\partial \dot{q}_j \partial \dot{q}_k} \ddot{q}_k + \frac{\partial^2 T}{\partial \dot{q}_j \partial t}$$

$$= \sum_k \left[\frac{\partial^2 T}{\partial \dot{q}_j \partial q_k} \dot{q}_k + \frac{\partial^2 T}{\partial \dot{q}_j \partial \dot{q}_k} \ddot{q}_k\right] + \frac{\partial T}{\partial q_j} + \frac{\partial^2 T}{\partial \dot{q}_j \partial t} \quad \text{——(3)}$$

d(uv) $\dot{T} = \sum_k \left[\frac{\partial T}{\partial q_k} \cdot \dot{q}_k\right] \Rightarrow \frac{\partial \dot{T}}{\partial \dot{q}_j} = \sum_k \frac{\partial^2 T}{\partial \dot{q}_j \partial q_k} \dot{q}_k + \frac{\partial T}{\partial q_j}$

$\dot{T} = \frac{\partial T}{\partial \dot{q}_k} \ddot{q}_k \Rightarrow \frac{\partial \dot{T}}{\partial \dot{q}_j} = \frac{\partial^2 T}{\partial \dot{q}_j \partial \dot{q}_k} \cdot \ddot{q}_k$

consider,

key step

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) = \sum_k \left[\frac{\partial^2 T}{\partial \dot{q}_j \partial q_k} \dot{q}_k + \frac{\partial^2 T}{\partial \dot{q}_j \partial \dot{q}_k} \cdot \ddot{q}_k\right] + \frac{\partial^2 T}{\partial \dot{q}_j \partial t} \quad \text{——(4)}$$

Substracting eq^n (4) from (3)

14. $\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} = Q_j \Rightarrow \frac{\partial \dot{T}}{\partial \dot{q}_j} - 2\frac{\partial T}{\partial q_j} = Q_j$

$$\frac{\partial \dot{T}}{\partial \dot{q}_j} - \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) = \frac{\partial T}{\partial q_j}$$

$T = T(q_j, \dot{q}_j, t)$.

$\dot{T}$

$\frac{\partial \dot{T}}{\partial \dot{q}_j}$

from eq^n (1) we have

$$\frac{\partial \dot{T}}{\partial \dot{q}_j} - Q_j - \frac{\partial T}{\partial q_j} = \frac{\partial T}{\partial q_j}$$

$\frac{\partial \dot{T}}{\partial \dot{q}_j} - \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right)$

$$\frac{\partial \dot{T}}{\partial \dot{q}_j} - 2\frac{\partial T}{\partial q_j} = Q_j$$

Hence, the proof.

2. show that the lagranges eq^n of motion can also be written as $\frac{\partial L}{\partial t} - \frac{d}{dt}\left(L - \sum_j \dot{q}_j \frac{\partial L}{\partial \dot{q}_j}\right) = 0$

proof $\Rightarrow$ We have lagrangian

$$L = L(q_j, \dot{q}_j, t) \quad \text{——— (1)}$$

Diff. eq^n (1) w.r.t. t.

$$\dot{L} = \frac{dL}{dt} = \sum_j \left[\frac{\partial L}{\partial q_j} \cdot \dot{q}_j + \frac{\partial L}{\partial \dot{q}_j} \ddot{q}_j\right] + \frac{\partial L}{\partial t} \quad \text{——— (2)}$$

consider the expression:

$$\frac{d}{dt}\left[\sum_j \dot{q}_j \frac{\partial L}{\partial \dot{q}_j}\right] = \sum_j \left[\dot{q}_j \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) + \ddot{q}_j \frac{\partial L}{\partial \dot{q}_j}\right] \quad \text{——— (3)}$$

substracting eq^n (3) from eq^n (2)

$$\frac{dL}{dt} - \frac{d}{dt}\left(\sum_j \dot{q}_j \frac{\partial L}{\partial \dot{q}_j}\right) = \frac{\partial L}{\partial t} + \sum_j \dot{q}_j \left[\frac{\partial L}{\partial q_j} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right)\right]$$

as L is a lagrangian

$$\therefore \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0$$

$\therefore$ above eq^n becomes

$$\frac{dL}{dt} - \frac{d}{dt}\left(\sum_j \dot{q}_j \frac{\partial L}{\partial \dot{q}_j}\right) = \frac{\partial L}{\partial t}$$

$$\frac{d}{dt}\left(L - \sum_j \dot{q}_j \frac{\partial L}{\partial \dot{q}_j}\right) = \frac{\partial L}{\partial t}$$

$$\therefore \quad \frac{\partial L}{\partial t} - \frac{d}{dt}\left(L - \sum_j \dot{q}_j \frac{\partial L}{\partial \dot{q}_j}\right) = 0$$

Hence, the proof.

Que 3) IMP — Show that if the force acting on the particle is conservative then the total energy of that particle is conserved.

Solution → Consider a particle of mass m. Let $\bar{F}$ be the conservative force acting on the particle. Suppose that, the particle is displaced from position P_1 to position P_2 under the action of $\bar{F}$.

∴ The work done is,

$$W = \int_{P_1}^{P_2} \bar{F}\, d\bar{r} \quad ---(1)$$

By Newton's 2nd law of motion.

$$\bar{F} = \dot{\bar{p}} = m\ddot{\bar{r}}$$

∴ eqⁿ (1) becomes,

$$W = \int_{P_1}^{P_2} m\ddot{\bar{r}}\, d\bar{r}$$

$$= \int_{P_1}^{P_2} m\ddot{\bar{r}} \frac{d\bar{r}}{dt} . dt$$

$$= \int_{P_1}^{P_2} m\ddot{\bar{r}} . \dot{\bar{r}} . dt \qquad \frac{1}{2}\frac{d}{dt}\dot{\bar{r}}^2 = \frac{1}{2} 2\dot{\bar{r}} . \ddot{\bar{r}}$$

$$= \int_{P_1}^{P_2} m\frac{1}{2}\left(\frac{d}{dt}\dot{\bar{r}}^2\right) dt$$

$$= \int_{P_1}^{P_2} \frac{d}{dt}\left(\frac{1}{2}m\dot{\bar{r}}^2\right) dt$$

$$= \int_{P_1}^{P_2} \frac{d}{dt}(T)\,dt \quad ; \; T = \frac{1}{2} m \dot{\bar{r}}^2$$

$$= [T]_{P_1}^{P_2}$$

$$W = T_2 - T_1 \quad \text{---} (2)$$

Where, T_i is kinetic energy at P_i

Also, $\bar{F}$ is a conservative

$\Rightarrow \bar{F} = -\nabla V$; V is the potential energy of particle.

i.e. $\bar{F} = -\frac{\partial V}{\partial \bar{r}}$

$\therefore$ Eqⁿ (1) becomes,

$$W = \int_{P_1}^{P_2} -\frac{\partial V}{\partial \bar{r}} \cdot d\bar{r}$$

$$= -\int_{P_1}^{P_2} \frac{\partial V}{\partial \bar{r}} \cdot d\bar{r} \qquad V = V(r_1, r_2)$$

$$dV = \frac{\partial V}{\partial r_1} dr_1 + \frac{\partial V}{\partial r_2} dr_2$$

$$= -\int_{P_1}^{P_2} dV$$

$$= -(V_2 - V_1)$$

$$W = -V_2 + V_1 \quad \text{----} (3)$$

Where, V_i is potential energy at P_i

using (2) and (3)

$$T_2 - T_1 = -V_2 + V_1$$

$$T_2 + V_2 = T_1 + V_1$$

since, P_1 and P_2 are arbitrary point.

$\therefore$ Total energy is constant everywhere.

Hence, the proof.

Total energy const. दाखवायची असल्यास दोन्ही pt. ना energy same दाखवायची at pt. $P_1 = T_1 + V_1$ & $P_2 = T_2 + V_2$

- Kinetic energy in polar form.

consider a particle of mass m & position vector $\bar{r}$ moving in a plane. Let (x,y) be a cartesian co-ordinates of the particle.

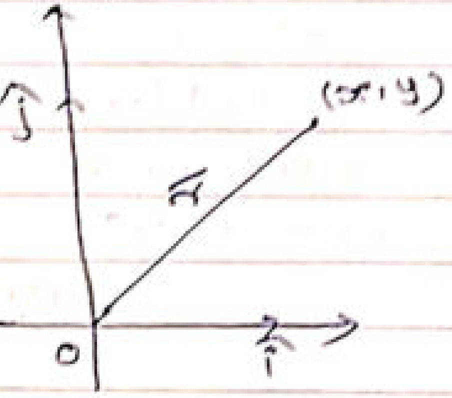


fig (1)

The kinetic energy is given by.

$$T = \frac{1}{2}mv^2 = \frac{1}{2}m\dot{\bar{r}}^2$$

$$= \frac{1}{2}m(\dot{x}^2+\dot{y}^2) \text{ --- (1) } (\because \bar{r} = \bar{r}(x,y))$$

Which is kinetic energy in ~~kart~~ cartesian co-ordinate form.

If we consider polar co-ordinate then

$x = \bar{r}\cos\theta$ & $y = \bar{r}\sin\theta$

$\dot{x} = \dot{\bar{r}}\cos\theta - \bar{r}\sin\theta\dot{\theta}$ and $\dot{y} = \dot{\bar{r}}\sin\theta + \bar{r}\cos\theta\dot{\theta}$

consider,

$$\dot{x}^2+\dot{y}^2 = [\dot{\bar{r}}\cos\theta - \bar{r}\sin\theta\cdot\dot{\theta}]^2 + [\dot{\bar{r}}\sin\theta + \bar{r}\cos\theta\cdot\dot{\theta}]^2$$

$$= (\dot{\bar{r}}\cos\theta)^2 - 2\dot{\bar{r}}\bar{r}\,\cancel{\sin\theta\cdot\cos\theta\dot{\theta}} + (\bar{r}\sin\theta\dot{\theta})^2$$

$$+ (\dot{\bar{r}}\sin\theta)^2 + 2\dot{\bar{r}}\bar{r}\,\cancel{\sin\theta\cdot\cos\theta\dot{\theta}} + (\bar{r}\cos\theta\dot{\theta})^2$$

$$= (\dot{\bar{r}}^2)(\cos^2\theta + \sin^2\theta) + \bar{r}^2\cdot\dot{\theta}^2(\cos^2\theta+\sin^2\theta)$$

$$= (\dot{\bar{r}})^2 + \bar{r}^2\cdot\dot{\theta}^2$$

$$= \dot{\bar{r}}^2 + \bar{r}^2\cdot\dot{\theta}^2$$

$\therefore$ eq^n (1) becomes,

16. $\frac{1}{2}m(\dot{x}^2+\dot{y}^2)$

$\frac{1}{2}m(\dot{\bar{r}}^2+\bar{r}^2\dot{\theta}^2)$

$$T = \frac{1}{2}m(\dot{\bar{r}}^2 + \bar{r}^2\dot{\theta}^2)$$

which is required kinetic energy in polar co-ordinate form.

Ex. A particle of mass m moves on a plane in the field of force given by $\bar{F} = -\hat{i}_r kr\cos\theta$, where k is const. and $\hat{i}_r$ is the unit radial vector obtain the d.E. of the orbit of the particle.

$\Rightarrow$ The force is given by,

$$\bar{F} = \bar{F}_r\hat{i}_r + \bar{F}_\theta\hat{i}_\theta$$
$$= -\hat{i}_r kr\cos\theta$$

$\therefore \bar{F}_r = -kr\cos\theta$

$\bar{F}_\theta = 0$

In this case θ and r are generalized co-ordinate i.e. $q_1 = r$, $q_2 = \theta$

$\therefore$ The Lagrange's eq^n of motion is given by,

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} = Q_j \quad ; j = 1, 2$$

$q_1 = \theta, Q = 0$

Here, $q_1 = r$, $q_2 = \theta$

and $Q_1 = \bar{F}_r$, $Q_2 = \bar{F}_\theta$

Where, $\bar{F}_r = -kr\cos\theta$, $\bar{F}_\theta = 0$

Here, $T = \frac{1}{2}m(\dot{\bar{r}}^2 + \bar{r}^2\dot{\theta}^2)$

$\frac{d}{dt}(\bar{r}^2 m\dot{\theta}) = 0$

$\bar{r}^2 m.\dot{\theta} = c$

$q_2 - r.$

case i>: $q_1 = r$, $Q_1 = \bar{F}_r = -kr\cos\theta$

$\therefore \frac{\partial T}{\partial \dot{r}} \cancel{\frac{\partial T}{\partial \dot{r}}} = \frac{1}{2}m(2\dot{r} + 0) = m\dot{\bar{r}}$

$m\ddot{\bar{r}} - m\dot{\theta}^2\bar{r}$

$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{r}}\right) \cancel{\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{r}}\right)} = m\ddot{\bar{r}}$

$\frac{\partial T}{\partial r} = m\bar{r}\dot{\theta}^2$

Therefore, Lagranges equation of motion is given by,

$$m\ddot{\bar{r}} - m\bar{r}\dot{\theta}^2 = -kr\cos\theta \quad \text{——(1)}$$

case (ii): $q_2 = \theta$, $Q_2 = \bar{F}_\theta = 0$

$$\therefore \frac{\partial T}{\partial \dot{\theta}} = \frac{1}{2} m (2\dot{\theta}\bar{r}^2) = m\dot{\theta}\bar{r}^2$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right) = \frac{d}{dt}(m\dot{\theta}\bar{r}^2)$$

$$\frac{\partial T}{\partial \theta} = 0$$

Therefore, Lagranges eqn of motion is given by

$$\frac{d}{dt}(m\dot{\theta}\bar{r}^2) = 0$$

$$\Rightarrow m\dot{\theta}\bar{r}^2 = \text{const.} \quad \text{——(2)}$$

eqn (1) & (2) are required eqn of orbit of motion.

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Ex. A particle of mass m moves in a plane under the action of conservation force $\bar{F}$ with components

$\bar{F}_x = -k^2(2x+y)$ and

$\bar{F}_y = -k^2(x+2y)$; k is const.

Find the total energy, Lagrangian & eqⁿ of motion

⇒ The force is given by,

$$\bar{F} = \bar{F}_x \cdot \hat{i}_x + \bar{F}_y \cdot \hat{i}_y \quad \text{—— (1)}$$

We have

$E = V + T$

$L = T - V$

$\bar{F}_x = -k^2(2x+y)$

$\bar{F}_y = -k^2(x+2y)$

since force is conservative.

$\therefore \bar{F} = -\nabla V$; V is potential energy

$$= -\left[\frac{\partial V}{\partial x}\hat{i}_x + \frac{\partial V}{\partial y}\hat{j}_y\right]$$

$\nabla \phi$

$$\bar{F} = -\frac{\partial V}{\partial x}\hat{i}_x - \frac{\partial V}{\partial y}\hat{i}_y \quad \text{—— (2)}$$

$\therefore$ from eqⁿ (1) & (2) we have

$$-\frac{\partial V}{\partial x} = \bar{F}_x = -k^2(2x+y) \Rightarrow \frac{\partial V}{\partial x} = k^2(2x+y)$$

$$-\frac{\partial V}{\partial y} = \bar{F}_y = -k^2(x+2y) \Rightarrow \frac{\partial V}{\partial y} = k^2(x+2y)$$

Also,

$V = V(x, y)$

$$dV = \frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy$$

$$dV = \frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y} \cdot dy$$

$$= k^2(2x+y)dx + k^2(x+2y)dy$$

$$= k^2[(2x+y)dx + (x+2y)dy]$$

$$= k^2[2x\,dx + y\,dx + x\,dy + 2y\,dy]$$

$$\underline{dV = k^2 d(x^2+y^2+xy)}$$

Taking integration on both side,

$$V = k^2(x^2+y^2+xy) + c$$

Take $c = 0$

$\therefore V = V(x, y)$

$$dV = \frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy$$

$$V = k^2(x^2 + y^2 + xy)$$

This is potential energy.

Now K.E of particle is

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2)$$

I) $\therefore$ Total energy is $= T + V$

$$= \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + k^2(x^2 + y^2 + xy)$$

II) Lagrangian $L = T - V$

$$= \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) - k^2(x^2 + y^2 + xy)$$

III) Eqⁿ of motion :-

case i) :

Here, $q_1 = x$, $Q_1 = F_x = -k^2(2x + y)$

$\therefore$ Lagranges eqⁿ corresponding to $q_1 = x$ is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0 \quad \text{——(3)}$$

$$\therefore \frac{\partial L}{\partial \dot{x}} = m\dot{x}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = m\ddot{x}$$

$$\frac{\partial L}{\partial x} = -k^2(2x + y)$$

$\therefore$ eqⁿ (3) becomes,

$$m\ddot{x} + k^2(2x + y) = 0 \quad \text{——(4)}$$

case (ii)

Here $q_2 = y$

$\therefore$ Lagranges eqⁿ corresponding to $q_2 = y$ is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) - \frac{\partial L}{\partial y} = 0 \quad \text{——(5)}$$

$\therefore \frac{\partial L}{\partial \dot{y}} = m\dot{y}$

$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) = m\ddot{y}$

$\frac{\partial L}{\partial y} = -k^2(2y+x)$

$\therefore$ eq^n (5) becomes

$m\ddot{y} + k^2(2y+x) = 0$ ——— (6)

$\therefore$ eq^n (4) and (6) are required eq^n of motion

VIMP 4

Theo. Kinetic energy is a homogeneous quadratic function of generalized velocities. $\dot{q}_k$

statement:

Find the expression for kinetic energy as the quadratic fun^n of generalized velocity.

Furthershow that,

I) When the constraints are Scleronomic (not contain time), the kinetic energy is homogeneous fun^n of generalized velocities & $\sum_j \dot{q}_j \frac{\partial T}{\partial \dot{q}_j} = 2T$

II) When the constraints are Rheonomic (contain time), then

$\sum_j \dot{q}_j \frac{\partial T}{\partial \dot{q}_j} = 2T_2 + T_1$

Where, T_1 & T_2 have usual meaning.

proof:

Let us consider a system of a particle of masses m_i and position vector $\bar{r}_i$.

suppose that q_j's are generalized co-ordinates

The kinetic energy is given by

$T = \frac{1}{2}\sum_i m_i \dot{r}_i^2$

$\left[\sum_{i=1}^{2} x_i\right]\left[\sum_{j=1}^{2} x_i\right]$ but $\left[\sum_{i=1}^{2} x_i\right]\left[\sum_{i=1}^{2} x_i\right]$

$\Rightarrow x_1^2 + 2x_1x_2 + x_2^2 \quad = \sum_{i=1}^{2} x_i^2 \Rightarrow x_1^2 + x_2^2$

i.e. $T = \frac{1}{2}\sum_i m_i \dot{\bar{r}}_i \cdot \dot{\bar{r}}_i$

Now,

$\bar{r}_i = \bar{r}_i(q_j, t)$

$$\therefore \dot{\bar{r}}_i = \sum_j \frac{\partial \bar{r}_i}{\partial q_j} \cdot \dot{q}_j + \frac{\partial \bar{r}_i}{\partial t} \quad \text{——(1)}$$

$$\therefore T = \frac{1}{2}\sum_i m_i \left(\sum_j \frac{\partial \bar{r}_i}{\partial q_j}\dot{q}_j + \frac{\partial \bar{r}_i}{\partial t}\right)\left(\sum_k \frac{\partial \bar{r}_i}{\partial q_k}\dot{q}_k + \frac{\partial \bar{r}_i}{\partial t}\right)$$

$$T = \frac{1}{2}\sum_i m_i \left\{\sum_{j,k} \frac{\partial \bar{r}_i}{\partial q_j}\frac{\partial \bar{r}_i}{\partial q_k}\dot{q}_j\dot{q}_k + \sum_j \frac{\partial \bar{r}_i}{\partial q_j}\cdot\frac{\partial \bar{r}_i}{\partial t}\dot{q}_j + \sum_k \frac{\partial \bar{r}_i}{\partial q_k}\frac{\partial \bar{r}_i}{\partial t}\dot{q}_k + \frac{\partial \bar{r}_i}{\partial t}\cdot\frac{\partial \bar{r}_i}{\partial t}\right\}$$

$$T = \frac{1}{2}\sum_i m_i \left\{\sum_{j,k} \frac{\partial \bar{r}_i}{\partial q_j}\frac{\partial \bar{r}_i}{\partial q_k}\dot{q}_j\dot{q}_k + 2\sum_j \frac{\partial \bar{r}_i}{\partial q_j}\frac{\partial \bar{r}_i}{\partial t}\dot{q}_j + \left(\frac{\partial \bar{r}_i}{\partial t}\right)^2\right\}$$

$$T = \sum_{j,k}\left(\frac{1}{2}\sum_i m_i \frac{\partial \bar{r}_i}{\partial q_j}\frac{\partial \bar{r}_i}{\partial q_k}\right)\dot{q}_j\dot{q}_k + \sum_j \left(\sum_i m_i \frac{\partial \bar{r}_i}{\partial q_j}\frac{\partial \bar{r}_i}{\partial t}\right)\dot{q}_j + \frac{1}{2}\sum_i m_i\left(\frac{\partial \bar{r}_i}{\partial t}\right)^2$$

$$T = \sum_{j,k} \underline{a_{j,k}}\, \underline{\dot{q}_j\dot{q}_k} + \sum_j \underline{a_j}\,\dot{q}_j + \underline{a} \quad \text{——(2)}$$

Where,

$$a_{j,k} = \frac{1}{2}\sum_i m_i \frac{\partial \bar{r}_i}{\partial q_j}\frac{\partial \bar{r}_i}{\partial q_k}$$

$$a_j = \sum_i m_i \frac{\partial \bar{r}_i}{\partial q_j}\frac{\partial \bar{r}_i}{\partial t}$$

$$a = \frac{1}{2}\sum_i m_i\left(\frac{\partial \bar{r}_i}{\partial t}\right)^2$$

$\therefore$ from eq^n (2) the kinetic energy is quadratic fun^n of generalized velocities $\dot{q}_1, \dot{q}_2, \cdots \dot{q}_n$

eq^n (2) can be re-write as

$$T = T_2 + T_1 + T_0 \text{ ——— } (4)$$

Where, $T_2 = \sum_{j,k} a_{j,k} \dot{q}_j \dot{q}_k$

$$T_1 = \sum_j a_j \dot{q}_j$$

$$T_0 = a$$

This are functions of generalized velocities of degree 2, 1 and 0 respe.

case I) : suppose that the system is scleronomic.

$\therefore$ Time t is not explicitily in constraints

$\therefore$ The transformation relation will not involve t explicitily.

Therefore,

$$\frac{\partial \bar{r}_i}{\partial t} = 0$$

[Euler's formula $f(x, y)$

$x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} = n f(x, y)$

n is homogeneous degree of fun^n f]

$\therefore \quad a_j = 0$ and $a = 0$

$\therefore \quad T = T_2 = \sum_{j,k} a_{j,k} \dot{q}_j \dot{q}_k$ —(3)

Thus, kinetic energy is ~~qud~~ homogeneous quadratic fun^n of generalized velocities.

$\therefore$ By Euler's theo.

$$\therefore \sum_j \frac{\partial T}{\partial \dot{q}_j} \dot{q}_j = 2T$$

Which is required eq^n.

case II) suppose that the system is rheonomic

In this case $a_j \neq 0$ & $a \neq 0$, because $\frac{\partial \bar{r}_i}{\partial t} \neq 0$

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$\therefore$ Eqⁿ (*) becomes,

$$T = T_2 + T_1 + T_0$$

Now, T_2, T_1 and T_0 are homogeneous functions of generalized of degree 2, 1 and 0 respe.

$\therefore$ By Euler's theo. we have

$$\sum_j \dot{q}_j \frac{\partial T_2}{\partial \dot{q}_j} = 2T_2$$

$$\sum_j \dot{q}_j \frac{\partial T_1}{\partial \dot{q}_j} = 1 \cdot T_1 = T_1$$

$$\sum_j \dot{q}_j \frac{\partial T_0}{\partial \dot{q}_j} = 0 \cdot T_0 = 0$$

Now, Differentiating eqⁿ (*) w.r.t. $\dot{q}_j$, multiplying it by $\dot{q}_j$ and summing we get

$$\sum_j \dot{q}_j \frac{\partial T}{\partial \dot{q}_j} = \sum_j \dot{q}_j \frac{\partial T_2}{\partial \dot{q}_j} + \sum_j \dot{q}_j \frac{\partial T_1}{\partial \dot{q}_j} + \sum_j \dot{q}_j \frac{\partial T_0}{\partial \dot{q}_j}$$

$$= 2T_2 + T_1 + 0$$

$$= 2T_2 + T_1$$

$$\therefore \sum_j \dot{q}_j \frac{\partial T}{\partial \dot{q}_j} = 2T_2 + T_1$$

Hence, the proof.

Theo. If the Lagrangian does not contain time t explicitly then the total energy of the conservative system is conserved / constant.

proof: For the conservative system the eqⁿ of motion are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0 \quad \text{——(1)}$$

Where, $L = T - V$

~~since, the system is conservative~~

$$L = L(q_j, \dot{q}_j, t) \quad \text{---} (2)$$

Diff. eqⁿ (2) w.r.t. t

$$\frac{dL}{dt} = \sum_k \left[\frac{\partial L}{\partial q_k}\dot{q}_k + \frac{\partial L}{\partial \dot{q}_k}\ddot{q}_k\right] + \frac{\partial L}{\partial t}$$

since, $\frac{\partial L}{\partial t} = 0$

$$\Rightarrow \frac{dL}{dt} = \sum_k \left[\frac{\partial L}{\partial q_k}\dot{q}_k + \frac{\partial L}{\partial \dot{q}_k}\ddot{q}_k\right]$$

from eqⁿ (1) we have

$$\frac{\partial L}{\partial q_k} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right)$$

$$\frac{dL}{dt} = \sum_k \left[\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right)\dot{q}_k + \frac{\partial L}{\partial \dot{q}_k}\ddot{q}_k\right]$$

$$\frac{dL}{dt} = \frac{d}{dt}\left[\sum_k \frac{\partial L}{\partial \dot{q}_k}\dot{q}_k\right]$$

$$\therefore \frac{dL}{dt} - \frac{d}{dt}\left(\sum_k \frac{\partial L}{\partial \dot{q}_k}\dot{q}_k\right) = 0$$

i.e. $\frac{d}{dt}\left(L - \sum_k \frac{\partial L}{\partial \dot{q}_k}\dot{q}_k\right) = 0$ (Derivative is linear operator)

$$\Rightarrow L - \sum_k \frac{\partial L}{\partial \dot{q}_k}\dot{q}_k = \text{constant.} \quad \text{---} (3)$$

since, L does not contain t explicitly and $L = T - V$.

∴ The kinetic energy and potential energy do not contain t explicitly.

∴ The transformation eqⁿ do not contain t explicitly.

∴ The system is scleronomic

$$\therefore \sum_k \dot{q}_k \frac{\partial T}{\partial \dot{q}_k} = 2T \quad \text{---} (4) \text{ - (By theo. case-f)}$$

since, the system is conservative.

$\therefore$ ~~He does not~~ V does not contain generalized velocity $\dot{q}_k$

$$\therefore \frac{\partial V}{\partial \dot{q}_k} = 0$$

$$\therefore \frac{\partial L}{\partial \dot{q}_k} = \frac{\partial}{\partial \dot{q}_k}(T-V) = \frac{\partial T}{\partial \dot{q}_k} - \frac{\partial V}{\partial \dot{q}_k} = \frac{\partial T}{\partial \dot{q}_k}$$

$$\therefore \frac{\partial L}{\partial \dot{q}_k} = \frac{\partial T}{\partial \dot{q}_k} \quad \text{---} (5)$$

Using (3), (4), (5)

$$L - 2T = \text{const.}$$

i.e. $T - V - 2T = \text{const.}$

$$\therefore T + V = \text{const.}$$

Hence, the proof.

Example: show that the new Lagrangian 'L'' defined by

$$L' = L + \frac{d}{dt} f(q_j, t),$$

(where f is arbitrary diff. funⁿ of q_j, t & L is Lagrangian) satisfies Lagranges eqⁿ of motion.

$\Rightarrow$ Given that $L' = L + \frac{d}{dt} f(q_j, t)$ — (1) where $j = 1, 2, \dots n$

Where, L satisfies $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0$ ----- (2)

Then we have to prove that $\frac{d}{dt}\left(\frac{\partial L'}{\partial \dot{q}_j}\right) - \left(\frac{\partial L'}{\partial q_j}\right) = 0$

since, $f = f(q_j, t)$

Differentiating above expression w.r.t. t

$\cancel{\partial f}$ $\frac{df}{dt} = \sum_k \frac{\partial f}{\partial q_k} \dot{q}_k + \frac{\partial f}{\partial t}$ ----(3)

Diff. eqⁿ (3) partially w.r.t. q_j

$$\frac{\partial}{\partial q_j}\left(\frac{df}{dt}\right) = \sum_k \frac{\partial}{\partial q_j}\left(\frac{\partial f}{\partial q_k}\dot{q}_k\right) + \frac{\partial^2 f}{\partial q_j \partial t} \quad \text{----(4)}$$

Now, Diff. eqⁿ (3) partially w.r.t. $\dot{q}_j$

$$\frac{\partial}{\partial \dot{q}_j}\left(\frac{df}{dt}\right) = \frac{\partial f}{\partial q_j}$$

consider,

$$\frac{d}{dt}\left(\frac{\partial}{\partial \dot{q}_j}\left(\frac{df}{dt}\right)\right) = \frac{d}{dt}\left(\frac{\partial f}{\partial q_j}\right) \cancel{\frac{d}{dt}}\left(\frac{\partial}{\partial q_j} f\right)$$

$$= \sum_k \frac{\partial^2 f}{\partial q_k \partial q_j}\dot{q}_k + \frac{\partial^2 f}{\partial q_j \partial t} \quad \text{----(5)}$$

substracting eqⁿ (4) from eqⁿ (5)

$$\frac{d}{dt}\left[\frac{\partial}{\partial \dot{q}_j}\left(\frac{df}{dt}\right)\right] - \frac{\partial}{\partial q_j}\left(\frac{df}{dt}\right) = 0 \quad \text{-----(6)}$$

consider,

$$\frac{d}{dt}\left(\frac{\partial L'}{\partial \dot{q}_j}\right) - \frac{\partial L'}{\partial q_j} = \frac{d}{dt}\left[\frac{\partial}{\partial \dot{q}_j}\left(L + \frac{df}{dt}\right)\right] - \frac{\partial}{\partial q_j}\left(L + \frac{df}{dt}\right)$$

$$\left(\because L' = L + \frac{df}{dt}\right)$$

$$= \frac{d}{dt}\left[\frac{\partial L}{\partial \dot{q}_j} + \underline{\frac{\partial}{\partial \dot{q}_j}\left(\frac{df}{dt}\right)}\right] - \frac{\partial L}{\partial q_j} - \underline{\frac{\partial}{\partial q_j}\left(\frac{df}{dt}\right)}$$

$$= \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} + \underbrace{\frac{d}{dt}\left[\frac{\partial}{\partial \dot{q}_j}\left(\frac{df}{dt}\right)\right] - \frac{\partial}{\partial q_j}\left(\frac{df}{dt}\right)}$$

$= 0 + 0$ ----(by eqⁿ (2) & (6))

$= 0$

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$$\therefore \frac{d}{dt}\left(\frac{\partial L'}{\partial \dot{q}_j}\right) - \frac{\partial L'}{\partial q_j} = 0$$

IMP $\therefore$ New L' satisfies Lagranges eqⁿ of motion.

Ex. 2) A particle is constreined to move on a plane curve $xy = c$ (where c is const.) under gravity. Find lagrangian & hence eqⁿ of motion.

$\Rightarrow$ Given that particle is constrained to move on the plane curve $xy = c$ .... (1)

The kinetic energy of the particle is given by,

$$T = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2) \quad \ldots\ldots (2)$$

The potential energy $V = mgy$ (y is verticle)

We see that x & y are not linearly independent as they are related by the eqⁿ (1).

$y = \frac{c}{x}$

Hence they are not generalized co-ordinates.

x is g.c.

However, we eliminate the variable y by putting $y = \frac{c}{x}$ .... from (1)

$$\therefore \dot{y} = \frac{-c}{x^2} \cdot \dot{x}$$

$$\therefore \dot{y}^2 = \frac{c^2}{x^4} \cdot \dot{x}^2$$

$$T = \frac{1}{2} m \left(\dot{x}^2 + \frac{c^2}{x^4} \cdot \dot{x}^2 \right)$$

$$T = \frac{1}{2} m \left(1 + \frac{c^2}{x^4} \right) \dot{x}^2$$

and $V = \frac{mgc}{x}$

Here, x is the generalized co-ordinates.

Hence, the lagrangian of the particle becomes

22.

$L = L(x, \dot{x}, t)$

$= T - V$

$= \frac{1}{2} m \left(1 + \frac{c^2}{x^4}\right) \dot{x}^2 - \frac{mgc}{x}$

The lagrange's eqn of motion corresponding to generalized co-ordinates x is given by is given by

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0 \text{ - - - - - - } (*)$$

Now,

$$\frac{\partial L}{\partial \dot{x}} = m\left(1 + \frac{c^2}{x^4}\right) \dot{x}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = m\left(1 + \frac{c^2}{x^4}\right) \ddot{x} - \frac{m4c^2}{x^5} \dot{x}^2 \quad \frac{1}{x^4} : -4x^{-5}$$

$$\frac{\partial L}{\partial x} = \frac{1}{2} m\left(\frac{-4c^2}{x^5}\right) \dot{x}^2 + \frac{mgc}{x^2}$$

$$= \frac{-2mc^2\dot{x}^2}{x^5} + \frac{mgc}{x^2}$$

put these values in eqn (*)

$$\frac{d}{dt} \; m\left(1 + \frac{c^2}{x^4}\right) \ddot{x} - 4\frac{mc^2 \dot{x}^2}{x^5} - \left[\frac{-2mc^2\dot{x}^2}{x^5} + \frac{mgc}{x^2}\right] = 0$$

$$m\left(1 + \frac{c^2}{x^4}\right) \ddot{x} - \frac{4mc^2\dot{x}^2}{x^5} + \frac{2mc^2\dot{x}^2}{x^5} - \frac{mgc}{x^2} = 0$$

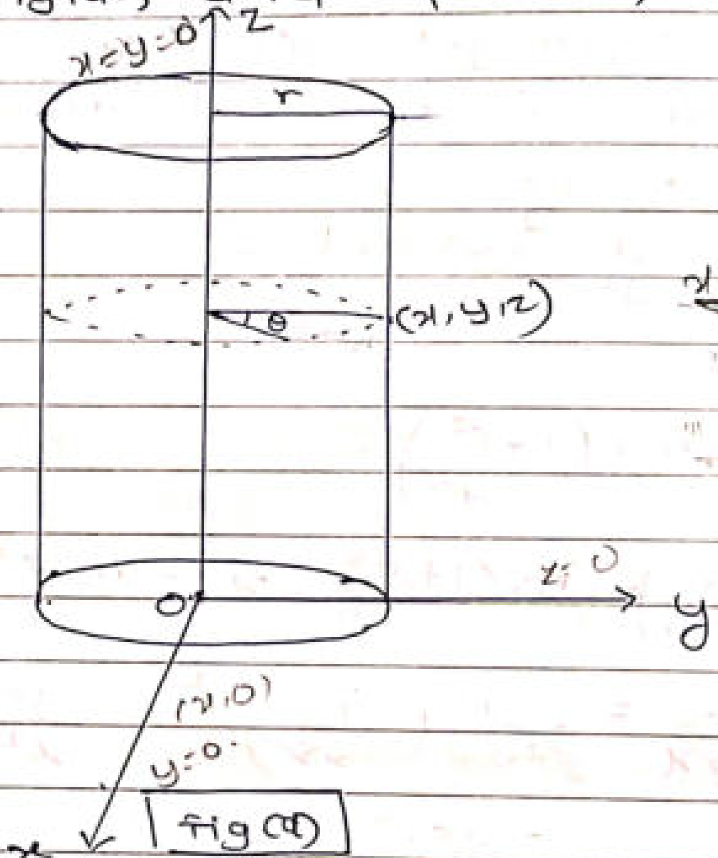
$$m\left(1 + \frac{c^2}{x^4}\right) \ddot{x} - \frac{2mc^2\dot{x}^2}{x^5} - \frac{mgc}{x^2} = 0$$

Which is required lagranges equation of motion

Que.3) A particle is constraints to move on a ~~surp~~ surface of cylinder. of a fixed radius find lagrangian and equation of motion.

$x = r\cos\theta$
$y = r\sin\theta$
$z = z$

g.c. are z and θ



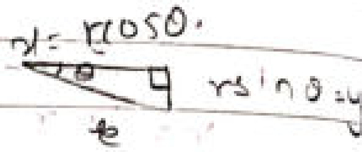


fig (a)

The surface of the cylinder is characterised by the parametric eqn given by

$x = r\cos\theta,\ y = r\sin\theta,\ z = z$ ...... (1)

However x, y are not generalized co-ordinates and x & y are releated by the eqn $x^2+y^2=r^2$ Where r is constant.

Hence, the generalized co-ordinates are z & θ

$\therefore\ T = \frac{1}{2}m(\dot{x}^2+\dot{y}^2+\dot{z}^2)$

$= \frac{1}{2}m(\dot{x}^2+\dot{y}^2) + \frac{1}{2}m\dot{z}^2$

$= \frac{1}{2}m(\underset{const}{\dot{r}^2} + r^2\dot{\theta}^2) + \frac{1}{2}m\dot{z}^2$

$T = \frac{1}{2}m(r^2\dot{\theta}^2+\dot{z}^2)$ .... (2) ; r is const.

$V = mgz$ .... (3)

Hence, the lagrangian is given by

$L = T - V$

$$\boxed{L = \frac{1}{2}m(\dot{r}^2\dot{\theta}^2 + \dot{z}^2) - mgz} \quad \ldots\ldots (4)$$

Now, lagranges eq^n of motion corresponding to generalized coordinate z & θ resp. are given by.

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{z}}\right) - \frac{\partial L}{\partial z} = 0 \quad \ldots\ldots (5)$$

and $$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0 \quad \ldots\ldots (6)$$

Now,

$$\frac{\partial L}{\partial \dot{z}} = m\dot{z}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{z}}\right) = m\ddot{z}$$

$$\frac{\partial L}{\partial z} = -mg$$

$\therefore$ eq^n (5) becomes,

$$m\ddot{z} + mg = 0$$

$$m(\ddot{z} + g) = 0$$

$$\ddot{z} + g = 0$$

$$\ddot{z} = -g$$

$$\dot{z} = -gt + c_1$$

$$\boxed{z = \frac{-gt^2}{2} + c_1t + c_2}$$

Now,

$$\frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta} \quad , \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = mr^2\ddot{\theta}$$

$$\frac{\partial L}{\partial \theta} = 0$$

$\therefore$ ~~eq^n~~ (6) becomes,

$$\therefore \frac{d}{dt}(mr^2\dot{\theta}) = 0$$

$$\Rightarrow mr^2\dot{\theta} = \text{const.}$$

$$\Rightarrow mr^2\dot{\theta} = l \qquad (l \text{ is const.})$$

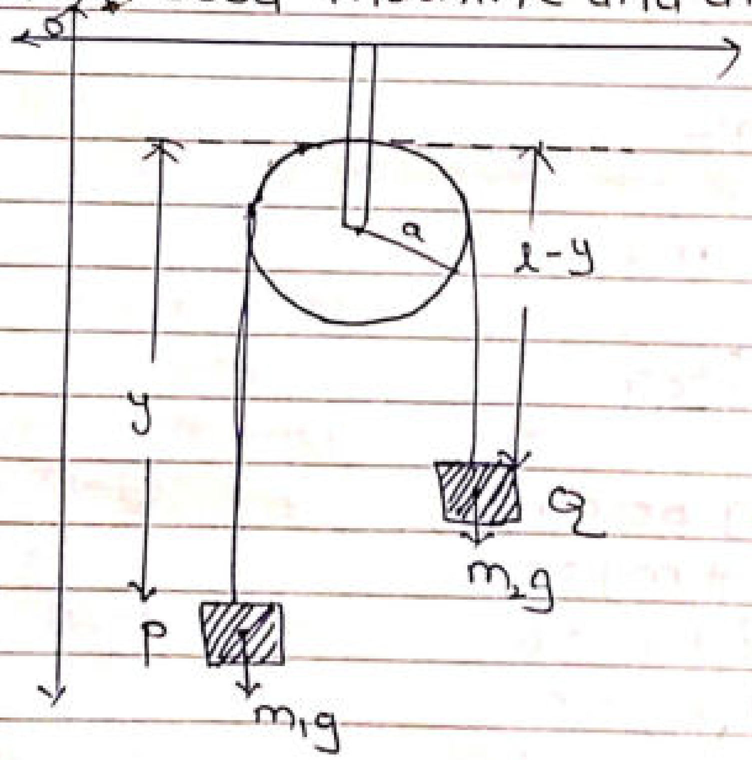
$$\Rightarrow \dot{\theta} = \frac{l}{mr^2} = k \qquad (k \text{ is const})$$

IM.P $$\Rightarrow \boxed{\theta = kt + c_3}$$

Que.24) Explain atwood machine and discuss it's motion

$\Rightarrow$

(4) mark



g.c = y

fig.(a).

Atwood machine consist of two masses m_1 & m_2 joined by spring of fixed length l.
These spring is on ~~pro~~ p rotating ~~pulleg~~ pulley of radius a whose axis is fixed as shown in fig. If we fix m_1 then m_2 also get fixed.
Thus the distance y from horizontal x-axis to

m_1 is generalized co-ordinate.

$\therefore$ DOF = 1

The distance of m_2 from horizontal axis will be then $(l-y)$.

The total kinetic energy of system is

$$T = \frac{1}{2} m_1 \dot{y}^2 + \frac{1}{2} m_2 \left\{ \underbrace{\frac{d}{dt}(l-y)}_{} \right\}^2 \quad l \text{ const}$$

$$= \frac{1}{2}(m_1 + m_2)\dot{y}^2 \quad \cdots\cdots (1)$$

The total potential energy of system is, (Where x-axis is reference line)

$$V = -m_1 g y - m_2 g(l-y)$$

('−' sign is because the particle below the reference line)

$\therefore$ The lagrangian is

$$L(y, \dot{y}, t) = T - V$$

$$\boxed{L = \frac{1}{2}(m_1 + m_2)\dot{y}^2 + m_1 g y + m_2 g(l-y)}$$

$\therefore$ Eq^n of motion corresponding generalized co-ordinate is,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) - \frac{\partial L}{\partial y} = 0$$

$$\frac{\partial L}{\partial \dot{y}} = \frac{1}{2}(m_1 + m_2) \cdot 2 \cdot \dot{y} +$$

$$= (m_1 + m_2)\dot{y}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) = (m_1 + m_2)\ddot{y}$$

$$\frac{\partial L}{\partial y} = m_1 g - m_2 g$$

$$(m_1 + m_2)\ddot{y} - m_1 g + m_2 g = 0$$

$$\cancel{(m_1 + m_2)\ddot{y} + (m_1)}$$

$$(m_1 + m_2)\ddot{y} - m_1 g + m_2 g.$$

$$\ddot{y} = \frac{(m_1 - m_2)g}{(m_1 + m_2)}$$

$$\dot{y} = \frac{(m_1 - m_2)g t}{(m_1 + m_2)} + c_1$$

$$y = \frac{(m_1 - m_2)g t^2}{2(m_1 + m_2)} + c_1 t + c_2$$

Where c_1, c_2 are const. of integration

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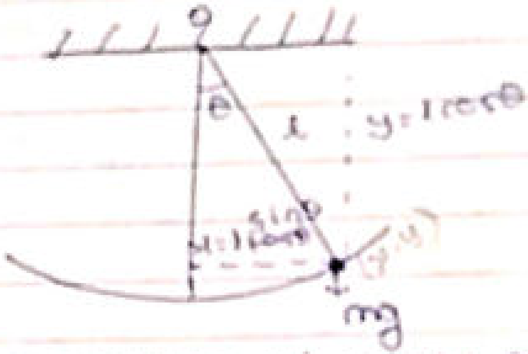
Ex. 5) Set up lagrangian and equation of motion for simple pendulum

⇒



consider a particle of mass m ~~attached to a fixed support (at O) by a light inex~~ and position vector $\bar{r}$ is (x, y) attach to a fixed support by a rigid string of a length l

The motion is in a plane.

If order pair (x, y) are cartesian co-ordinate of particle & θ is a angle made by string with a fixed verticle line then $x = l\sin\theta$, $y = l\cos\theta$

∴ DOF = 1

And hence generalized co-ordinate is θ.

∴ kinetic energy $= T = \frac{1}{2}m(\dot{l}^2 + l^2\dot{\theta}^2)$

$= \frac{1}{2}ml^2\dot{\theta}^2$; where l is const

potential energy $V = -mgy$

$= -mgl\cos\theta$

∴ Lagrangian $L = L(\theta, \dot{\theta}, t)$

$= T - V$

$$L = \frac{1}{2}ml^2\dot{\theta}^2 + mgl\cos\theta$$

∴ The eqn of motion is given by,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$$

$$\frac{\partial L}{\partial \dot\theta} = ml^2\dot\theta$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot\theta}\right) = ml^2\ddot\theta$$

$$\frac{\partial L}{\partial \theta} = -mgl\sin\theta.$$

$$ml^2\ddot\theta + mgl\sin\theta = 0$$

$$ml^2\ddot\theta = -mgl\sin\theta$$

$$\ddot\theta = \frac{-g}{l}\sin\theta.$$

Theo. Show that ~~can~~ non-conservation of total energy is directly associated with the existance of non-conservative force even if the transformation eq^n does not contain time t.

proof: We know that the lagranges eq^n of motion for a system in which conservative forces $\overline{F_i}$ and non-conservative forces $\overline{F_i}^{(d)}$ are ~~prs~~ present are given by.

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot q_j}\right) - \frac{\partial L}{\partial q_j} = Q_j^{(d)} \quad \cdots (1) \quad j = 1, 2, \cdots n$$

Where, lagrangian L contains the potential of the conservative forces and the forces which are not arrising ~~&~~ from potential V are represented by $Q_j^{(d)}$.

Now,

$$L = L(q_j, \dot q_j, t)$$

$$\frac{dL}{dt} = \sum_j \frac{\partial L}{\partial q_j}\cdot \dot q_j + \sum_j \frac{\partial L}{\partial \dot q_j}\ddot q_j + \frac{\partial L}{\partial t}$$

$$\frac{dL}{dt} = \sum_j \left[\frac{\partial L}{\partial q_j}\dot q_j + \frac{\partial L}{\partial \dot q_j}\ddot q_j\right] + \frac{\partial L}{\partial t} \quad \cdots\cdots (2)$$

from eq^n (1) we have,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - Q_j^{(d)} = \frac{\partial L}{\partial q_j}$$

$\therefore$ Eq^n (2) becomes,

$$\frac{dL}{dt} = \sum_j \left[\left(\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - Q_j^{(d)}\right)\dot{q}_j + \frac{\partial L}{\partial \dot{q}_j}\ddot{q}_j\right] + \frac{\partial L}{\partial t}$$

$$= \sum_j \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right)\dot{q}_j - \sum_j Q_j^{(d)}\dot{q}_j + \sum_j \frac{\partial L}{\partial \dot{q}_j}\ddot{q}_j + \frac{\partial L}{\partial t}$$

$$= \sum_j \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right)\dot{q}_j + \sum_j \frac{\partial L}{\partial \dot{q}_j}\ddot{q}_j - \sum_j Q_j^{(d)}\dot{q}_j + \frac{\partial L}{\partial t}$$

$$\frac{dL}{dt} = \frac{d}{dt}\left(\sum_j \frac{\partial L}{\partial \dot{q}_j}\cdot\dot{q}_j\right) - \sum_j Q_j^{(d)}\dot{q}_j + \frac{\partial L}{\partial t} \quad \ldots\ldots (3)$$

Since,

L contains potential of conservative forces.

$$\Rightarrow \frac{\partial L}{\partial \dot{q}_j} = \frac{\partial T}{\partial \dot{q}_j} \quad ; \quad \frac{\partial V}{\partial \dot{q}_j} = 0$$

$\therefore$ eq^n (3) becomes,

$$\frac{dL}{dt} = \frac{d}{dt}\left(\sum_j \frac{\partial T}{\partial \dot{q}_j}\dot{q}_j\right) - \sum_j Q_j^{(d)}\dot{q}_j + \frac{\partial L}{\partial t} \quad \ldots\ldots (4)$$

Here, T is a quadratic fun^n of generalized velocity and hence in this case we have

$$\sum_j \dot{q}_j \frac{\partial T}{\partial \dot{q}_j} = 2T \quad \ldots \text{By theo.} \quad \ldots (5)$$

using (5) in (4) we have

$$\frac{dL}{dt} = \frac{d}{dt}(2T) - \sum_j Q_j^{(d)}\dot{q}_j + \frac{\partial L}{\partial t}$$

$$\frac{d}{dt}(L-2T) = -\sum_j Q_j^{(d)}\dot{q}_j + \frac{\partial L}{\partial t}$$

consider,

$$\begin{aligned} L-2T &= T-V-2T \\ &= -T-V \\ &= -(T+V) \\ &= -E \end{aligned}$$

$$\therefore \frac{d}{dt}(-E) = -\sum_j Q_j^{(d)}\dot{q}_j + \frac{\partial L}{\partial t}$$

$$\Rightarrow \frac{dE}{dt} = \sum_j Q_j^{(d)}\dot{q}_j - \frac{\partial L}{\partial t} \quad \cdots\cdots (6)$$

If the transformation eq^n do not contain time t explicitly then the K.E. does not contain time t.

i.e. $\frac{\partial T}{\partial t} = 0$

Also lagrangian contain potential of conservative forces, We have.

$V = V(q_j)$

$\Rightarrow \frac{\partial V}{\partial t} = 0$

Hence,

$\Rightarrow \frac{\partial L}{\partial t} = 0$

$\therefore$ Eq^n (6) becomes.

$$\Rightarrow \frac{dE}{dt} = \sum_j Q_j^{(d)}\dot{q}_j$$

These shows that the non-conservation of total energy is directly associated with

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the existance of non-conservative forces
Hence, the proof.

Ex. 1) Two mass pt.s of masses m_1 & m_2 are connected by a string a passing through a hole in a smooth table so that m_1 is on the table surface & m_2 hangs suspended. Assuming m_2 moss only in a verticle line, find the generalised coordinate of the system. Write down the lagrangian & eq^n of motion. Reduce th problem to a single 2nd order differential eq^n & find its 1st integral.

$\Rightarrow$

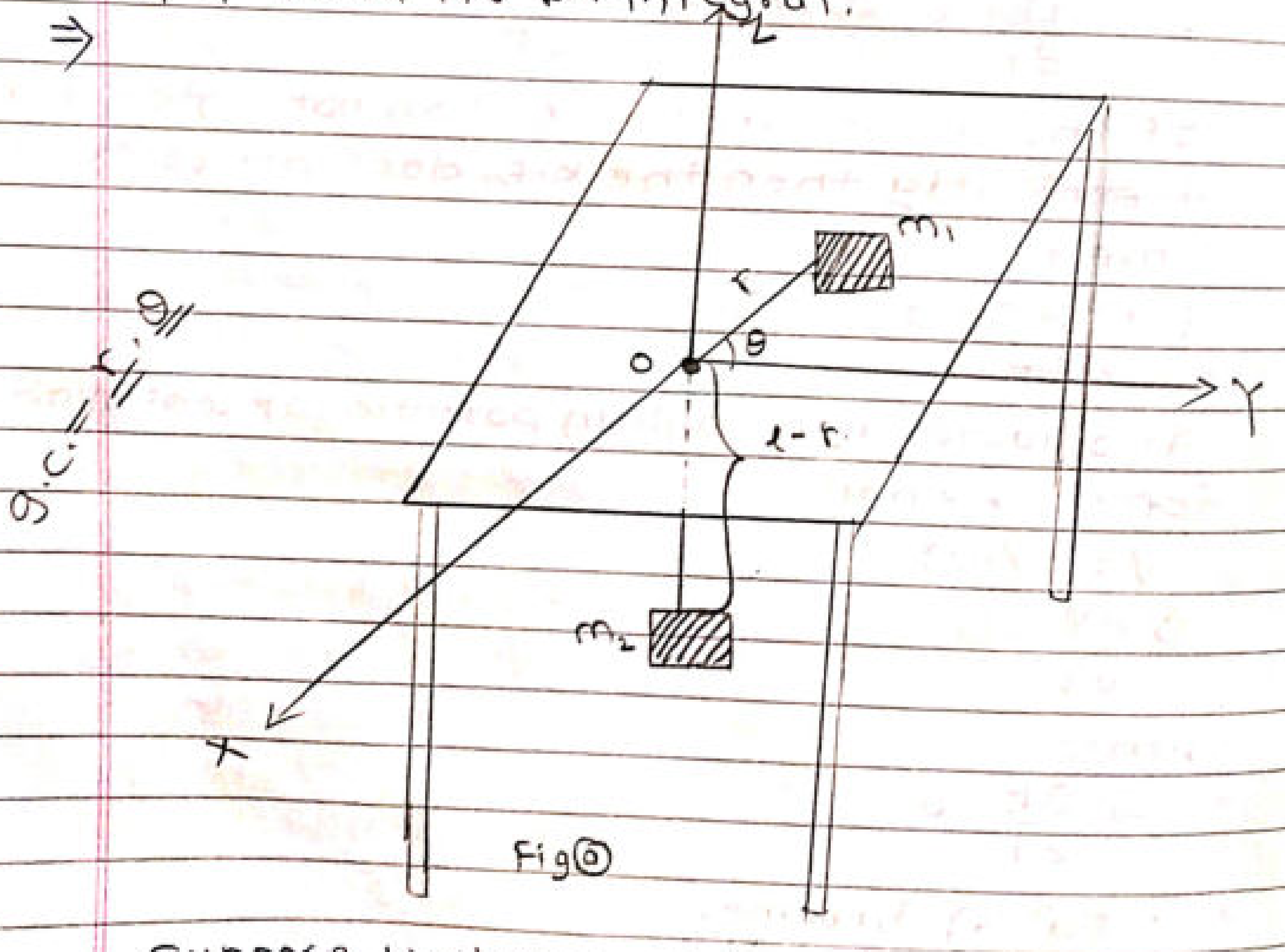


Fig ⓪

Suppose that the string joining m_1 and m_2 passes through a hole on the table at origin. The table surface is assumed as X-Y plane. If the length of the string from origin O to m_1

is r then the distance of m_2 from table surface is $l-r$ (l is the total length of the string). since m_2 moves only in the ~~or~~ vertical direction its position get fixed with the knowledge of $(l-r)$.

Now to fix m_1 we need one more co-ordinate say θ which is angle made by r with some fixed line passing through origin o.

The generalized co-ordinates are 'r' and 'θ'.

The kinetic energy of the system is the sum of K.E. of the two masses and is given.

$$T = \frac{1}{2} m_1 (\dot{x}^2 + \dot{y}^2) + \frac{1}{2} m_2 \left(\frac{d}{dt}(l-r)\right)^2$$

$$= \frac{1}{2} m_1 (\dot{x}^2 + \dot{y}^2) + \frac{1}{2} m_2 \dot{r}^2$$

$$T = \frac{1}{2} m_1 (\dot{r}^2 + r^2\dot{\theta}^2) + \frac{1}{2} m_2 \dot{r}^2 \quad \cdots\cdots (1)$$

potential energy of mass m_1 is zero while that of mass m_2 is $-m_2 g(l-r)$

$$\therefore V = \underline{-m_2 g(l-r)} \quad \cdots\cdots (2)$$

Now, lagrangian $L = L(r, \theta, \dot{r}, \dot{\theta}, t)$

$$\therefore L = T - V$$

$$= \frac{1}{2} m_1 (\dot{x}^2 + \dot{y}^2) + \frac{1}{2} m_2 \dot{r}^2 + m_2 g(l-r)$$

$$\boxed{L = \frac{1}{2} m_1 (\dot{r}^2 + r^2\dot{\theta}^2) + \frac{1}{2} m_2 \dot{r}^2 + m_2 g(l-r)}$$

Now, the lagrange's eqn corresponding to generalized co-ordinates r & θ are given by.

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} = 0 \quad \cdots\cdots (4)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0 \quad \cdots\cdots (5)$$

Now, $\frac{\partial L}{\partial \dot{r}} = m_1\dot{r} + m_2\dot{r} = (m_1+m_2)\dot{r}$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) = (m_1+m_2)\ddot{r}$$

$$\frac{\partial L}{\partial r} = m_1\dot{\theta}^2 r - m_2 g$$

Eq^n (4) becomes,

$$(m_1+m_2)\ddot{r} - (m_1\dot{\theta}^2 r - m_2 g) = 0$$

$$(m_1+m_2)\ddot{r} - m_1\dot{\theta}^2 r + m_2 g = 0 \quad \cdots\cdots (6)$$

Now, $\frac{\partial L}{\partial \dot{\theta}} = m_1 r^2\dot{\theta}$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = m_1 r^2\ddot{\theta}$$

$$\frac{\partial L}{\partial \theta} = 0$$

eq^n (5) becomes

$$m_1 r^2\ddot{\theta} = 0$$

$$\Rightarrow m_1 r^2\dot{\theta} = c \quad \cdots\cdots (7)$$

Eq^n (6) and (7) are required eq^{n}s of motion

from eq^n (7) we have,

$$\dot{\theta} = \frac{c}{m_1 r^2}$$

Substituting these value of $\dot{\theta}^2$ in eq^n(6)

$$(m_1+m_2)\ddot{r} - m_1\left(\frac{c}{m_1 r^2}\right)^2 r + m_2 g = 0$$

$$\Rightarrow (m_1+m_2)\ddot{r} - m_1 \cdot \frac{c^2}{m_1^2 \cdot r^4} \cdot r + m_2 g = 0$$

$$\Rightarrow (m_1+m_2)\ddot{r} - \frac{c^2}{m_1 r^3} + m_2 g = 0 \quad \cdots\cdots (8)$$

Eq^n (8) is the required single second order differencial eq^n.

Now, to find 1st integral of eqn (8).
Multiply eqn (8) by $2\dot{r}$ and integrating it w.r.t. t. We get.

$$2(m_1+m_2)\dot{r}\ddot{r} - \frac{c^2}{m_1 r^3}\cdot 2\dot{r} + 2\dot{r}m_2 g = 0$$

$$(m_1+m_2)(2\dot{r}\ddot{r}) - \frac{c^2 2\dot{r}}{m_1 r^3} + m_2 g(2\dot{r}) = 0$$

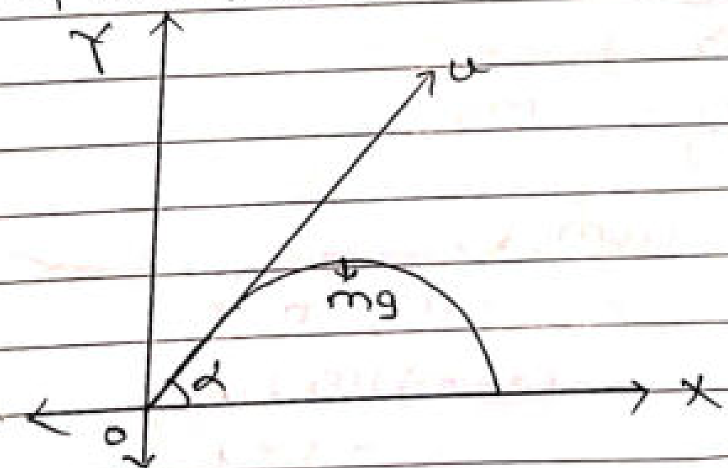
$$\text{i.e. } \frac{d}{dt}\left[(m_1+m_2)\dot{r}^2 + \frac{c^2}{m_1}\cdot\frac{1}{r^2} + 2m_2 g r\right] = 0$$

$$\Rightarrow (m_1+m_2)\dot{r}^2 + \frac{c^2}{m_1}\cdot\frac{1}{r^2} + 2m_2 g r = c_1$$

Where c and c_1 are integrating const.

Ex. 2) A particle of mass m projected with initial velocity u at an angle α with the horizontal. use lagranges eqn of motion to determine the motion of projectile.
IMP

$\Rightarrow$ Let the particle of mass m be projected with an initial velocity u making an angle α with x-axis.



x, y → g.c.

If (x, y) is possition of particle then x & y are independent.

∴ We take x and y are generalized co-ordinate.

∴ the kinetic energy $T = \frac{1}{2}m(\dot{x}^2+\dot{y}^2)$

potential energy $V = mgy$

Thus,

lagrangian $L = L(x, y, \dot{x}, \dot{y}, t)$

$= T - V$

$$\boxed{L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) - mgy} \quad \ldots\ldots (1)$$

$\therefore$ The lagrange's eqn of motion correspond to generalised co-ordinate x & y is given by

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0 \quad \ldots\ldots (2)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) - \frac{\partial L}{\partial y} = 0 \quad \ldots\ldots (3)$$

Now, $\frac{\partial L}{\partial \dot{x}} = m\dot{x}$, $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = m\ddot{x}$

$\frac{\partial L}{\partial x} = 0$.

eqn (2) becomes,

$m\ddot{x} = 0$

$\Rightarrow m\dot{x} = c$ (const. c)

$\Rightarrow \dot{x} = \frac{c}{m}$

$\Rightarrow x = \frac{c}{m}t + c_2$

$\Rightarrow x = c_1 t + c_2$

$m\dot{x} = c$.

$\dot{x} = \frac{c}{m}$

$x = \frac{c}{m}t + c_2$

$x = c_1 t + c_2$ — (4)

Now, $\frac{\partial L}{\partial \dot{y}} = m\dot{y}$, $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) = m\ddot{y}$

$\frac{\partial L}{\partial y} = -mg$

$\therefore$ eqn (3) becomes,

$m\ddot{y} + mg = 0$

$(\ddot{y} + g)m = 0$

$\ddot{y} + g = 0$

$\ddot{y} = -g$

$\dot{y} = -gt + c_3$

$y = -\frac{gt^2}{2} + c_3 t + c_4 \quad \ldots\ldots (5)$

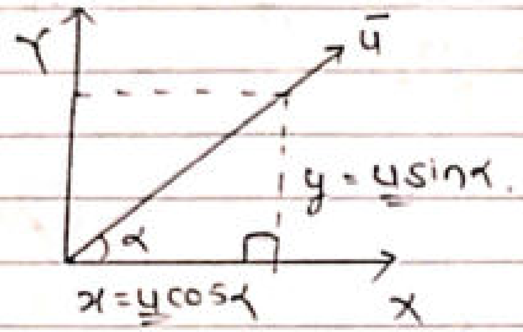
29. $x(t)=0$ $\quad \bar{u}=(u\cos\alpha, u\sin\alpha)$

$x(0)=0,\ y(0)=0$ $\quad \bar{u}=(\dot{x}(0), \dot{y}(0))$

Now, at initial particle at origin

$\therefore\ x(0)=0$ and $y(0)=0$ ...... (6)

The initial velocity vector $\bar{u}$ is making an α with x-axis.



$\therefore\ \bar{u}=(u\cos\alpha, u\sin\alpha)$ $\quad \bar{u}=(\dot{x},\dot{y}) \rightarrow$ u-velocity

$\bar{u}=(\dot{x}(0), \dot{y}(0))$

$\therefore\ \dot{x}(0)=u\cos\alpha$ and $\dot{y}(0)=u\sin\alpha$ ~~using~~ (7)

using eq^n (4), (5), (6) and (7) $\quad x=c_1t+c_2$

we obtain. $\quad y=\frac{-gt^2}{2}+c_3t+c_4$

$c_2=0\ ,\ c_4=0$

$c_1=u\cos\alpha,\ c_3=u\sin\alpha$ $\quad \dot{x}=c_1$

$\dot{y}=-gt+c_3$

putting these values of c_1, c_2, c_3, c_4 in eq^n (4) & (5) we get.

$$\left.\begin{aligned} x &= u\cos\alpha \cdot t \\ y &= \frac{-gt^2}{2}+u\sin\alpha \cdot t \end{aligned}\right\} \text{— (8)}$$

Eliminating 't' from eq^n (8) we get

$t=\frac{x}{u\cos\alpha}$

$\therefore\ y=\frac{-g}{2}\left(\frac{x}{u\cos\alpha}\right)^2+u\sin\alpha\cdot\frac{x}{u\cos\alpha}$

$=\frac{-g}{2}\cdot\frac{x^2}{u^2\cos^2\alpha}+x\cdot\tan\alpha$

$y=\frac{-g}{2}\cdot\frac{x^2}{u^2}\cdot\sec^2\alpha+x\tan\alpha$

Ex. 8) A body of mass m is thrown up and inclined plane which is moving horizontaly with const velocity. use lagrangian eqn to find the locus of position of the body at any time 't' after the motion sets in.

IMP

$\Rightarrow$

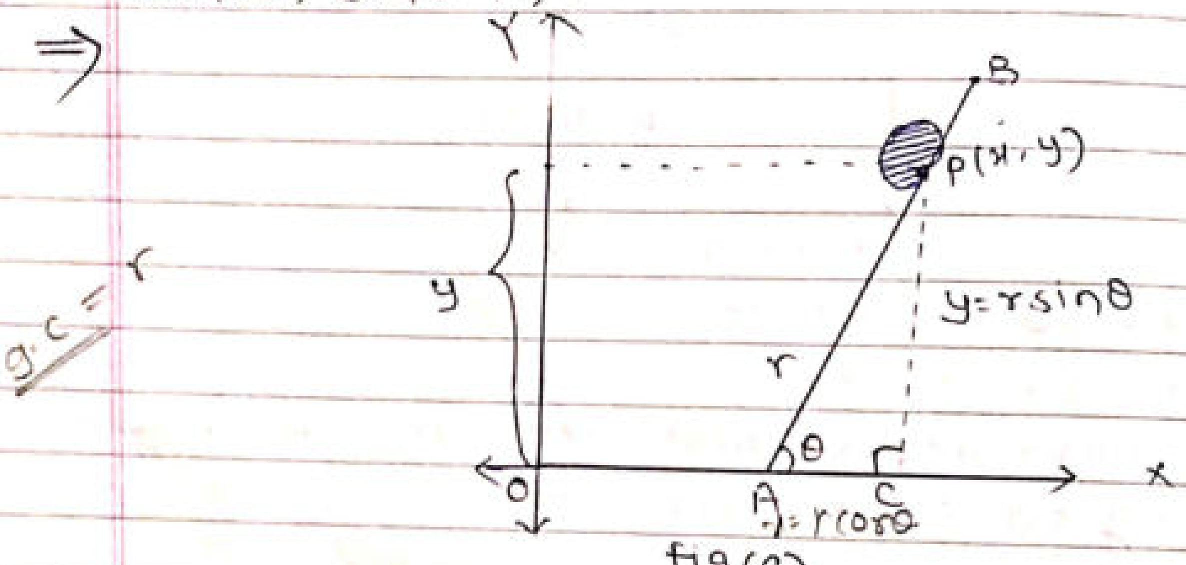


fig (a)

Let AB be a inclined plane moving horizontaly with const. velocity v.

$\therefore$ At instant t the distance moved by plane AB is given by

$OA = vt$

Let at $t=0$ a body of mass m be thrown up on and inclined plane AB.

Let P be the position of particle at that instant t. where $AP = r$.

If (x,y) are the co-ordinates of the particle at P then we have

$x = OA + AP\cos\theta$

$x = vt + r\cos\theta$

& $y = r\sin\theta$

The kinetic energy of particle is given by.

$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2)$

We notice that x & y are not free. Hence will not be the generalized co-ordinates.

The only generalized co-ordinate is 'r'.

$x = vt + r\cos\theta$ and $y = r\sin\theta$

$\dot{x} = v + \cos\theta\,\dot{r}$ & $\dot{y} = \dot{r}\sin\theta$

$\therefore \dot{x}^2 + \dot{y}^2 = (v + \dot{r}\cos\theta)^2 + \dot{r}^2\sin^2\theta$

$= v^2 + 2v\dot{r}\cos\theta + \dot{r}^2\cos^2\theta + \dot{r}^2\sin^2\theta$

$= v^2 + 2v\dot{r}\cos\theta + \dot{r}^2(\cos^2\theta + \sin^2\theta)$

$= v^2 + 2v\dot{r}\cos\theta + \dot{r}^2$

$\therefore T = \frac{1}{2}m(v^2 + 2v\dot{r}\cos\theta + \dot{r}^2)$ ...... (1)

potential energy $= mgy$

$= mgr\sin\theta$ ...... (2)

$\therefore$ Lagrangian $L = L(r, \dot{r}, t)$

$= T - V$

$L = \frac{1}{2}m(v^2 + 2v\dot{r}\cos\theta + \dot{r}^2) - mgr\sin\theta$

The lagranges eq^n of motion corresponding to generalised co-ordinate 'r' is,

$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} = 0$ .... (3)

Now $\frac{\partial L}{\partial \dot{r}} = mv\cos\theta + \dot{r}.m$

$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) = \cancel{mv\cos\theta\dot{r}} + \ddot{r}m$

$\frac{\partial L}{\partial r} = -mg\sin\theta$

eq^n (3) becomes,

$\cancel{mv\cos\theta\dot{r}} + \ddot{r}m + mg\sin\theta = 0$

$\ddot{r}m + mg\sin\theta = 0$

$\ddot{r} + g\sin\theta = 0$

$\dot{r}$ = velocity

$\Rightarrow \ddot{r} = -g\sin\theta$ $\Rightarrow \dot{r} = -g\sin\theta . t + c_1$

$\Rightarrow \ddot{r} = -g\sin\theta$

at $t=0$.

Let $\dot{r} = u$ be the initial velocity of the particle with which it projected

This gives, $\dot{r}(0) = u = c_1$

i.e. $\underline{u = c_1}$

$\dot{r} = u - g\sin\theta . t$

Integrating,

$r(0) = 0$ $r = ut - g\sin\theta . \frac{t^2}{2} + c_2$

at $t=0$ we get,

$\underline{c_2 = 0}$

$r^2 = (x-ut)^2 + y^2$ $\therefore r = ut - g\sin\theta . \frac{t^2}{2}$

* Hence, the locus of the position of the particle is given by u

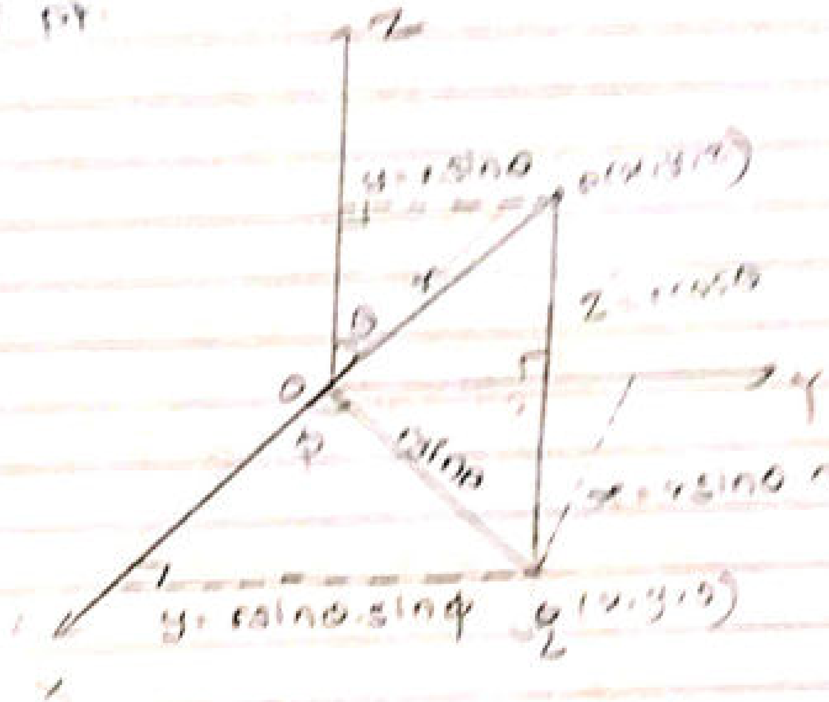
$r^2 = (ut - g\sin\theta . \frac{t^2}{2})^2$

$r^2 = u^2t^2 - \cancel{2}ut . g . \sin\theta . \frac{t^2}{\cancel{2}} + g^2\sin^2\theta . \frac{t^4}{4}$

$r^2 = u^2t^2 - ugt^3\sin\theta + \frac{g^2t^4 . \sin^2\theta}{4}$

$r^2 = (x - ut)^2 + y^2$

- Spherical pendulum:
Consider a point P in a space $\mathbb{R}^3$, suppose and order triple (x, y, z) are cartesian coordinates of this pt.



If r = distance of pt from origin O
θ = angle betn $\overline{OP}$ and z axis
ϕ = angle made by projection of $\overline{OP}$ in x-y plane with x-axis.

then $x = r\sin\theta \cdot \cos\phi$
$y = r\sin\theta \cdot \sin\phi$
$z = r\cos\theta$

In a spherical pendulum a pt mass is constrained to move on the surface of sphere.

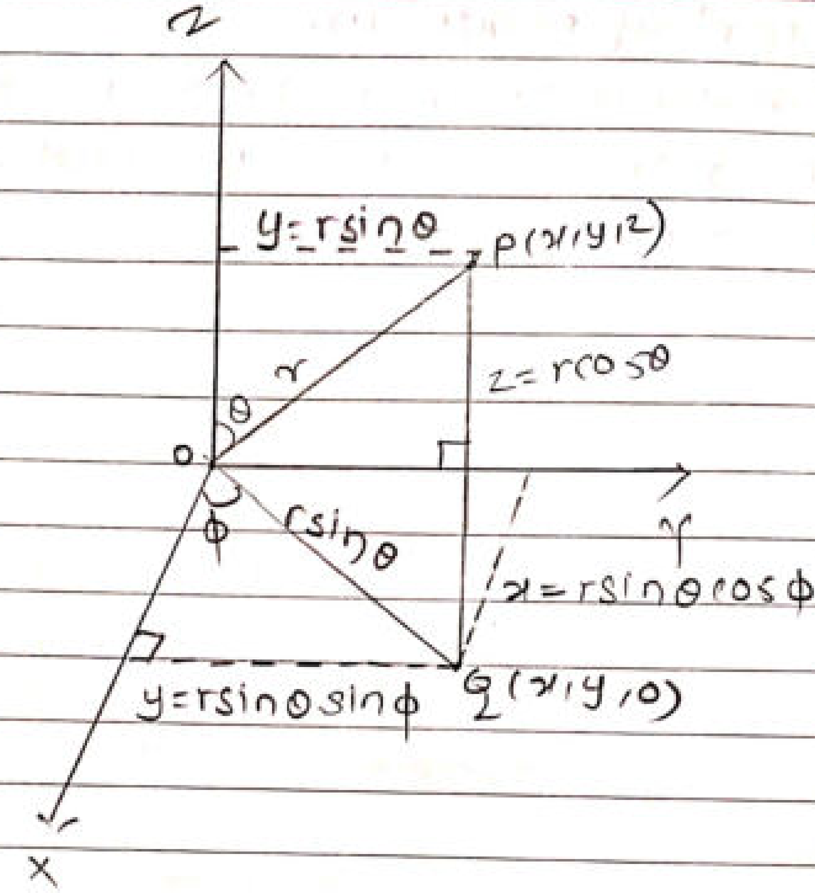
que. To find lagrangian & eq of motion for a particle moving on surface of sphere.

VIMP

or

Suppose that a particle of mass m is constraint to move on surface of sphere. Find lagrangian & eq of motion.

⇒



g.c = θ φ

Let $p(x, y, z)$ be the position co-ordinate of the particle moving on the surface sphere of radius r. ~~#~~

If (r, θ, ϕ) are ~~it~~ is its spherical co-ordinat
Then we have

$$\left.\begin{aligned} x &= r\sin\theta.\cos\phi \\ y &= r\sin\theta.\sin\phi \\ z &= r\cos\theta \end{aligned}\right\} \text{-----} (1)$$

It clearly shows that x, y, z are not the generalized co-ordinates as they are related by constraint relation (1).

The generalised co-ordinates are ~~θ and~~ φ (θ, ϕ)

Hence, kinetic energy & potential energy of particle are respe. given by,

$$T = \frac{1}{2} m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) \text{------} (2)$$

32.

$x = r\sin\theta.\cos\phi$

$\dot{x} = r\cos\theta\dot{\theta}\cos\phi + r\sin\theta.(-\sin\phi).\dot{\phi}$

$\dot{x} = r[\dot{\theta}\cos\theta.\cos\phi - \dot{\phi}\sin\theta.\sin\phi]$

$\dot{x}^2 = r^2[\dot{\theta}\cos\theta.\cos\phi - \dot{\phi}\sin\theta.\sin\phi]^2$

$= r^2[\dot{\theta}^2\cos^2\theta.\cos^2\phi - 2\dot{\theta}\dot{\phi}\cos\theta.\cos\phi.\sin\theta.$
$\sin\phi + \dot{\phi}^2\sin^2\theta.\sin^2\phi]$

$= r^2[$

$= r^2\dot{\theta}^2\cos^2\theta.\underline{\cos^2\phi} - 2\dot{\theta}\dot{\phi}.r^2\underbrace{\cos\theta.\cos\phi.\sin\theta.}_{\text{similar}}$
$\sin\phi + r^2\dot{\phi}^2\sin^2\theta.\underline{\sin^2\phi}$

$y = r\sin\theta.\sin\phi$

$\dot{y} = r\cos\theta.\dot{\theta}\sin\phi + r\sin\theta.\cos\phi.\dot{\phi}$

$\dot{y}^2 = r[\cos\theta.\dot{\theta}\sin\phi + \sin\theta.\cos\phi.\dot{\phi}]^2$

$= r^2[\cos^2\theta.\dot{\theta}^2\sin^2\phi + 2\dot{\theta}\dot{\phi}\cos\theta.\sin\phi.\sin\theta.$
$\cos\phi + \dot{\phi}^2\sin^2\theta.\cos^2\phi]$

$= r^2\cos^2\theta.\dot{\theta}^2\underline{\sin^2\phi} + 2\dot{\theta}\dot{\phi}r^2\underbrace{\sin\theta.\cos\theta.\sin\phi.}_{\text{similar}}$
$\cos\phi + r^2\dot{\phi}^2\sin^2\theta.\underline{\cos^2\phi}$

$z = r\cos\theta$

$\dot{z} = -r\sin\theta.\dot{\theta}$

$\dot{z}^2 = r^2\sin^2\theta.\dot{\theta}^2$

consider,

$\dot{x}^2 + \dot{y}^2 + \dot{z}^2 = r^2\dot{\theta}^2\cos^2\theta.\cos^2\phi - \cancel{2\dot{\theta}\dot{\phi}r^2\cos\theta.\sin\theta}$
$\cancel{\cos\phi.\sin\phi} + r^2\dot{\phi}^2\sin^2\theta.\sin^2\phi.$
$+ r^2\cos^2\theta\,\dot{\theta}^2\sin^2\phi + \cancel{2\dot{\theta}\dot{\phi}r^2\sin\theta.\cos\theta.}$
$\cancel{\sin\phi\cos\phi} + r^2\dot{\phi}^2\sin^2\theta.\cos^2\phi +$
$r^2\sin^2\theta.\dot{\theta}^2$

$= r^2\dot{\theta}^2\cos^2\theta.\cos^2\phi + r^2\dot{\phi}^2\sin^2\theta.\sin^2\phi$
$+ r^2\cos^2\theta.\dot{\theta}^2\sin^2\phi + r^2\dot{\phi}^2\sin^2\theta.\cos^2\phi$
$+ r^2\sin^2\theta\,\dot{\theta}^2$

$= r^2\dot{\theta}^2\cos^2\theta(\cos^2\phi + \sin^2\phi) + r^2\dot{\phi}^2\sin^2\theta.$
$(\sin^2\phi + \cancel{\sin}\cos^2\phi) + r^2\sin^2\theta\,\dot{\theta}^2$

$\dot{x}^2+\dot{y}^2+\dot{z}^2 = r^2\dot{\theta}^2\cos^2\theta + r^2\dot{\phi}^2\sin^2\theta + r^2\sin^2\theta\,\dot{\theta}^2$

$\therefore$ eq^n (2) becomes, $r^2[\dot{\theta}^2\cos^2\theta + \dot{\phi}^2\sin^2\theta + \sin^2\theta\dot{\theta}^2]$

$\therefore T = \frac{1}{2}m(\dot{x}^2+\dot{y}^2+\dot{z}^2)$

$= \frac{1}{2}mr^2(\dot{\theta}^2\cos^2\theta + \dot{\phi}^2\sin^2\theta + \dot{\theta}^2\sin^2\theta)$

$= \frac{1}{2}mr^2[\dot{\theta}^2(\cos^2\theta+\sin^2\theta) + \dot{\phi}^2\sin^2\theta]$

$T = \frac{1}{2}mr^2[\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta]$

potential energy $V = mgr\cos\theta$.

Hence, Lagrangian becomes,

$L = L(\theta, \phi, \dot{\theta}, \dot{\phi}, t)$

$= T - V$

$$\boxed{L = \frac{1}{2}mr^2[\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta] - mgr\cos\theta} \cdots$$

The lagrange's eq^n of motion corresponding to generalised co-ordinates are given by. $(\theta + \phi)$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0 \quad \cdots\cdots (4)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right) - \frac{\partial L}{\partial \phi} = 0 \quad \cdots\cdots (5)$$

Now, $\frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta}$, $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = mr^2\ddot{\theta}$

$\frac{\partial L}{\partial \theta} = mr^2\dot{\phi}^2\cos\theta\cdot\sin\theta\cdot\dot{\theta} + mgr\sin\theta$

$= \frac{1}{2}mr^2\dot{\phi}^2\,2\sin\theta\cdot\cos\theta\cdot\dot{\theta}^2 + mgr\sin\theta$

Eq^n (4) becomes,

$$mr^2\ddot{\theta} - mr^2\dot{\phi}^2\sin\theta\cdot\cos\theta \pm mgr\sin\theta = 0$$

$\therefore \cancel{mr\{\ddot{\theta} -}$

$$\therefore r\ddot{\theta} - r\dot{\phi}^2\sin\theta\cdot\cos\theta - g\sin\theta = 0$$

$$\therefore \ddot{\theta} - \dot{\phi}^2\sin\theta\cdot\cos\theta - \frac{g}{r}\sin\theta = 0 \quad \cdots\cdots (6)$$

Now,

$$\frac{\partial L}{\partial \dot{\phi}} = mr^2\sin^2\theta\cdot\dot{\phi}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right) = \frac{d}{dt}\left(mr^2\sin^2\theta\cdot\dot{\phi}\right)$$

$$\frac{\partial L}{\partial \phi} = 0$$

so using these value in eq^n (5).

$$\frac{d}{dt}\left[mr^2\sin^2\theta\cdot\dot{\phi}\right] = 0$$

$$mr^2\sin^2\theta\cdot\dot{\phi} = \text{const.}$$

$$mr^2\sin^2\theta\,\dot{\phi} = c_1 \text{ (say)} \quad \cdots\cdots (7)$$

$$\dot{\phi} = \frac{c_1}{mr^2\sin^2\theta}$$

substitute these value in eq^n (6)

$$\ddot{\theta} - \frac{c_1^2}{m^2r^4\sin^4\theta}\cdot\sin\theta\cdot\cos\theta - \frac{g}{r}\sin\theta = 0.$$

$$\ddot{\theta} - \frac{c_1^2\cdot\cos\theta}{m^2r^4\sin^3\theta} - \frac{g}{r}\sin\theta = 0$$

$$\boxed{\ddot{\theta} - \frac{c_1^2\cdot\cos\theta}{m^2r^4\cdot\sin^3\theta} - \frac{g}{r}\sin\theta = 0}$$

VIMP

Ex. find Lagrangian and eq^n of motion for double pendulum.

⇒ VIMP

$g.c \to \theta_1, \theta_2$

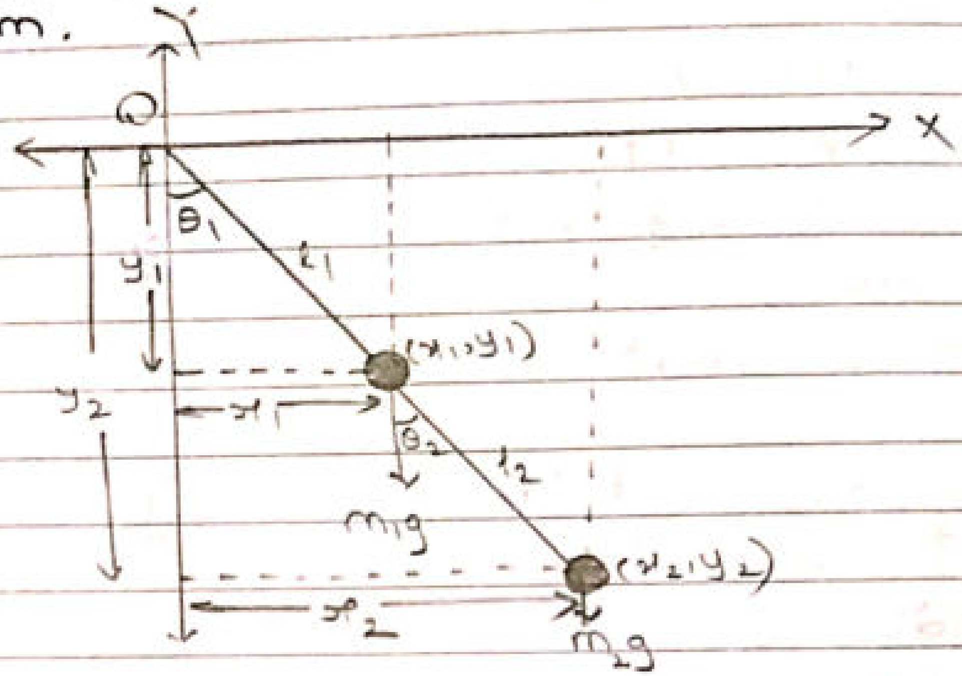


fig.(a) construction of double pendulum

The double is described as below.

consider a particle of mass m_1 attach to some fixed pt. O by a rigid string of length l_1. suppose further that, another particle of mass m_2 is attach to 1st particle by a rigid string of length l_2.

If θ_1 and θ_2 are angles made by l_1 and l_2 with verticles respe. are of then θ_1 and θ_2 are generalized co-ordinates.

If (x_1, y_1) and (x_2, y_2) are cartesian co-ordina- of 1st & 2nd particle then,

$x_1 = l_1 \sin\theta_1$

$y_1 = l_1 \cos\theta_1$

and from fig (a) we have,

$x_2 = l_1 \sin\theta_1 + l_2 \sin\theta_2$

$y_2 = l_1 \cos\theta_1 + l_2 \cos\theta_2$

∴ The total K.E. is given by,

$T = \frac{1}{2} m_1 (\dot{x}_1^2 + \dot{y}_1^2) + \frac{1}{2} m_2 (\dot{x}_2^2 + \dot{y}_2^2)$

* $\dot{x}_1 = l_1 \cos\theta_1 \dot{\theta}_1$ and $\dot{y}_1 = -l_1 \sin\theta_1 \cdot \dot{\theta}_1$

$\dot{x}_1^2 + \dot{y}_1^2 = l_1^2 \cos^2\theta_1 \cdot \dot{\theta}_1^2 + l_1^2 \sin^2\theta_1 \cdot \dot{\theta}_1^2$

$= l_1^2 \dot{\theta}_1^2$

$\dot{x}_2 = l_1 \cos\theta_1 \dot{\theta}_1 + l_2 \cos\theta_2 \dot{\theta}_2$

$\dot{y}_2 = -l_1 \sin\theta_1 \dot{\theta}_1 - l_2 \sin\theta_2 \cdot \dot{\theta}_2$

$\dot{x}_2^2 + \dot{y}_2^2 = (l_1 \cos\theta_1 \dot{\theta}_1 + l_2 \cos\theta_2 \dot{\theta}_2)^2 + [(-1)(l_1 \sin\theta_1 \dot{\theta}_1 + l_2 \sin\theta_2 \dot{\theta}_2)^2]$

$= l_1^2 \cos^2\theta_1 \dot{\theta}_1^2 + 2 l_1 l_2 \cos\theta_1 \cos\theta_2 \dot{\theta}_1 \dot{\theta}_2 + l_2^2 \cos^2\theta_2 \dot{\theta}_2^2 + l_1^2 \sin^2\theta_1 \dot{\theta}_1^2 + 2 l_1 \sin\theta_1 \dot{\theta}_1 \cdot l_2 \sin\theta_2 \dot{\theta}_2 + l_2^2 \sin^2\theta_2 \dot{\theta}_2^2$

$= l_1^2 \dot{\theta}_1^2 (\cos^2\theta_1 + \sin^2\theta_1) + l_2^2 \dot{\theta}_2^2 (\cos^2\theta_2 + \sin^2\theta_2) + 2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 (\cos\theta_1 \cos\theta_2 + \sin\theta_1 \sin\theta_2)$

$= l_1^2 \dot{\theta}_1^2 + l_2^2 \dot{\theta}_2^2 + 2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2)$

$\therefore T = \frac{1}{2} m_1 l_1^2 \dot{\theta}_1^2 + \frac{1}{2} m_2 (l_1^2 \dot{\theta}_1^2 + l_2^2 \dot{\theta}_2^2 + 2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2))$

Taking the refrence level as a horizontal plane through the pt. suspension O, the total potential energy is given by.

* $V = \underline{-m_1 g y_1 - m_2 g y_2}$

$= -m_1 g l_1 \cos\theta_1 - m_2 g (l_1 \cos\theta_1 + l_2 \cos\theta_2)$

$= -m_1 g l_1 \cos\theta_1 - m_2 g l_1 \cos\theta_1 - m_2 g l_2 \cos\theta_2$

$\therefore$ The lagrangian L is

$L = L(\theta_1, \theta_2, \dot{\theta}_1, \dot{\theta}_2, t)$

$= T - V$

★ $L = \frac{1}{2} m_1 l_1^2 \dot{\theta}_1^2 + \frac{1}{2} m_2 (l_1^2 \dot{\theta}_1^2 + l_2^2 \dot{\theta}_2^2 + 2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2))$

$+ m_1 g l_1 \cos\theta_1 + m_2 g l_1 \cos\theta_1 + m_2 g l_2 \cos\theta_2$

$g l_1 \cos\theta_1 (m_1 + m_2) + m_2 g l_2 \cos\theta_2$

Now, the lagrange's eq^n of motion correspondi[ng] to generalised co-ordinates θ_1 & θ_2 are respe. given by,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_1}\right) - \frac{\partial L}{\partial \theta_1} = 0 \quad \cdots\cdots (1)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_2}\right) - \frac{\partial L}{\partial \theta_2} = 0 \quad \cdots\cdots (2)$$

Now

$$\frac{\partial L}{\partial \dot{\theta}_1} = m_1 l_1^2 \dot{\theta}_1 + m_2 l_1^2 \dot{\theta}_1 + 2 l_1 l_2 \cdot \frac{m_2}{2} \dot{\theta}_2 \cos(\theta_1 - \theta_2)$$

$$\frac{\partial L}{\partial \dot{\theta}_1} = m_1 l_1^2 \dot{\theta}_1 + m_2 l_1^2 \dot{\theta}_1 + \frac{m_2}{2} l_1 l_2 \dot{\theta}_2 \cos(\theta_1 - \theta_2)$$

$$\cancel{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_1}\right) = m_1 l_1^2 \ddot{\theta}_1 + m_2 l_2^2} \qquad \theta = \theta(t).$$

$$\frac{\partial L}{\partial \dot{\theta}_1} = (m_1 + m_2) l_1^2 \dot{\theta}_1 + m_2 l_2 l_1 \underset{u}{\dot{\theta}_2} \underset{v}{\cos(\theta_1 - \theta_2)}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_1}\right) = (m_1 + m_2) l_1^2 \ddot{\theta}_1 + m_2 l_2 l_1 \ddot{\theta}_2 \cos(\theta_1 - \theta_2) - m_2 l_1 l_2 \dot{\theta}_2 \sin(\theta_1 - \theta_2)(\dot{\theta}_1 - \dot{\theta}_2)$$

$$= (m_1 + m_2) l_1^2 \ddot{\theta}_1 + m_2 l_1 l_2 \ddot{\theta}_2 \cos(\theta_1 - \theta_2) - m_2 l_1 l_2 \dot{\theta}_2 \sin(\theta_1 - \theta_2)\dot{\theta}_1 + m_2 l_1 l_2 \dot{\theta}_2 \sin(\theta_1 - \theta_2)\dot{\theta}_2$$

$$\frac{\partial L}{\partial \theta_1} = -m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2) - g l_1 \sin\theta_1 (m_1 + m_2)$$

$\therefore$ eq^n (1) becomes

$$(m_1 + m_2) l_1^2 \ddot{\theta}_1 + m_2 l_1 l_2 \ddot{\theta}_2 \cos(\theta_1 - \theta_2) + m_2 l_1 l_2 \dot{\theta}_2 \sin(\theta_1 - \theta_2)\dot{\theta}_2 + g l_1 \sin\theta_1 (m_1 + m_2) = 0$$

$$\frac{\partial L}{\partial \dot{\theta}_2} = m_2 l_2^2 \dot{\theta}_2 + m_2 l_1 l_2 \dot{\theta}_1 \cos(\theta_1 - \theta_2)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_2}\right) = m_2 l_2^2 \ddot{\theta}_2 + m_2 l_1 l_2 \dot{\theta}_1 \cos(\theta_1 - \theta_2) - m_2 l_2 l_1 \dot{\theta}_1 \sin(\theta_1 - \theta_2)(\dot{\theta}_1 - \dot{\theta}_2)$$

$$= m_2 l_2^2 \ddot{\theta}_2 + m_2 l_2 l_1 \dot{\theta}_1 \cos(\theta_1 - \theta_2) - m_2 l_1 l_2 \dot{\theta}_1 \sin(\theta_1 - \theta_2)\dot{\theta}_1 + \underline{m_2 l_1 l_2 \dot{\theta}_1 \sin(\theta_1 - \theta_2)\dot{\theta}_2}$$

$\times$

$$\frac{\partial L}{\partial \theta_2} = m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 (-\sin(\theta_1 - \theta_2))(-1) - m_2 g l_2 \sin\theta_2$$

$$= m_2 l_2 l_1 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2) - m_2 l_2 g \sin\theta_2$$

$\times$

$\therefore$ eq^n (2) becomes,

$$m_2 l_2^2 \ddot{\theta}_2 + m_2 l_1 l_2 \dot{\theta}_1 \cos(\theta_1 - \theta_2) - m_2 l_1 l_2 \dot{\theta}_1 \sin(\theta_1 - \theta_2)\dot{\theta}_1 + \underline{m_2 l_1 l_2 \dot{\theta}_1 \sin(\theta_1 - \theta_2)}\dot{\theta}_2 - \cancel{m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2)} + m_2 l_2 g \sin\theta_2 = 0$$

$$\Rightarrow m_2 l_2^2 \ddot{\theta}_2 + m_2 l_1 l_2 \dot{\theta}_1 \cos(\theta_1 - \theta_2) - m_2 l_1 l_2 \dot{\theta}_1^2 \sin(\theta_1 - \theta_2) + m_2 l_2 g \sin\theta_2 = 0$$

$$\Rightarrow m_2 l_2^2 \ddot{\theta}_2 + m_2 l_1 l_2 \dot{\theta}_1 [\cos(\theta_1 - \theta_2) - \theta_1 \sin(\theta_1 - \theta_2)] + m_2 l_2 g \sin\theta_2 = 0$$

$\therefore$ Which required equation of lagrange's motion.

- compound pendulum:

A rigid body capable of oscillating in a verticle plane about a fixeed horizontal axis under the action of gravity is called a compound pendulum.

Que. Set up the lagrangian and find eqⁿ of motion for a compound pendulum.

VIMP

VIMP

g.c - θ

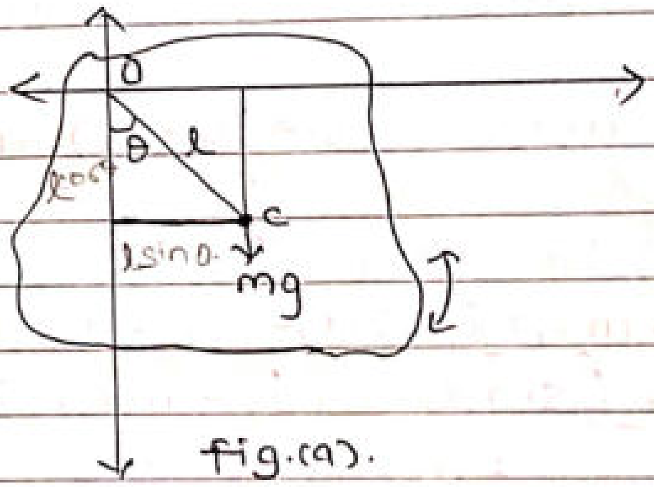


fig.(a).

consider a rigid body oscillating about a fixed horizontal axis passing through O. Let 'c' be the centre of mass of these body & $OC = l$.

Let m be the mass of these body and I be the moment of intertia about axis of rotation. If θ is angle of diplection then the rotational K.E. of these body is

* $T = \frac{1}{2} I \dot{\theta}^2$

The pontial (potential) energy relative to horizontal plane passing through O is $V = -mgl\cos\theta$

∴ The lagrangion $L = L(\theta, \dot{\theta}, t)$

$$= T - V$$

$$L = \frac{1}{2} I \dot{\theta}^2 + mgl\cos\theta$$

36. $T = 2\pi\sqrt{\dfrac{I}{mgl}}$

Now the eqⁿ of motion corresponding to generalised co-ordinate θ is,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$$

Now, $\dfrac{\partial L}{\partial \dot{\theta}} = I\dot{\theta}$, $\dfrac{d}{dt}\left(\dfrac{\partial L}{\partial \dot{\theta}}\right) = I\ddot{\theta}$

$\dfrac{\partial L}{\partial \theta} = -mgl\sin\theta$

$\therefore\ I\ddot{\theta} + mgl\sin\theta = 0$

i.e. $\ddot{\theta} + \dfrac{mgl\sin\theta}{I} = 0$ $\quad \ddot{\theta} = -\frac{g}{l}\sin\theta$

Which is required eqⁿ of motion.

M.C.Q

Note. The ~~perod~~ periodic time of oscillation of compound pendulum is given by,

* $T = 2\pi\sqrt{\dfrac{I}{mgl}}$ $\quad T = \frac{1}{2}I\dot{\theta}^2 \quad V = -mgl\cos\theta$

- Generalized momentum:

consider a particle moving in a plane under gravity, if m is mass of these particle then K.E $T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2)$

observe that,

$\dfrac{\partial T}{\partial \dot{x}} = m\dot{x}$

$= p_x$ (momentum along X-axis)

$\dfrac{\partial T}{\partial \dot{y}} = m\dot{y}$

$= p_y$ (momentum along Y-axis)

p_j corresponding to generalised co-ordinate.

We generalized this concept when T is a funⁿ of generalized velocity $\dot{q}_j$.
consider a system with generalized co-ordinate q_j we define $p_j = \frac{\partial T}{\partial \dot{q}_j}$ as

$\frac{\partial T}{\partial \dot{q}}$ generalized momentum corresponding to generalized co-ordinate q_j.

Note: If system is conservative. then $\frac{\partial U}{\partial \dot{q}_j} = 0$

$\Rightarrow p_j = \frac{\partial L}{\partial \dot{q}_j}$ $\quad p_j = \frac{\partial T}{\partial q_j}$ $\quad p_j = \frac{\partial L}{\partial q_j}$

- ~~cyclic~~ cyclic co-ordinate / Ignorable co-ordinate:-

Defiⁿ: The co-ordinate which is absent in lagrangian is called as cyclic or ignorable co-ordinate. (Note that corresponding generalised velocity may be present in L)

Ex. ϕ is cyclic co-ordinate for lagrangian in case of spherical pendulum.

Result: show that the generalized momentum corresponding to cyclic co-ordinate is conserved or constant.

$\Rightarrow$ consider conservative system with lagrangian L. suppose that generalized co-ordinate q_k is cyclic in L:

$\therefore$ We have, $\frac{\partial L}{\partial q_k} = 0 \; \ldots\ldots (1)$

The Lagranges eqⁿ of motion corresponding to generalized coordinate q_k is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = 0 \quad \cdots\cdots (2)$$

Using (1) in (2), we have,

$$\Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) = 0 \qquad p_j = \frac{\partial T}{\partial \dot{q}_j}$$

$$\Rightarrow \frac{\partial L}{\partial \dot{q}_k} = \text{const.} \qquad Q_j \leftarrow F$$

$$p_j, q_j \leftarrow \bar{P}$$

$$\Rightarrow p_k = \text{const.} \rightarrow Q_j \leftarrow \bar{N}$$

$$p_j, q_j \leftarrow \bar{L}$$

Hence, the proof.

UFMP UFMP

Theo. 5 If the cyclic generalized co-ordinate q_j is such that dq_j represents the translation of the system then prove that total linear momentum is conserved. $\bar{L}$

OR

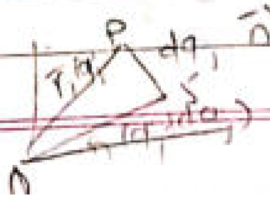
Show that Q_j represents the compound component of total force acting along the action of translation of q_j and P_j is the component of total linear momentum along this direction.

proof. consider a conservative system so that P.E V is a funⁿ of generalized co-ordinate q_j only.

i.e. $V = V(q_j)$ $\qquad Q_j = \bar{F}\cdot\hat{n}$

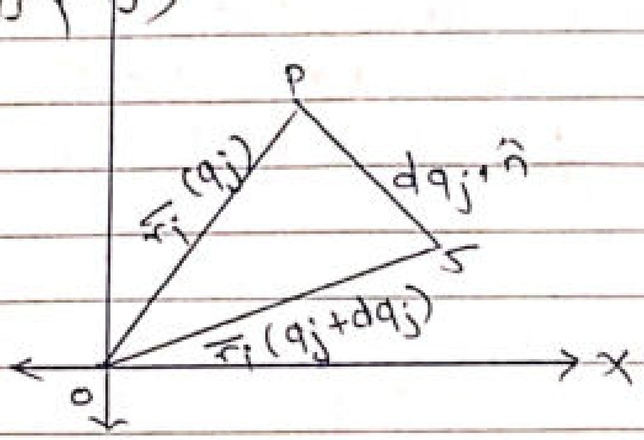
$$\Rightarrow \frac{\partial V}{\partial \dot{q}_j} = 0 \quad \cdots\cdots\cdots (1) \qquad P_j = \bar{P}\cdot\hat{n}$$

$$\bar{P} = \text{const}$$



$P = \bar{r}_i(q_j)$

$S = \bar{r}_i(q_j + dq_j)$



Let $P = \bar{r}_i(q_j)$ be the initial position and $S = \bar{r}_i(q_j + dq_j)$ be the position after the translation dq_j

Let $\hat{n}$ be the unit vector along the direction of translation dq_j.

* Step. I:

claim: Q_j is component of total force along $\hat{n}$

We have,

$$\frac{\partial \bar{r}_i}{\partial q_j} = \lim_{dq_j \to 0} \frac{\bar{r}_i(q_j + dq_j) - \bar{r}_i(q_j)}{dq_j}$$

$$= \lim_{dq_j \to 0} \frac{\overline{PS}}{dq_j}$$

$$= \lim_{dq_j \to 0} \frac{dq_j \cdot \hat{n}}{dq_j}$$

$$= \hat{n} \quad \cdots\cdots\cdots (2)$$

Now the component of generalized force is,

$$Q_j = \sum_i \bar{F}_i \frac{\partial \bar{r}_i}{\partial q_j}$$

$$= \sum_i \bar{F}_i \, \hat{n} \quad \cdots \text{ from (2)}$$

$$= \bar{F} \, \hat{n} \qquad ; \bar{F} = \sum_i \bar{F}_i$$

$$\therefore Q_j = \bar{F} \cdot \hat{n} \quad \cdots (3)$$

$Q_j \rightarrow F$ $\quad$ $P_j \rightarrow$ ge. momentum

$P_j \rightarrow P$ $\quad$ $p \rightarrow$ linear moment

$q_p \rightarrow$ cyclic $\rightarrow P \rightarrow$ cost.

Where F is the total force acting on the system.

$\therefore$ from eq^n (3) we have,

Q_j are component of total force in the direction of $\hat{n}$.

step. II:

* claim: P_j is component of total linear momentum along $\hat{n}$. P

We have,

$$P_j = \frac{\partial T}{\partial \dot{q}_j}$$

$$= \frac{\partial}{\partial \dot{q}_j}\left(\sum_i \frac{1}{2} m_i \dot{\bar{r}}_i^2\right)$$

$$= \cancel{\sum_i} \cancel{\tfrac{1}{2}} \sum_i m_i \dot{\bar{r}}_i \cdot \frac{\partial \dot{\bar{r}}_i}{\partial \dot{q}_j}$$

$$= \sum_i m_i \dot{\bar{r}}_i \frac{\partial \bar{r}_i}{\partial q_j} \quad ; \quad \frac{\partial \dot{\bar{r}}_i}{\partial \dot{q}_j} = \frac{\partial \bar{r}_i}{\partial q_j}$$

from eq^n (2) we have

$$P_j = \sum_i m_i \dot{\bar{r}}_i \cdot \hat{n}$$

$$= \left(\sum_i m_i \dot{\bar{r}}_i\right) \hat{n}$$

$$= \bar{P} \cdot \hat{n} \quad \ldots\ldots (4)$$

Where, $\bar{P} = \sum_i m_i \dot{\bar{r}}_i$ is the total linear momentum of the system.

step-III:

If q_j is cyclic then

$$\frac{\partial L}{\partial q_j} = 0$$

$\therefore$ Lagrange's eq^n of motion corresponding to q_j is given by,

Q_j - zero.
$p \rightarrow$ conserved

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) = 0$$

i.e. $$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) = 0$$

$$\frac{d}{dt}(p_j) = 0$$

$\Rightarrow \cancel{\dot{p}_j \neq 0}$

$\Rightarrow p_j = \text{const.}$

$\Rightarrow \bar{p} \cdot \hat{n} = \text{const.}$ ... from (4)

$\Rightarrow \cancel{\bar{p} = \text{const.}}$

$\Rightarrow \bar{p}$ is conserved along $\hat{n}$.

Hence, the proof.

Note: If the component of total force Q_j are zero. then the total linear momentum is const.

V.IMP.

Theo. If the cyclic generalized co-ordinate q_j is such that dq_j represents the rotation of the system around some axis $\hat{n}$ then the total angular momentum is conserved along $\hat{n}$.

6

V.IMP

OR

show that Q_j are component of total torque along the axis of rotation $\hat{n}$ and p_j are component of total angular momentum is const along $\hat{n}$.

proof:

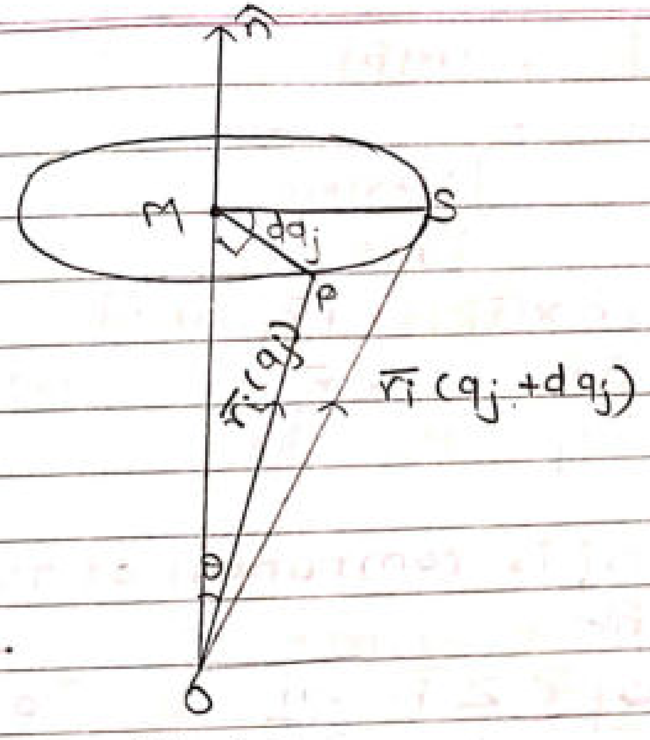


fig.(a)

consider a ~~system~~ conservative system so that the P.E. is a funn of generalized co-ordinate only.

$$V = V(q_j)$$

$$\therefore \frac{\partial V}{\partial \dot{q}_j} = 0 \quad \cdots\cdots (1)$$

Suppose that P is a initial position and system is rotated through an angle dq_j about unit vector $\hat{n}$. The final position of system i.e. S.

$\therefore \overline{OP} = \bar{r}_i(q_j)$

and $\overline{OS} = \bar{r}_i(q_j + dq_j)$

Now,

$$\left|\frac{\partial \bar{r}_i}{\partial q_j}\right| = \left|\lim_{dq_j \to 0} \frac{\bar{r}_i(q_j+dq_j) - \bar{r}_i(q_j)}{dq_j}\right|$$

$$= \left|\lim_{dq_j \to 0} \frac{\overline{PS}}{dq_j}\right|$$

$$= \left|\lim_{dq_j \to 0} \frac{\overline{MP}\, dq_j}{dq_j}\right| \quad ; S = r\theta$$

$\bar{a}\cdot(\bar{b}\times\bar{c}) = \bar{b}\cdot(\bar{c}\times\bar{a}) = \bar{c}\cdot(\bar{a}\times\bar{b})$

scalar triple product

$$\left|\frac{\partial \bar{r}_i}{\partial q_j}\right| = |\overline{mp}| \qquad |\bar{a}\times\bar{b}| = |\bar{a}||\bar{b}|\sin\alpha$$

$$= |\overline{op}\sin\theta|$$

$$= |\bar{r}_i \sin\theta|$$

Also, $|\hat{n}\times\bar{r}_i| = |\bar{r}_i \sin\theta|$ $\quad |\hat{n}\times\bar{r}_i|$

$$\therefore \frac{\partial \bar{r}_i}{\partial q_j} = \hat{n}\times\bar{r}_i \quad \cdots\cdots (2)$$

* step. I

<u>claim</u>: Q_j is component of total torque along $\hat{n}$.

$$\therefore Q_j = \sum_i \bar{F}_i \frac{\partial \bar{r}_i}{\partial q_j}$$

$$= \sum_i \bar{F}_i\,(\hat{n}\times\bar{r}_i) \quad \cdots\cdots (\text{by } (2))$$

$$= \sum_i \hat{n}\cdot(\bar{r}_i\times\bar{F}_i) \qquad L = \bar{r}\times\bar{p}$$

$$\qquad\qquad N = \bar{r}\times\bar{F}$$

$$= \hat{n}\sum_i(\bar{r}_i\times\bar{F}_i)$$

$$Q_j = \hat{n}N \quad \cdots\cdots\cdots (3)$$

Where, $N = \sum \bar{r}_i\times\bar{F}_i$ is the total torque acting on the system, eq^n(3) shows that Q_j are the component of the total torque along axis of rotation.

* step. II:

<u>claim</u>: P_j is component of total angular momentum. along $\hat{n}$.

We have,

$$P_j = \frac{\partial T}{\partial \dot{q}_j}$$

$$= \frac{\partial}{\partial \dot{q}_j}\left(\frac{1}{2}m_i\dot{\bar{r}}_i^2\right)$$

40. $q_j \to N$ L - angular momentum
$p_j \to L$ N - Torque.
$q_j \to L$ is const. p_j = gene. momentum

$$= \sum_i m_i \dot{\bar{r}}_i \frac{\partial \dot{\bar{r}}_i}{\partial \dot{q}_j}$$

$q_j : \hat{n}\bar{N}$
$p_j : \hat{n}\bar{L}$

$$= \sum_i m_i \dot{\bar{r}}_i \frac{\partial \bar{r}_i}{\partial q_j} \quad \cdots \quad \frac{\partial \dot{\bar{r}}_i}{\partial \dot{q}_j} = \frac{\partial \bar{r}_i}{\partial q_j}$$

$q_j : \hat{n}\bar{F}$
$p_j : \hat{n}\bar{P}$

$$= \sum_i m_i \dot{\bar{r}}_i (\hat{n} \times \bar{r}_i)$$

$$= \sum_i \hat{n}\cdot(\bar{r}_i \times m_i \dot{\bar{r}}_i) \quad \sum_p \hat{n}(\bar{r}_i \times \bar{p}_i)$$

$$p_j = \hat{n}\bar{L} \cdots\cdots (4)$$

Where, $L = \sum_i \bar{r}_i \times m_i \dot{\bar{r}}_i$ is total angular momentum of system.
eq^n (4) shows that p_j are component of total angular momentum of the system along the axis of rotation.

* Step III:

Claim: If q_j are cyclic then $\frac{\partial L}{\partial q_j} = 0$

$\therefore$ Lagrange's eq^n of motion becomes,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) = 0$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) = 0 \text{, system is conservative}$$

$$\therefore \frac{\partial U}{\partial \dot{q}_j} = 0$$

i.e. $\frac{\partial T}{\partial \dot{q}_j}$ = constant.

$\Rightarrow p_j$ = constant

from eq^n (4) we have,

$\hat{n}\bar{L}$ = constant

$\therefore \bar{L}$ is constant along $\hat{n}$.

Hence, the proof.