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unit-III - Hamilton's principle and Hamilton's formulation

Introduction:

In this chapter, we discuss Hamilton's principle (which is used to find eq^ns of motion) further, we defined Hamiltonian ~~H~~ fun^n H derived Halmilton's eq^n of motion and solve examples.

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- Hamilton's principle (for non-conservative system)

Hamilton's principle for non-conservative system states that " The motion of dynamical system betn two points in the intervals t_0 to t_1 is such that the line integral

$$I = \int_{t_0}^{t_1} (T + W)\, dt$$

is extremum for the actual path followed by the system". Where T is K.E & W is the work done by the particle

i.e. mathematically,

$$\delta I = 0$$

$$\text{i.e.}\quad \delta \int_{t_0}^{t_1} (T + W)\, dt = 0$$

Hamilton's principle (for conservative system):

" of all possible paths betn two pt.s along which a dynamical system may more from one pt. to another within a given time interval from t_0 to t_1 the actual path followed by the system is the one which minimizes the line integral of Lagrangian."

This means that the motion of a dynamical system from t_0 to t_1 is such that the line integral $\int_{t_0}^{t_1} L.dt$ is extremum for actual path.

i.e. mathematically,

$$\delta \int_{t_0}^{t_1} L.dt = 0$$

Note:
The integrals $\int_{t_0}^{t_1} (T+W)dt$ and $\int_{t_0}^{t_1} L.dt$ involved in above principles are called as "action" or "action integrals".

Theo. 1)
Que. Derive Hamilton's principle for a non-conservative system from D' Alembert's principle. further derive the Hamilton's principle for conservative system from it.
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proof: The D'Alembert's principle is

$$\sum_i (\bar{F}_i - \dot{\bar{P}}_i)\delta \bar{r}_i = 0 \text{ ----- } (1)$$

Which holds for all types of system (i.e. conservative as well as non-conservative
We write eq^n (1) as

$$\sum_i \bar{F}_i.\delta\bar{r}_i = \sum_i \dot{\bar{P}}_i \delta \bar{r}_i \text{ -------- } (2)$$

$$\delta W = \sum_i \dot{\bar{P}}_i \delta \bar{r}_i \text{ -------- } (3)$$

Where, $\delta W = \sum_i \bar{F}_i \delta \bar{r}_i$ is virtual work.
Now, consider

$$\sum_i \dot{\bar{P}}_i \delta \bar{r}_i = \sum_i m_i \ddot{\bar{r}}_i \delta \bar{r}_i \qquad ; \bar{P}_i = m_i \dot{\bar{r}}_i$$

$$= \frac{d}{dt}\Big(\sum_i m_i \dot{\bar{r}}_i \underbrace{\delta r_i}_{u.v.}\Big) - \sum_i m_i \dot{\bar{r}}_i \frac{d}{dt}\delta \bar{r}_i$$

$$\sum_i m_i \ddot{\bar{r}}_i \delta \bar{r}_i + \sum_i m_i \dot{\bar{r}}_i \frac{d}{dt}\delta \bar{r}_i$$

$$= \frac{d}{dt}\left[\sum_i m_i \dot{\bar{r}}_i \delta \bar{r}_i\right] - \sum_i m_i \dot{\bar{r}}_i \cdot \delta \dot{\bar{r}}_i \quad ; \frac{d}{dt}\delta = \delta\frac{d}{dt}$$

$$= \frac{d}{dt}\left[\sum_i m_i \dot{\bar{r}}_i \delta \bar{r}_i\right] - \sum_i \delta\left(\frac{1}{2} m_i \dot{\bar{r}}_i^{\,2}\right)$$

$$= \frac{d}{dt}\left[\sum_i m_i \dot{\bar{r}}_i \delta \bar{r}_i\right] - \delta T$$ Where T is K.E.

$\therefore$ eqⁿ (3) becomes,

$$\delta W = \frac{d}{dt}\left[\sum_i m_i \dot{\bar{r}}_i \delta \bar{r}_i\right] - \delta T$$

$$\delta W + \delta T = \frac{d}{dt}\left[\sum_i m_i \dot{\bar{r}}_i \delta \bar{r}_i\right]$$

$$\delta (W + T) = \frac{d}{dt}\left[\sum_i m_i \dot{\bar{r}}_i \delta \bar{r}_i\right]$$

$$\delta (T + W) = \frac{d}{dt}\left(\sum_i m_i \dot{\bar{r}}_i \delta \bar{r}_i\right)$$

$\delta\int_{t_0}^{t_1}(T+W)dt=0$

integrating w.r.t. t from t_0 to t_1

$$\delta\int_{t_0}^{t_1}(T+W)\,dt = \left[\sum_i m_i \dot{\bar{r}}_i \delta \bar{r}_i\right]_{t_0}^{t_1}$$

Since $\delta\bar{r}_i = 0$ at end pt.s because all the paths are fixed at end pt.

$$\therefore \quad \delta\int_{t_0}^{t_1}(T+W)\,dt = 0 \quad \cdots\cdots (4)$$

eqⁿ (4) is the required Hamilton's principle for non-conservative system.

Now, if the system is conservation then

$\bar{F} = -\nabla V$

For some scalar potential V.

$$\delta W = \sum_i \bar{F}_i \delta \bar{r}_i$$

$$= \sum_i -\nabla V_i \delta \bar{r}_i$$

$$\delta W = -\sum_i \frac{\partial V_i}{\partial r_i} \cdot \delta \bar{r_i} \qquad \delta z = \frac{\partial z}{\partial x}\delta x - \frac{\partial z}{\partial y}\delta y$$

$$\delta W = -\delta V \quad \ldots\ldots (5)$$

Using (5) in (4).

$$\delta \int_{t_0}^{t_1} (T-V)\, \delta t = 0$$

i.e.

$$\delta \int_{t_0}^{t_1} L \cdot dt = 0 \quad \ldots\ldots (6)$$

Where $L = T - V$ is lagrangian.

$\therefore$ Eq^n (6) is the required eq^n of Hamilton's principle for conservative system.

Theo. 2) state Hamilton's principle for non-conservative system and hence derive lagranges equation

proof: consider a holonomic non-conservative system whose configaration is defined by generalized co-ordinate $q_1, q_2, \ldots q_n$

Now, Hamilton's principle for non-conservative system is given by,

$$\delta \int_{t_0}^{t_1} (T+W)\, dt = 0 \quad \ldots\ldots (1)$$

Now,

$$T = T(q_j, \dot{q}_j, t)$$

$$\therefore \delta T = \sum_j \left[\frac{\partial T}{\partial q_j}\delta q_j + \frac{\partial T}{\partial \dot{q}_j}\delta \dot{q}_j \right] + \frac{\partial T}{\partial t} \cdot \delta t$$

$$\delta T = \sum_j \left[\frac{\partial T}{\partial q_j}\delta q_j + \frac{\partial T}{\partial \dot{q}_j}\delta \dot{q}_j \right] \quad ; \delta t = 0 \text{ for virtual mot}$$

$$\delta T = \sum_j \frac{\partial T}{\partial q_j}\delta q_j + \sum_j \frac{\partial T}{\partial \dot{q}_j}\delta \dot{q}_j$$

Now integrating above eq^n w.r.t. t from t_0 to t_1

$$\int_{t_0}^{t_1} \delta T\, dt = \sum_j \int_{t_0}^{t_1} \frac{\partial T}{\partial q_j}\delta q_j\, dt + \sum_j \int_{t_0}^{t_1} \frac{\partial T}{\partial \dot{q}_j}\underline{\delta \dot{q}_j}\, dt$$

$$= \sum_j \int_{t_0}^{t_1} \frac{\partial T}{\partial q_j}\delta q_j\, dt + \sum_j \int_{t_0}^{t_1} \overset{I}{\frac{\partial T}{\partial \dot{q}_j}}\ \overset{II}{\frac{d}{dt}(\delta q_j)}\, dt$$

$$\because \left(\delta \frac{d}{dt} = \frac{d}{dt}\delta\right)$$

Evaluate the 2nd integral by using LIATE rule.

$$= \sum_j \int_{t_0}^{t_1} \frac{\partial T}{\partial q_j}\delta q_j\, dt + \sum_j \left\{ \left[\frac{\partial T}{\partial \dot{q}_j}\cdot \delta q_j\right]_{t_0}^{t_1} - \int_{t_0}^{t_1} \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right)\delta q_j\, dt \right\}$$

$$= \sum_j \int_{t_0}^{t_1} \frac{\partial T}{\partial q_j}\delta q_j\, dt - \sum_j \int_{t_0}^{t_1} \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right)\delta q_j \cdot dt \quad ; \text{since}$$

in δ-variation there is no change in the co-ordinates at the end pt.s

$$\therefore \left(\delta q_j\right)_{t_0}^{t_1} = 0$$

$$\therefore \int_{t_0}^{t_1} \delta T \cdot dt = \int_{t_0}^{t_1} \sum_j \left[\frac{\partial T}{\partial q_j} - \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right)\right]\delta q_j\, dt \quad \cdots (2)$$

Now, virtual work

$$\delta W = \sum_i \bar{F}_i \,\delta \bar{r}_i$$

$$= \sum_i \bar{F}_i \left(\sum_j \frac{\partial \bar{r}_i}{\partial q_j}\delta q_j\right)$$

$r_i = (q_1, q_2, q_3 \cdot$

$\delta r_i = \frac{\partial r_i}{\partial q_j}\delta q_j$

$$= \sum_{i,j} \bar{F}_i \frac{\partial \bar{r}_i}{\partial q_j} \delta q_j$$

$$= \sum_j \left(\sum_i \bar{F}_i \frac{\partial \bar{r}_i}{\partial q_j} \right) \delta q_j$$

$$\delta W = \sum_j Q_j \cdot \delta q_j \qquad ; Q_j = \sum_i \bar{F}_i \frac{\partial \bar{r}_i}{\partial q_j}$$

$$\int_{t_0}^{t_1} \delta W \cdot dt = \int_{t_0}^{t_1} \sum_j Q_j \cdot \delta q_j \, dt \quad \ldots\ldots (3)$$

Using (2) and (3) in eq^n (1).

$$\int_{t_0}^{t_1} \sum_j \left[\frac{\partial T}{\partial q_j} - \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j} \right) \right] \delta q_j \, dt + \int_{t_0}^{t_1} \sum_j Q_j \, \delta q_j \, dt$$

i.e.

$$\int_{t_0}^{t_0} \sum_j \left[\frac{\partial T}{\partial q_j} - \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j} \right) + Q_j \right] \delta q_j \cdot dt = 0$$

$$\Rightarrow \sum_j \left[\frac{\partial T}{\partial q_j} - \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j} \right) + Q_j \right] \delta q_j = 0$$

since, q_j are L.I and hence δq_j also

$$\therefore \frac{\partial T}{\partial q_j} - \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j} \right) + Q_j = 0$$

$$\Rightarrow \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j} \right) - \frac{\partial T}{\partial q_j} = Q_j$$

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Which is required equation.

Theo. State Hamilton's principle for conservative system. Derive lagranges eqn of the motion for conservative system from Hamilton's principle.

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Show that the Lagranges eq^n's are necessary and sufficient condition for the action to have stationary value

proof: We know the action of particle is defined by

$$I = \int_{t_0}^{t_1} L.dt$$

Where L is the Lagrangian of the system.

consider $\delta I = \delta \int_{t_0}^{t_1} L.dt$ ....(1)

$$= \int_{t_0}^{t_1} \left\{ \sum_j \left[\frac{\partial L}{\partial q_j} . \delta q_j + \frac{\partial L}{\partial \dot{q}_j} \delta \dot{q}_j \right] + \frac{\partial L}{\partial t} . \delta t \right\} dt$$

But $\delta t = 0$

$$\therefore \delta I = \int_{t_0}^{t_1} \left\{ \sum_j \left[\frac{\partial L}{\partial q_j} \delta q_j + \frac{\partial L}{\partial \dot{q}_j} . \delta \dot{q}_j \right] dt \right.$$

$$= \int_{t_0}^{t_1} \sum_j \frac{\partial L}{\partial q_j} \delta q_j . dt + \int_{t_0}^{t_1} \frac{\partial L}{\partial \dot{q}_j} \delta \dot{q}_j . dt$$

We know that $\quad \delta \dot{q}_j = \delta . \frac{d}{dt} q_j$

$$\frac{d}{dt} . \delta = \delta . \frac{d}{dt}$$

$$= \int_{t_0}^{t_1} \sum_j \frac{\partial L}{\partial q_j} . \delta q_j . dt + \int_{t_0}^{t_1} \sum_j \frac{\partial L}{\partial \dot{q}_j} . \frac{d}{dt} (\delta q_j) . dt \quad \text{......(2)}$$

Integrating the 2nd integral in the R.H.S. of eq^n (2) we get..

$$= \int_{t_0}^{t_1} \sum_j \frac{\partial L}{\partial q_j} \delta q_j . dt + \left\{ \left[\sum_j \frac{\partial L}{\partial \dot{q}_j} \delta q_j \right]_{t_0}^{t_1} - \int_{t_0}^{t_1} \sum_j \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_j} \right) \delta q_j . dt \right\}$$

$$= \int_{t_0}^{t_1} \sum_j \frac{\partial L}{\partial q_j} . \delta q_j . dt - \int_{t_0}^{t_1} \sum_j \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_j} \right) \delta q_j . dt$$

since there is no variation in co-ordinates along any path at the end point.

$$\therefore \left(\delta q_j\right)_{t_0}^{t_1} = 0$$

$$\delta\int_{t_0}^{t_1} L.dt = \int_{t_0}^{t_1} \sum_j \left[\frac{\partial L}{\partial q_j} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right)\right]\delta q_j . dt$$

$$\therefore \delta\int_{t_0}^{t_1} L.dt = 0 \Leftrightarrow \int_{t_0}^{t_1} \sum_j \left[\frac{\partial L}{\partial q_j} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right)\right]\delta q_j . dt = 0$$

$$\therefore \int_{t_0}^{t_1} \sum_j \left[\frac{\partial L}{\partial q_j} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right)\right]\delta q_j . dt = 0$$

$$\Rightarrow \sum_j \left[\frac{\partial L}{\partial q_j} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right)\right]\delta q_j = 0$$

If the system is holonomic then all the generalized co-ordinate are L.I.

$$\therefore \frac{\partial L}{\partial q_j} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) = 0$$

$$\delta\int_{t_0}^{t_1} L.dt = 0 \Leftrightarrow \frac{\partial L}{\partial q_j} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) = 0$$

There are required Lagranges equations obtrained from Hamilton's principle of conservative system.

Ex. Use Hamilton's principle to find the eq^n of motion of a simple pendulum.

$\Rightarrow$ We know that the lagrangian fun^n for the simple pendulum is given by

$$L = \frac{1}{2}ml^2\dot{\theta}^2 + mgl\cos\theta$$

Now Hamilton's principle for conservative system is $\delta\int_{t_0}^{t_1} L.dt = 0$

$$\delta\int_{t_0}^{t_1}\left[\frac{1}{2}ml^2\dot{\theta}^2 + mgl\cos\theta\right]dt = 0$$

$$\Rightarrow \int_{t_0}^{t_1}\left[\delta\left(\frac{1}{2}ml^2\dot{\theta}^2\right) + \delta\left(mgl\cos\theta\right)\right]dt = 0.$$

$$\Rightarrow \int_{t_0}^{t_1}\left[\frac{1}{2}ml^2 \cdot 2\dot{\theta}\,\delta\dot{\theta} + mgl(-\sin\theta)\cdot\delta\theta\right]dt = 0.$$

$$\Rightarrow \int_{t_0}^{t_1}\left[ml^2\dot{\theta}\cdot\delta\dot{\theta} + mgl(-\sin\theta)\delta\theta\right]dt = 0$$

We know that

$$\delta\cdot\frac{d}{dt} = \frac{d}{dt}\cdot\delta.$$

$$\Rightarrow \int_{t_0}^{t_1}\left[ml^2\dot{\theta}\frac{d}{dt}(\delta\theta) - mgl\sin\theta\cdot\delta\theta\right]dt = 0.$$

$$\Rightarrow \int_{t_0}^{t_1} \underset{\mathrm{I}}{ml^2\dot{\theta}}\,\underset{\mathrm{II}}{\frac{d}{dt}(\delta\theta)}dt - \int_{t_0}^{t_1} mgl\sin\theta\,\delta\theta\cdot dt = 0.$$

Integrating 1st term by LIATE rule.

$$ml^2\left\{\left[\dot{\theta}\,\delta\theta\right]_{t_0}^{t_1} - \int_{t_0}^{t_1}\ddot{\theta}\,\delta\theta\cdot dt\right\} - \int_{t_0}^{t_1} mgl\sin\theta\cdot\delta\theta\cdot dt = 0$$

$$(\delta\theta)_{t_0}^{t_1} = 0.$$

$$-ml^2\int_{t_0}^{t_1}\ddot{\theta}\,\delta\theta\cdot dt - \int_{t_0}^{t_1} mgl\sin\theta\cdot\delta\theta\cdot dt = 0.$$

$$ml^2\int_{t_0}^{t_1}\ddot{\theta}\,\delta\theta\cdot dt + \int_{t_0}^{t_1} mgl\sin\theta\cdot\delta\theta\cdot dt = 0.$$

$$\int_{t_0}^{t_1}\left[ml^2\ddot{\theta} + mgl\sin\theta\right]\delta\theta\cdot dt = 0.$$

since $\left[ml^2\ddot{\theta} + mgl\sin\theta\right]\delta\theta = 0.$

since θ is generalized co-ordinate, θ is L.I. and hence $\delta\theta$ is also L.I.

$$\therefore \quad ml^2\ddot{\theta} + mgl\sin\theta = 0$$

$$ml^2\ddot{\theta} \cancel{A} +$$

$$lm(l^2\ddot{\theta} + g\sin\theta) = 0.$$

$$ml(l\ddot{\theta} + g\sin\theta) = 0.$$

$$l\ddot{\theta} = -g\sin\theta$$

$$\ddot{\theta} = -\frac{g}{l}\sin\theta.$$

Ex. Use Hamilton's principle to find the eq^n of motion of a particle of unit mass moving on a plane in a conservative ~~force~~ field.

$\Rightarrow$ Let $\bar{F} = F_x\hat{i} + F_y\hat{j}$ .... (1) be a conservative force applied on a particle of mass $m=1$ since $\bar{F}$ is conservative.

$$\therefore \bar{F} = -\nabla V$$

$$F_x\hat{i} + F_y\hat{j} = -\frac{\partial V}{\partial x}\hat{i} - \frac{\partial V}{\partial y}\hat{j}$$

$$\Rightarrow \left.\begin{aligned} F_x &= -\frac{\partial V}{\partial x} \\ F_y &= -\frac{\partial V}{\partial y} \end{aligned}\right\} \quad \cdots\cdots (2)$$

The K.E. of a particle is given by

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2)$$

Here $m=1$

$$T = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) \quad \cdots\cdots (3)$$

The P.E. of a particle is given by $V = V(x,y)$ ......(4)

The Lagrangian L is given by

$$l = T - V$$

$$L = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) - V(x, y) \quad \cdots\cdots (5)$$

Now Hamilton's principle for conservative system is

$$\delta \int_{t_0}^{t_1} l \cdot dt = 0.$$

$$\delta \int_{t_0}^{t_1} \left[\frac{1}{2}(\dot{x}^2 + \dot{y}^2) - V(x, y)\right] dt = 0$$

$$\int_{t_0}^{t_1} \left[\delta\left(\frac{1}{2}(\dot{x}^2 + \dot{y}^2)\right) - \delta(V(x, y))\right] \cdot dt = 0$$

$$\int_{t_0}^{t_1} \left[\frac{1}{2}(2\dot{x}\delta\dot{x} + 2\dot{y}\delta\dot{y}) - \delta V\right] dt = 0.$$

$$\int_{t_0}^{t_1} \left[(\dot{x} \cdot \delta\dot{x} + \dot{y}\delta\dot{y}) - \left(\frac{\partial V}{\partial x}\delta x + \frac{\partial V}{\partial y}\delta y\right)\right] dt = 0.$$

$$\int_{t_0}^{t_1} \left\{\dot{x}\delta\dot{x} + \dot{y}\delta\dot{y} - \frac{\partial V}{\partial x}\delta x - \frac{\partial V}{\partial y}\delta y\right\} dt = 0.$$

$$\int_{t_0}^{t_1} \left[\dot{x}\delta\dot{x} + \dot{y}\delta\dot{y} + F_x\delta x + F_y\delta y\right] dt = 0$$

$$* \int_{t_0}^{t_1} \dot{x}\delta\dot{x}\, dt + \int_{t_0}^{t_1} \dot{y}\delta\dot{y} \cdot dt + \int_{t_0}^{t_1} F_x \cdot \delta x \cdot dt + \int_{t_0}^{t_1} F_y \delta y\, dt = 0$$

$$\delta\dot{x} = \delta \frac{d}{dt} x = \frac{d}{dt}\delta x$$

$$* \int_{t_0}^{t_1} \dot{x} \cdot \frac{d}{dt} \cdot \delta x \cdot dt + \int_{t_0}^{t_1} \dot{y} \cdot \frac{d}{dt} \cdot \delta y\, dt + \int_{t_0}^{t_1} F_x \cdot \delta x \cdot dt + \int_{t_0}^{t_1} F_y \delta y\, dt = 0$$

$$\left\{\left[\dot{x} \cdot \delta x\right]_{t_0}^{t_1} - \int_{t_0}^{t_1} \ddot{x} \cdot \delta x \cdot dt\right\} + \left\{\left[\dot{y} \cdot \delta y\right]_{t_0}^{t_1} - \int_{t_0}^{t_1} \ddot{y}\delta y\, dt\right\}$$

$$+ \int_{t_0}^{t_1} F_x \cdot \delta x\, dt + \int_{t_0}^{t_1} F_y \cdot \delta y\, dt = 0 \quad \cdots \text{(by LIATE rule use in 1st + 2nd term)}$$

$$(\delta x)_{t_0}^{t_1} = 0 \quad \text{and} \quad (\delta y)_{t_0}^{t_1} = 0.$$

$$-\int_{t_0}^{t_1} \ddot{x}\,\delta x\,dt - \int_{t_0}^{t_1} \ddot{y}\,\delta y.dt + \int_{t_0}^{t_1} F_x.\delta x.dt$$

$$+\int_{t_0}^{t_1} F_y.\delta y.dt = 0$$

$$\Rightarrow \int_{t_0}^{t_1} (F_x - \ddot{x})\,\delta x.dt + \int_{t_0}^{t_1} (F_y - \ddot{y})\,\delta y.dt = 0$$

$$\Rightarrow \int_{t_0}^{t_1} (F_x - \ddot{x})\,\delta x.dt = 0 \text{ and } \int_{t_0}^{t_1} (F_y - \ddot{y})\,\delta y.dt = 0$$

$\Rightarrow (F_x - \ddot{x})\delta x = 0$ and $(F_y - \ddot{y})\delta y = 0$

since x and y are generalized coordinate

$\Rightarrow x$ & y are L.I also δx & δy are L.I.

$\therefore F_x - \ddot{x} = 0$ and $F_y - \ddot{y} = 0$

$\Rightarrow F_x = \ddot{x}$ and $F_y = \ddot{y}$

Which is required eqns of motion.

Ex. A particle of mass m is moving on the surface of the sphere of radius r in the gravitational field. Use Hamilton's principle to show that the eqn of motion is

$$\ddot{\theta} - \frac{P_\phi^2 \cos\theta}{m^2 r^4 \sin^3\theta} - \frac{g}{r}\sin\theta = 0$$

Where P_ϕ^2 is const. of angular momentum

<u>OR</u>

<u>By</u> using Hamilton's principle find eqn of motion of spherical pendulum.

$\Rightarrow$ We know that the lagrangian funn for the spherical pendulum is given by

$$L = \frac{1}{2} m r^2 (\dot{\theta}^2 + \sin^2\theta.\dot{\phi}^2) - mgr\cos\theta.$$

Now Hamilton's principle for conservative system is,

$$\left(2\sin\theta\cdot\cos\theta\cdot\delta\theta\,\dot\phi^2 + \sin^2\theta\cdot 2\dot\phi\,\delta\dot\phi\right)$$

$$\delta\int_{t_0}^{t_1} L\cdot dt = 0.$$

$$\delta\int_{t_0}^{t_1}\left[\frac{1}{2}mr^2(\dot\theta^2+\sin^2\theta\,\dot\phi^2) - mgr\cos\theta\right]dt = 0$$

$$\Rightarrow \int_{t_0}^{t_1}\left\{\delta\left[\frac{1}{2}mr^2(\dot\theta^2+\sin^2\theta\cdot\dot\phi^2)\right] - \delta\left[mgr\cos\theta\right]\right\}dt = 0$$

$$\Rightarrow \int_{t_0}^{t_1}\left\{\frac{1}{2}mr^2\left(2\dot\theta\cdot\delta\dot\theta + 2\sin\theta\cdot\cos\theta\cdot\delta\theta\,\dot\phi^2 + \sin^2\theta\cdot 2\cdot\dot\phi\,\delta\dot\phi\right) - mgr(-\sin\theta)\delta\theta\right\}dt = 0.$$

$$\Rightarrow \int_{t_0}^{t_1}\left\{mr^2\left(\dot\theta\,\delta\dot\theta + \sin\theta\cdot\cos\theta\cdot\delta\theta\cdot\dot\phi^2 + \sin^2\theta\cdot\dot\phi\,\delta\dot\phi\right) + mgr\sin\theta\cdot\delta\theta\right\}dt = 0$$

$$\Rightarrow \int_{t_0}^{t_1} mr^2\dot\theta\,\delta\dot\theta\cdot dt + mr^2\int_{t_0}^{t_1}\sin\theta\cdot\cos\theta\cdot\delta\theta\cdot\dot\phi^2\cdot dt + mr^2\int_{t_0}^{t_1}\sin^2\theta\,\dot\phi\cdot\delta\dot\phi\cdot dt + mgr\int_{t_0}^{t_1}\sin\theta\cdot\delta\theta\cdot dt = 0.$$

$$\Rightarrow \int_{t_0}^{t_1} mr^2\cdot\dot\theta\cdot\delta\dot\theta\cdot dt + mr^2\int_{t_0}^{t_1}\sin^2\theta\cdot\dot\phi\cdot\delta\dot\phi\cdot dt + mr^2\int_{t_0}^{t_1}\sin\theta\cdot\cos\theta\cdot\delta\theta\cdot\dot\phi^2\cdot dt + mgr\int_{t_0}^{t_1}\sin\theta\cdot\delta\theta\cdot dt = 0$$

$$\Rightarrow mr^2\left\{\int_{t_0}^{t_1}\dot\theta\,\delta\dot\theta\cdot dt + \int_{t_0}^{t_1}\sin^2\theta\cdot\dot\phi\cdot\delta\dot\phi\cdot dt + \int_{t_0}^{t_1}\sin\theta\cdot\cos\theta\cdot\delta\theta\cdot\dot\phi^2\cdot dt + \frac{g}{r}\int_{t_0}^{t_1}\sin\theta\cdot\delta\theta\cdot dt\right\} = 0.$$

We have. $\delta\cdot\frac{d}{dt} = \frac{d}{dt}\,\delta\cdot$

$$\Rightarrow \int_{t_0}^{t_1}\dot\theta\cdot\frac{d(\delta\theta)}{dt}\cdot dt + \int_{t_0}^{t_1}\underbrace{\sin^2\theta\cdot\dot\phi}_{\mathrm{I}}\;\underbrace{\frac{d}{dt}(\delta\phi)}_{\mathrm{II}}\cdot dt + \int_{t_0}^{t_1}\sin\theta\cdot\cos\theta\cdot\delta\theta\cdot\dot\phi^2 dt + \frac{g}{r}\int_{t_0}^{t_1}\sin\theta\cdot\delta\theta\cdot dt = 0.$$

Integrating 1st & 2nd term by using LIATE rule.

$$\Rightarrow \left\{ \left[\dot{\theta}\cdot\delta\theta\right]_{t_0}^{t_1} - \int_{t_0}^{t_1} \ddot{\theta}\,\delta\theta\,dt \right\} + \left\{ \left[\sin^2\theta\cdot\dot{\phi}\cdot\delta\phi\right]_{t_0}^{t_1} - \int_{t_0}^{t_1} \Big(\underbrace{\cancel{2\sin\theta\cos\theta\,\dot{\phi}\dot{\theta} + \sin^2\theta\cdot\ddot{\phi}}}_{\frac{d}{dt}(\sin^2\theta\cdot\dot{\phi})} \Big)\,\delta\phi\,dt \right\}$$

$$+ \int_{t_0}^{t_1} \sin\theta\cos\theta\,\dot{\phi}^2\cdot\delta\theta\cdot dt + \frac{g}{r}\int_{t_0}^{t_1}\sin\theta\cdot\delta\theta\cdot dt = 0$$

$\Rightarrow$ since $(\delta\theta)_{t_0}^{t_1} = 0$ and $(\delta\phi)_{t_1}^{t_0} = 0$

$$\Rightarrow -\int_{t_0}^{t_1} \ddot{\theta}\,\delta\theta\,dt - \int_{t_0}^{t_1} \Big(\underbrace{\cancel{2\sin\theta\cos\theta\cdot\dot{\phi}\dot{\theta} + \sin^2\theta\cdot\ddot{\phi}}}_{\frac{d}{dt}(\sin^2\theta\cdot\dot{\phi})}\Big)\,\delta\phi\cdot dt$$

$$+ \int_{t_0}^{t_1} \sin\theta\cdot\cos\theta\cdot\dot{\phi}^2\cdot\delta\theta\cdot dt + \frac{g}{r}\int_{t_0}\sin\theta\cdot\delta\theta\cdot dt = 0$$

$$\times \Rightarrow \int_{t_0}^{t_1} \Big(-\ddot{\theta} - \overbrace{\cancel{2\sin\theta\cos\theta\,\dot{\phi}\cdot\dot{\theta} - \sin^2\theta\cdot\ddot{\phi}}}^{\frac{d}{dt}(\sin^2\theta\cdot\dot{\phi})} + \sin\theta\cos\theta\,\dot{\phi}^2 + \frac{g}{r}\sin\theta\Big)\,\delta\theta\cdot dt = 0$$

$$\times \Rightarrow \left(\cancel{-}\ddot{\theta} - \cancel{2}\sin\theta\cdot\cos\theta\,\dot{\phi}\dot{\theta} - \sin^2\theta\cdot\ddot{\phi} \cancel{+} \sin\theta\cdot\cos\theta\cdot\dot{\phi}^2 + \frac{g}{r}\sin\theta\right)\delta\theta = 0$$

$$\times \Rightarrow \int_{t_0}^{t_1} \left(-\ddot{\theta} - \frac{d}{dt}\right.$$

$$\Rightarrow \int_{t_0}^{t_1} -\ddot{\theta}\,\delta\theta\,dt - \int_{t_0} \frac{d}{dt}(\sin^2\theta\cdot\dot{\phi})\,\delta\phi\,dt + \int_{t_0}^{t_1} \sin\theta\cdot\cos\theta\cdot\dot{\phi}^2 \ldots$$

$$+ \frac{g}{r}\int_{t_0}^{t_1}\sin\theta\cdot\delta\theta\cdot dt = 0.$$

$$\Rightarrow \int_{t_0}^{t_1} -mr^2\ddot{\theta}\,\delta\theta\cdot dt - \int_{t_0}^{t_1} mr^2 \frac{d}{dt}(\sin^2\theta\cdot\dot{\phi})\,\delta\phi\cdot dt$$

$$+\int_{t_0}^{t_1} mr^2\sin\theta\cdot\cos\theta\,\dot{\phi}^2\,\delta\theta\cdot dt + mgr\int_{t_0}^{t_1}\sin\theta\cdot\delta\theta\,dt = 0$$

$$mr^2\frac{d}{dt}(\sin^2\theta\cdot\dot{\phi}) = 0$$

$$\therefore mr^2\sin^2\theta\cdot\dot{\phi} = P_\phi \quad ; P_\phi \text{ const.}$$

$$\dot{\phi} = \frac{P_\phi}{mr^2\sin^2\theta}$$

$$\Rightarrow \int_{t_0}^{t_1} -mr^2\ddot{\theta}\,\delta\theta\cdot dt - \int_{t_0}^{t_1} mr^2\cdot 0 + \int_{t_0}^{t_1} mr^2\sin\theta\cdot\cos\theta\cdot$$

$$\frac{P_\phi^2}{m^2r^4\sin^4\theta}\cdot\delta\theta\cdot dt + mgr\int_{t_0}^{t_1}\sin\theta\cdot\delta\theta\cdot dt = 0.$$

$$\Rightarrow \int_{t_0}^{t_1}(-mr^2\ddot{\theta} + mgr\sin\theta)\,\delta\theta\cdot dt +$$

$$\int_{t_0}^{t_1}\cdot\cos\theta\cdot\frac{P_\phi^2}{mr^2\sin^3\theta}\cdot\delta\theta\cdot dt$$

since, θ and ϕ are generalized co-ordinate

$\Rightarrow$ θ and ϕ are L.I. also $\delta\theta$ & $\delta\phi$ are L.I.

$$\therefore -mr^2\ddot{\theta} + mgr\sin\theta + \cos\theta\cdot\frac{P_\phi^2}{mr^2\sin^3\theta} = 0.$$

$$\ddot{\theta} - \frac{g}{r}\sin\theta - \cos\theta\cdot\frac{P_\phi^2}{m^2r^4\sin^3\theta} = 0$$

$$\ddot{\theta} = \frac{g}{r}\sin\theta + \cos\theta\frac{P_\phi^2}{m^2r^4\sin^3\theta}.$$

; P_ϕ is const of angular momentum.

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- Hamilton's formulation.

In this section we defined hamiltonian fun^n H and derived eq^n of motion which are systems of 1st order O.D.E's.

- The Hamiltonian Function.

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The quantity $\sum_j p_j \dot{q}_j - L$ when expressed in terms of q_j, p_j and t is called hamiltonian & it is denoted by H.

$\therefore$ We have

$H(q_j, p_j, t) = \sum_j p_j \dot{q}_j - L$, where L are is the lagrangian.

- Hamilton's canonical equations of motion.

Que. Derive Hamilton's canonical equations of motion.

proof: We know the Hamiltonian 'H' is defined as

$H = H(p_j, q_j, t) = \sum_j p_j \dot{q}_j - L$ ------- (1)

consider,

$H = H(p_j, q_j, t)$ ..... (2)

$\therefore$ from eq^n (2) ; we have.

$$dH = \sum_j \left[\frac{\partial H}{\partial p_j} dp_j + \frac{\partial H}{\partial q_j} dq_j\right] + \frac{\partial H}{\partial t} \cdot dt \quad \cdots\cdots (3)$$

Now consider,

$H = \sum_j p_j \dot{q}_j - L$

$L = L(q_j, \dot{q}_j, t)$:

$$dH = \sum_j p_j \cdot d\dot{q}_j + \sum_j \dot{q}_j \, dp_j - \sum_j \left[\frac{\partial L}{\partial q_j} \cdot dq_j - \sum \frac{\partial L}{\partial \dot{q}_j} d\dot{q}_j\right] \frac{\partial L}{\partial t}$$

.... (4)

We know the generalized momentum is given by

$p_j = \frac{\partial L}{\partial \dot{q}_j}$ .... (5)

$$p_j = \frac{\partial L}{\partial \dot{q}_j}$$

$$\dot{p}_j = \frac{\partial L}{\partial q_j}$$

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$\therefore$ eq^n (4) becomes,

$$dH = \sum_j p_j d\dot{q}_j + \sum_j \dot{q}_j dp_j - \sum_j \frac{\partial L}{\partial q_j} \cdot dq_j - \sum_j p_j d\dot{q}_j - \frac{\partial L}{\partial t} \cdot dt$$

$$\therefore dH = \sum_j \dot{q}_j dp_j - \sum_j \frac{\partial L}{\partial q_j} \cdot dq_j - \frac{\partial L}{\partial t} \cdot dt \quad \cdots\cdots (6)$$

We have

$$\frac{\partial L}{\partial q_j} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) = 0$$

$$\frac{\partial L}{\partial q_j} - \frac{d}{dt}(p_j) = 0 \quad \cdots\cdots \text{from (5)}$$

$$\frac{\partial L}{\partial q_j} = \dot{p}_j$$

$\therefore$ eq^n (6) becomes,

$$dH = \sum_j \dot{q}_j dp_j - \sum_j \dot{p}_j dq_j - \frac{\partial L}{\partial t} \cdot dt \quad \cdots\cdots (7)$$

comparing eq^n (3) & (7) ; we get.

$$\dot{q}_j = \frac{\partial H}{\partial p_j}$$

$$-\dot{p}_j = +\frac{\partial H}{\partial q_j} \Rightarrow \dot{p}_j = -\frac{\partial H}{\partial q_j}$$

$$\therefore \dot{q}_j = \frac{\partial H}{\partial p_j} \quad \text{and} \quad \dot{p}_j = -\frac{\partial H}{\partial q_j}$$

Which are required Hamilton's canonical eq^{n}s of motion.

$H(p_j, q_j, t)$

$$\dot{q}_j = \frac{\partial H}{\partial p_j}$$

$$\dot{p}_j = \frac{-\partial H}{\partial q_j}$$

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Derivation of Hamilton's eq^ns of motion from Hamilton's principle

UTML*

Theorem:
obtain Hamilton's eq^n of motion from the Hamilton's principle.

proof:
We know action of a particle is defined

$$I = \int_{t_0}^{t_1} L.dt \quad \ldots\ldots(1)$$

Where L is lagrangian.

Now,

$$H = \sum_j p_j \dot{q}_j - L \quad \ldots\ldots(2)$$

$$\therefore L = \sum_j p_j \dot{q}_j - H$$

$\therefore$ eq^n (1) becomes,

$$I = \int_{t_0}^{t_1} L.dt = \int_{t_0}^{t_1} \left[\sum_j p_j \dot{q}_j - H\right].dt \quad \ldots\ldots(3)$$

$$\therefore \delta\int_{t_0}^{t_1} L.dt = 0 \Rightarrow \delta\int_{t_0}^{t_1} \left[\sum_j p_j \dot{q}_j - H\right]dt = 0 \ldots$$

Now consider,

$$\delta\int_{t_0}^{t_1} \left[\sum_j p_j \dot{q}_j - H\right].dt =$$

$$\int_{t_0}^{t_1} \left\{\sum_j \left[p_j \delta\dot{q}_j + \dot{q}_j \delta p_j\right] - \delta H\right\}.dt$$

$$= \int_{t_0}^{t_1} \left\{\sum_j \left[p_j \delta\dot{q}_j + \dot{q}_j \delta p_j\right] - \left[\sum_j \left(\frac{\partial H}{\partial p_j}\delta p_j + \frac{\partial H}{\partial q_j}\delta q_j\right) + \frac{\partial H}{\partial t}.\delta t\right]\right\}dt = 0$$

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$$= \int_{t_0}^{t_1} \left\{ \sum_j \left[p_j \delta \dot{q}_j + \dot{q}_j \delta p_j \right] - \sum_j \left[\frac{\partial H}{\partial p_j} \delta p_j + \frac{\partial H}{\partial q_j} \delta q_j \right] \right\} dt$$

; since $\delta t = 0$ along any path.

$$= \int_{t_0}^{t_1} \left[\sum_j \left(\dot{q}_j - \frac{\partial H}{\partial p_j} \right) \delta p_j + \underline{\sum_j p_j \delta \dot{q}_j} - \sum_j \frac{\partial H}{\partial q_j} \delta q_j \right] dt \quad \ldots\ldots (5)$$

Now, we evaluate,

$$\int_{t_0}^{t_1} \left(\sum_j p_j \delta \dot{q}_j \right) \cdot dt = \int_{t_0}^{t_1} \left[\sum_j p_j \cdot \delta \cdot \frac{d}{dt} q_j \right] \cdot dt$$

$$= \int_{t_0}^{t_1} \left[\sum_j p_j \cdot \frac{d}{dt} \delta q_j \right] \cdot dt$$

$$= \left[\sum_j p_j \delta q_j \right]_{t_0}^{t_1} - \int_{t_0}^{t_1} \left(\sum_j \dot{p}_j \delta q_j \right) dt$$

$$= - \int_{t_0}^{t_1} \left(\sum_j \dot{p}_j \delta q_j \right) dt \quad \ldots\ldots (6)$$

Using eqn (6) in eqn (5); we get

$$\delta \int_{t_0}^{t_1} \left[\sum_j p_j \dot{q}_j - H \right] dt =$$

$$\int_{t_0}^{t_1} \left[\sum_j \left(\dot{q}_j - \frac{\partial H}{\partial p_j} \right) \delta p_j - \sum_j \dot{p}_j \delta q_j - \sum_j \frac{\partial H}{\partial q_j} \delta q_j \right] dt$$

$$= \int_{t_0}^{t_1} \left[\sum_j \left(\dot{q}_j - \frac{\partial H}{\partial p_j} \right] \delta p_j - \sum_j \left(\dot{p}_j + \frac{\partial H}{\partial q_j} \right) \delta q_j \right] dt$$

from eqn (4). We have,

$$\int_{t_0}^{t_1} \left[\sum_j \left(\dot{q}_j - \frac{\partial H}{\partial p_j} \right) \delta p_j - \sum_j \left(\dot{p}_j + \frac{\partial H}{\partial q_j} \right) \delta q_j \right] dt = 0$$

For holonomic system we have p_j, q_j are linearly independent.

$\because \delta \int_{t_0}^{t_1} L \cdot dt = 0 \Rightarrow \dot{q}_j - \frac{\partial H}{\partial p_j} = 0$ and $-\left(\dot{p}_j + \frac{\partial H}{\partial q_j}\right) = 0$

$$\Rightarrow \dot{q}_j = \frac{\partial H}{\partial p_j} \text{ and } \dot{p}_j = -\frac{\partial H}{\partial q_j}$$

Which is required Hamilton's canonical eqn of motion.

Que. Show that the Lagrangian $L' = L + \frac{df}{dt}$, where $f = f(q_j, t)$ does not change the Hamilton's corresponding to new Lagrangian L'

OR

Show that the addition of the total time derivative of any funn of the form $f(q$ to the lagrangian of Holonomic system. The generalized momentum & Jacobian integral (i.e. Hamiltonian H) are respe. given by $p_j + \frac{\partial f}{\partial q_j}$ and $H - \frac{\partial f}{\partial t}$

$\Rightarrow$ We have

$$L' = L + \frac{df}{dt} \quad \cdots\cdots (1)$$

Where $f = f(q_j, t)$.

The generalized momentum p_j correspo to L' is

$$p_j' = \frac{\partial L'}{\partial \dot{q}_j} = \frac{\partial}{\partial \dot{q}_j}\left(L + \frac{df}{dt}\right)$$

$$= \frac{\partial L}{\partial \dot{q}_j} + \frac{\partial}{\partial \dot{q}_j}\left(\frac{df}{dt}\right)$$

$$p_j' = p_j + \frac{\partial}{\partial \dot{q}_j}\left[\sum_k \frac{\partial f}{\partial q_k} \dot{q}_k + \frac{\partial f}{\partial t}\right]$$

$$= p_j + \frac{\partial f}{\partial q_j}$$

$$\therefore \boxed{p_j' = p_j + \frac{\partial f}{\partial q_j}} \cdots\cdots (2)$$

The new Hamiltonian H' corresponding to L' is

$$H' = \sum_j p_j' \dot{q}_j - L'$$

$$= \sum_j \left(p_j + \frac{\partial f}{\partial q_j}\right)\dot{q}_j - \left(L + \frac{df}{dt}\right)$$

$$= \sum_j p_j \dot{q}_j + \sum_j \frac{\partial f}{\partial q_j}\dot{q}_j - L - \frac{df}{dt}$$

$$= \sum_j p_j \dot{q}_j - L + \sum_j \frac{\partial f}{\partial q_j}\dot{q}_j - \frac{df}{dt}$$

$$= H + \left(\sum_j \frac{\partial f}{\partial q_j}\dot{q}_j - \frac{df}{dt}\right)$$

$$* \boxed{H' = H - \frac{\partial f}{\partial t}} \cdots (3) \quad \left(\because f = f(q_j, t) \quad \frac{df}{dt} = \sum_j \frac{\partial f}{\partial q_j}\cdot\dot{q}_j + \frac{\partial f}{\partial t} \Rightarrow -\frac{\partial f}{\partial t} = \sum_j \frac{\partial f}{\partial q_j}\dot{q}_j - \frac{df}{dt}\right)$$

Now consider,

$$\delta\int_{t_0}^{t_1} L'\,dt = \delta\int_{t_0}^{t_1}\left[L + \frac{df}{dt}\right].dt$$

$$= \delta\int_{t_0}^{t_1} L.dt + \delta\int_{t_0}^{t_1}\frac{df}{dt}.dt$$

$$= \delta\int_{t_0}^{t_1}\frac{df}{dt}.dt \quad ; \delta\int_{t_0}^{t_1} L.dt = 0$$

$$\cancel{= \left[\delta f\right]_{t_0}^{t_1}}$$

$$\delta\int_{t_0}^{t_1} L' \cdot dt = \delta\int_{t_0}^{t_1} df$$

$$= \left[\delta f\right]_{t_0}^{t_1}$$

$$= \left[\sum_j \frac{\partial f}{\partial q_j}\delta q_j + \frac{\partial f}{\partial t}\delta t\right]_{t_0}^{t_1}$$

$$= \left[\sum_j \frac{\partial f}{\partial q_j}\delta q_j\right]_{t_0}^{t_1} \quad ; \delta t = 0$$

Since there is no variation at the ends pts

$\therefore \left[\delta q_j\right]_{t_0}^{t_1} = 0$

$\therefore \delta\int_{t_0}^{t_1} L' dt = 0$

$\therefore$ Hamilton's principle is unchanged.

VIMP

* Lagrangian from Hamiltonian and converse

Ex. obtain Lagrangian L from Hamiltonian H and show that it satisfies Lagrange's eqn of motion.

Soln: The Hamiltonian H is defined as,

$$H = \sum_j p_j \dot{q}_j - L \quad \text{------- (1)}$$

Which satisfies

$$\dot{q}_j = \frac{\partial H}{\partial p_j} \quad \text{and} \quad \dot{p}_j = -\frac{\partial H}{\partial q_j} \quad \text{------- (2)}$$

from (1) we have

$$L = \sum_j p_j \dot{q}_j - H \quad \text{——— (3)}$$

from (3)

$$\frac{\partial L}{\partial q_j} = -\frac{\partial H}{\partial q_j}$$

$$\frac{\partial L}{\partial \dot{q}_j} = p_j$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) = \dot{p}_j$$

consider,

$$\frac{\partial L}{\partial q_j} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) = -\frac{\partial H}{\partial q_j} - \dot{p}_j$$

$$= \dot{p}_j - \dot{p}_j \quad \cdots\cdots (\text{from eq}^n\ (2))$$

$$= 0$$

$$\therefore \frac{\partial L}{\partial q_j} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) = 0.$$

This shows eq^n (3) satisfies Lagranges eq^n of motion.

Ex. obtain the Hamiltonian H from the Lagrangian and show that it satisfies the Hamilton's canonical eq^n of motion.

sol^n: The Hamiltonian H in terms of Lagrangian L is defined as.

$$H = \sum_j p_j \dot{q}_j - L \quad \cdots\cdots\cdots (1)$$

Where L is Lagrangian which satisfies Lagrange's eq^n of motion.

$$\frac{\partial L}{\partial q_j} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) = 0 \quad \cdots\cdots\cdots (2)$$

$$\dot{q}_j = \frac{\partial H}{\partial p_j}$$

$$\frac{\partial L}{\partial q_j} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right)$$

$$\dot{p}_j = -\frac{\partial H}{\partial q_j}$$

$$= \frac{d}{dt}(p_j)$$

$$\frac{\partial L}{\partial q_j} = \dot{p}_j \quad \text{------- (3)}$$

Diff. eqⁿ (1) w.r.t. q_j

$$\frac{\partial H}{\partial q_j} = -\frac{\partial L}{\partial q_j} \quad \text{------ (4)}$$

from (3) ; we have

$$\frac{\partial H}{\partial q_j} = -\dot{p}_j$$

$$\boxed{\dot{p}_j = -\frac{\partial H}{\partial q_j}} \quad \text{------ (5)}$$

Diff. eqⁿ (1) w.r.t. p_j

$$\frac{\partial H}{\partial p_j} = \dot{q}_j$$

$$\Rightarrow \boxed{\dot{q}_j = \frac{\partial H}{\partial p_j}} \quad \text{------ (6)}$$

This shows that eqⁿ (5) & (6) satisfies Hamilton's canonical eqⁿ of motion.

Ex. obtain Hamilton H and the Hamilton's eqⁿs of motion of a simple pendulum. V.IMP

prove or disprove that H represents the const. of motion and total energy.

⇒ The lagrangian of the pendulum is given by

$$L = \frac{1}{2} m l^2 \dot{\theta}^2 + mgl\cos\theta \quad \text{------ (1)}$$

Here θ is generalized co-ordinate.

Where the generalized momentum is gi[ven] by.

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = ml^2\dot{\theta}$$

$$\Rightarrow p_\theta = ml^2\dot{\theta}$$

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$$\Rightarrow \dot{\theta} = \frac{P_\theta}{ml^2} \quad \ldots\ldots\ldots (2)$$

Now the Hamiltonian of the system is given by

$$H = P_\theta \cdot \dot{\theta} - L$$

$$= P_\theta \cdot \dot{\theta} - \frac{1}{2} ml^2 \dot{\theta}^2 - mgl\cos\theta$$

By using (2) we have

$$H = P_\theta \cdot \frac{P_\theta}{ml^2} - \frac{1}{2} ml^2 \left(\frac{P_\theta}{ml^2}\right)^2 - mgl\cos\theta$$

$$= \frac{P_\theta^2}{ml^2} - \frac{1}{2} ml^2 \frac{P_\theta^2}{m^2 l^4} - mgl\cos\theta$$

$$= \frac{P_\theta^2}{ml^2} - \frac{1}{2} \frac{P_\theta^2}{ml^2} - mgl\cos\theta \qquad H(p_j, q_j, t)$$

$$\boxed{H = \frac{1}{2} \frac{P_\theta^2}{ml^2} - mgl\cos\theta} \quad \ldots\ldots\ldots (3).$$

Now Hamilton's canonical eq^ns of motion are.

$$\dot{\theta} = \frac{\partial H}{\partial P_\theta} \quad \text{and} \quad \dot{P}_\theta = -\frac{\partial H}{\partial \theta}$$

$$\dot{\theta} = \frac{P_\theta}{ml^2} \quad \text{and} \quad \dot{P}_\theta = -(mgl\sin\theta)$$

$$= -mgl\sin\theta$$

$$\therefore \dot{\theta} = \frac{P_\theta}{ml^2} \ldots\ldots (*) \quad \dot{P}_\theta = -mgl\sin\theta \ldots\ldots (**)$$

from (*)

$$\ddot{\theta} = \frac{\dot{P}_\theta}{ml^2}$$

Using (**)

$$\ddot{\theta} = \frac{-mgl\sin\theta}{ml^2}$$

$$\Rightarrow \boxed{\ddot{\theta} = -\frac{g}{l}\sin\theta} \quad \ldots\ldots\ldots (4)$$

Now we claim that H represents constant

of motion.

$\frac{dH}{dt}$ Diff. eq^n (3) w.r.t. t.

$$\frac{dH}{dt} = \frac{1}{2}\cdot\frac{1}{ml^2}\cdot 2p_\theta\cdot\dot{p}_\theta - mgl(-\sin\theta)\cdot\dot{\theta}$$

$$= \frac{p_\theta\cdot\dot{p}_\theta}{ml^2} + mgl\sin\theta\cdot\dot{\theta}$$

$$= \frac{(ml^2\dot{\theta})(ml^2\ddot{\theta})}{ml^2} + mgl\sin\theta\cdot\dot{\theta} \qquad \because p_\theta = ml^2\dot{\theta},\ \dot{p}_\theta = ml^2\ddot{\theta}$$

$$= \dot{\theta}\cdot ml^2\ddot{\theta} + mgl\sin\theta\cdot\dot{\theta}$$

$$= ml^2\dot{\theta}\left(\ddot{\theta} + \frac{g}{l}\sin\theta\right)$$

$$= ml^2\dot{\theta}(0) \quad \ldots \text{from (4)}$$

$$\frac{dH}{dt} = 0$$

$\therefore$ H represents constant of motion.

* Now consider,

$$E = T+V = \frac{1}{2}ml^2\dot{\theta}^2 - mgl\cos\theta.$$

put $\dot{\theta} = \frac{p_\theta}{ml^2}$

$$= \frac{1}{2}ml^2\cdot\frac{p_\theta^2}{m^2l^4} - mgl\cos\theta$$

$$= \frac{1}{2}\frac{p_\theta^2}{ml^2} - mgl\cos\theta$$

$$= H \quad \ldots \text{from (3)}$$

$\therefore$ H represents total energy.

Ex. IITJAM Describe the motion of particle of mass m moving in the space under the earth constant gravitational field. By using Hamilton's eq^n.

$\ddot{x} = 0 \qquad \dot{z} = \text{const}$

$\ddot{y} = 0$

$L = T - V$

$T = \frac{1}{2}m(\dot{x}^2+\dot{y}^2+\dot{z}^2)$

$V = mgz$

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VIMP

⇒ consider a particle of mass 'm' and position vector $\bar{r}$ moving in space

Let (x, y, z) be a cartesian co-ordinate of the particle.

$\therefore T = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$ and

$V = mgz$

Lagrangian $L = T - V$

$$L = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - mgz \quad \ldots\ldots (1)$$

Here x, y, z are generalized co-ordinate

Where the generalized momentum is given by,

$$p_x = \frac{\partial L}{\partial \dot{x}} = m\dot{x} \Rightarrow \dot{x} = \frac{p_x}{m}$$

$$p_y = \frac{\partial L}{\partial \dot{y}} = m\dot{y} \Rightarrow \dot{y} = \frac{p_y}{m}$$

$$p_z = \frac{\partial L}{\partial \dot{z}} = m\dot{z} \Rightarrow \dot{z} = \frac{p_z}{m}$$

Now the Hamiltonian of the system is given by

$$H = p_x \cdot \dot{x} + p_y \dot{y} + p_z \cdot \dot{z} - L$$

$$= p_x \cdot \dot{x} + p_y \cdot \dot{y} + p_z \cdot \dot{z} - \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + mgz$$

$H(q_j, p_j, t)$

$$= \frac{p_x^2}{m} + \frac{p_y^2}{m} + \frac{p_z^2}{m} - \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + mgz$$

$$= \frac{1}{m}(p_x^2 + p_y^2 + p_z^2) - \frac{1}{2} m \left(\frac{p_x^2}{m^2} + \frac{p_y^2}{m^2} + \frac{p_z^2}{m^2} \right) + mgz$$

$$= \frac{1}{m}(p_x^2 + p_y^2 + p_z^2) - \frac{1}{2m}(p_x^2 + p_y^2 + p_z^2) + mgz$$

$$\boxed{H = \frac{1}{2m}(p_x^2 + p_y^2 + p_z^2) + mgz} \quad \ldots\ldots (2)$$

Now Hamilton's canonical eq^ns of motion is given by

i) $\dot{x} = \frac{\partial H}{\partial p_x}$ and $\dot{p}_x = -\frac{\partial H}{\partial x}$

$\dot{x} = \frac{p_x}{m}$ .... (*) and $\dot{p}_x = 0$ ......(**)

from (*)

$\ddot{x} = \frac{\dot{p}_x}{m}$

$= \frac{0}{m}$ ---- by (**)

$\ddot{x} = 0$

Integrating,

$\dot{x} = c_1$

$x = \cancel{c_1 x + c_2}\ c_1 \cdot t + c_2$

ii) $\dot{y} = \frac{\partial H}{\partial p_y}$ and $\dot{p}_y = -\frac{\partial H}{\partial y}$

$\dot{y} = \frac{p_y}{m}$ and $\dot{p}_y = 0$

$\Rightarrow \ddot{y} = \frac{\dot{p}_y}{m}$

$= \frac{0}{m}$ .... ; $\dot{p}_y = 0$

$\ddot{y} = 0$

Integrating,

$\dot{y} = c_3$

$y = c_3 t + c_4$

iii) $\dot{z} = \frac{\partial H}{\partial p_z}$ and $\dot{p}_z = -\frac{\partial H}{\partial z}$

$= mg$ and $\dot{p}_z = mg$

$\dot{z} = \frac{p_z}{m}$

$$\Rightarrow \ddot{z} = \frac{\dot{p}_z}{m}$$

$$= \frac{mg}{m}$$

$$\ddot{z} = g$$

Integrating,

$$\dot{z} = gt + c_5$$

$$z = \frac{gt^2}{2} + c_5 t + c_6$$

$$\therefore x = c_1 \cdot t + c_2, \quad y = c_3 \cdot t + c_4, \quad z = \frac{gt^2}{2} + c_5 \cdot t + c_6$$

$$\therefore x = c_1 \cdot t + c_2, \quad y = c_3 \cdot t + c_4, \quad z = \frac{gt^2}{2} + c_5 t + c_6$$

$$= c_1 t + c_2, \quad y = c_3 t + c_4, \quad z = \frac{gt^2}{2} + c_5 t + c_6$$

EX. The Lagrangian of a particle moving on a surface of a surface of a sphere of radius r is given by $L = \frac{1}{2} mr^2(\dot{\theta}^2 + \sin^2\theta \cdot \dot{\phi}^2) - mgr\cos\theta$

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Find H and show that it is const. of motion i.e. $\frac{dH}{dt} = 0$. prove or disprove that

H = Total energy. Find eqn of motion.

$\Rightarrow$ The Lagrangian is given by,

$$L = \frac{1}{2} mr^2(\dot{\theta}^2 + \sin^2\theta \cdot \dot{\phi}^2) - mgr\cos\theta \quad \ldots (1)$$

Here θ & ϕ are generalized co-ordinate. Where the generalized momentum is given by,

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta} \Rightarrow \dot{\theta} = \frac{p_\theta}{mr^2}$$

$$p_\phi = \frac{\partial L}{\partial \dot{\phi}} = mr^2\sin^2\theta \cdot \dot{\phi} \Rightarrow \dot{\phi} = \frac{p_\phi}{mr^2\sin^2\theta}$$

$$\Rightarrow \dot{p}_\theta = mr^2\dot{\theta}$$

Now the Hamiltonian of the system is given by

$$H = P_\theta \cdot \dot{\theta} + P_\phi \cdot \dot{\phi} - L$$

$$= P_\theta \cdot \dot{\theta} + P_\phi \cdot \dot{\phi} - \frac{1}{2} mr^2 (\dot{\theta}^2 + \sin^2\theta \cdot \dot{\phi}^2) + mgr$$

$$= \frac{P_\theta^2}{mr^2} + \frac{P_\phi^2}{mr^2\sin^2\theta} - \frac{1}{2} mr^2 \left(\frac{P_\theta^2}{m^2r^4} + \sin^2\theta \cdot \frac{P_\phi^2}{m^2r^4\sin^4\theta} \right)$$
$$+ mgr\cos\theta$$

$$= \frac{P_\theta^2}{mr^2} + \frac{P_\phi^2}{mr^2\sin^2\theta} - \frac{1}{2} mr^2 \left(\frac{P_\theta^2}{m^2r^4} + \frac{P_\phi^2}{m^2r^4\sin^2\theta} \right)$$
$$+ mgr\cos\theta$$

$$= \frac{P_\theta^2}{mr^2} - \frac{1}{2}\frac{P_\theta^2}{mr^2} + \frac{P_\phi^2}{mr^2\sin^2\theta} - \frac{1}{2}\frac{P_\phi^2}{mr^2\sin^2\theta}$$
$$+ mgr\cos\theta$$

$$= \frac{P_\theta^2}{2mr^2} + \frac{P_\phi^2}{2mr^2\sin^2\theta} + mgr\cos\theta$$

$$\boxed{H = \frac{1}{2mr^2}\left(P_\theta^2 + \frac{P_\phi^2}{\sin^2\theta} \right) + mgr\cos\theta} \quad \ldots (2)$$

Now Hamilton's canonical eq^ns of motion is given by

$$\dot{\theta} = \frac{\partial H}{\partial P_\theta} \quad \text{and} \quad \dot{P}_\theta = \frac{-\partial H}{\partial \theta}$$

$$\dot{\theta} = \frac{P_\theta}{mr^2} \quad \text{and} \quad \dot{P}_\theta = -\left[\frac{P_\phi^2}{2mr^2} 2\operatorname{cosec}\theta(-\cot\theta \operatorname{cosec}\theta) + mgr(-\sin\theta) \right]$$

$$\dot{P}_\theta = \frac{P_\phi^2}{mr^2} \cdot \frac{\cos\theta}{\sin^3\theta} + mgr\sin\theta$$

$$\dot{\theta} = \frac{P_\theta}{mr^2}$$

$$\Rightarrow \ddot{\theta} = \frac{\dot{P}_\theta}{mr^2}$$

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$$\ddot{\theta} = \frac{\frac{p_\phi^2}{mr^2} \cdot \frac{\cos\theta}{\sin^3\theta} + mgr\sin\theta}{mr^2}$$

$$\ddot{\theta} = \frac{p_\phi^2}{m^2r^4} \cdot \frac{\cos\theta}{\sin^3\theta} + \frac{g}{r}\sin\theta$$

$$\cancel{\ddot{\theta} = \frac{m^2r^4\sin^2\theta\dot{\phi}^2}{m^2r^4} \frac{\cos\theta}{\sin^3\theta} + \frac{g}{r}\cos\sin\theta}$$

$$\boxed{\ddot{\theta} = \frac{\dot{\phi}^2}{mr^2} \cdot \cot\theta + \frac{g}{r}\sin\theta} \quad \ldots\ldots (3)$$

Now we ~~represent~~ claim that H represents constant of motion.

Diff. eq^n (2) w.r.t. t.

$$\frac{dH}{dt} = \frac{1}{mr^2} \cdot \frac{1}{2}\left[2p_\theta \cdot \dot{p}_\theta + 2p_\phi \cdot \dot{p}_\phi \right.$$

$$\ddot{\theta} = \frac{(mr^2\sin^2\theta \cdot \dot{\phi})^2}{m^2r^4} \cdot \frac{\cos\theta}{\sin^3\theta} + \frac{g}{r}\sin\theta$$

$$\ddot{\theta} = \dot{\phi}^2 \sin\theta \cdot \cos\theta \cdot + \frac{g}{r}\sin\theta$$

$$\dot{\phi} = \frac{\partial H}{\partial p_\phi} = \frac{p_\phi}{mr^2\sin^2\theta}$$

$$\dot{p}_\phi = \frac{-\partial H}{\partial \phi} = 0. \Rightarrow \dot{p}_\phi = 0$$

$$\Rightarrow p_\phi = c_1$$

$$\ddot{\theta} = \frac{p_\phi^2}{m^2r^4\sin^4\theta} \cdot \sin\theta \cdot \cos\theta + \frac{g}{r}\sin\theta$$

$$\ddot{\theta} = \frac{p_\phi^2}{m^2r^4} \cdot \frac{\cos\theta}{\sin^3\theta} + \frac{g}{r}\sin\theta.$$

Spherical Pendulum

$$\ddot{\theta} = \frac{p_\phi^2}{m^2r^4} \frac{\cos\theta}{\sin^3\theta} + \frac{g}{r}\sin\theta. \quad \text{———} (4)$$

Now we claim H represents a constant of motion.

Diff. eq^n (2) w.r.t. H

$p_\phi \cdot \frac{1}{\sin^2\theta}$

$$\frac{dH}{dt} = \frac{d}{dt}\left[\frac{1}{2mr^2}\left(p_\theta^2 + \frac{p_\phi^2}{\sin^2\theta}\right) + mgr\cos\theta\right]$$

$$= \frac{1}{2mr^2}\left[2p_\theta \cdot \dot{p}_\theta + \frac{\sin^2\theta\, 2p_\phi \cdot \dot{p}_\phi + (-2)\sin\theta\cos\theta\, p_\phi^2\dot{\theta}}{\sin^4\theta}\right] - mgr\sin\theta \cdot \dot{\theta}$$

$$= \frac{1}{mr^2}\left[p_\theta \cdot \dot{p}_\theta + \frac{p_\phi \cdot \dot{p}_\phi - \sin\theta\, p_\phi^2 \cos\theta \cdot \dot{\theta}}{\sin^4\theta}\right] - mgr\sin\theta \cdot \dot{\theta}$$

$$\Rightarrow = \frac{1}{mr^2}\left[(mr^2\dot{\theta})(mr^2\ddot{\theta}) + 0 - \frac{\sin\theta \cdot \cos\theta\, \dot{\theta} p_\phi^2}{\sin^4\theta}\right] - mgr\sin\theta \cdot \dot{\theta}$$

$$= \dot{\theta} mr^2 \ddot{\theta} - \frac{1}{mr^2}\frac{\cos\theta\, p_\phi^2 \dot{\theta}}{\sin^3\theta} - mgr\sin\theta \cdot \dot{\theta}$$

$$\Rightarrow = \dot{\theta} mr^2\left[\ddot{\theta} - \frac{p_\phi^2 \cos\theta}{m^2r^4\sin^3\theta} - \frac{g}{r}\sin\theta \cdot \dot{\theta}\right]$$

$$= \dot{\theta} \cdot mr^2 (0) \text{ ------ by (4)}$$

$$\frac{dH}{dt} = 0$$

H represent consto of motion

Now,

$$E = T + V = \frac{1}{2}mr^2(\dot{\theta}^2 + \sin^2\theta \cdot \dot{\phi}^2) + mgr\cos\theta$$

put $\dot{\theta} = \frac{p_\theta}{mr^2}$ and

$$\dot{\phi} = \frac{p_\phi}{mr^2\sin^2\theta}$$

$$= \frac{1}{2}mr^2\left[\frac{p_\theta^2}{m^2r^4} + \frac{p_\phi^2}{m^2r^4\sin^2\theta}\right] + mgr\cos\theta$$

$$= \frac{1}{2mr^2}\left[p_\theta + \frac{p_\phi^2}{\sin^2\theta}\right] + mgr\cos\theta$$

$= H$ ... from ③

H represents total energy.

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*. Physical Meaning of Hamiltonian:

1) For conservative scleronomic (not contain time.) system, the Hamiltonian H represents both constant of motion and total energy.

2) For conservative rheonomic system H may represents a constant of motion but does not represent the total energy.

proof: The Hamiltonian H is given by

$$H(p_j, q_j, t) = \sum_j p_j \dot{q}_j - L \quad \text{--------(1)}$$

Where L is Lagrangian and which must satisfies equation of motion.

$$\frac{\partial L}{\partial q_j} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) = 0 \quad \text{.........(2)}$$

The generalised momentum p_j is given by.

$$p_j = \frac{\partial L}{\partial \dot{q}_j} \quad \text{...........(3)}$$

$$\Rightarrow \dot{p}_j = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right)$$

$\therefore$ From (2) We have

$$\dot{p}_j = \frac{\partial L}{\partial q_j} \quad \text{..........(4)}$$

Now, consider,

$$\frac{dH}{dt} = \sum_j p_j \ddot{q}_j + \sum_j \dot{p}_j \dot{q}_j - \left\{\sum_j \left[\frac{\partial L}{\partial q_j}\dot{q}_j + \frac{\partial L}{\partial \dot{q}_j}\ddot{q}_j\right] + \frac{\partial L}{\partial t}\cdot\frac{dt}{dt}\right\}$$

$$= \sum_j p_j \cdot \ddot{q}_j + \sum_j \dot{p}_j \dot{q}_j - \sum_j \frac{\partial L}{\partial q_j}\dot{q}_j - \sum_j \frac{\partial L}{\partial \dot{q}_j}\ddot{q}_j - \frac{\partial L}{\partial t}$$

Using (3) & (4) in above eqn.

$$= \sum_j p_j \ddot{q}_j + \sum_j \dot{p}_j \dot{q}_j - \sum_j \dot{p}_j \dot{q}_j - \sum_j p_j \ddot{q}_j - \frac{\partial L}{\partial t}$$

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$$\frac{dH}{dt} = \frac{-\partial L}{\partial t} \quad \ldots\ldots\ldots (5)$$

case II : conservative scleronomic system.

In this case t is not involved explicitly.

∴ In these case, we have,

$$\frac{\partial L}{\partial t} = 0$$

∴ from eqⁿ (5),

$$\frac{dH}{dt} = 0$$

$$\Rightarrow H = \text{constant}.$$

∴ H represents const. of motion.

Now for scleronomic system, the K.E. is,

$$T = \sum_{j,k} a_{jk}\, \dot{q}_j \dot{q}_k$$

and $\sum_j \dot{q}_j \frac{\partial T}{\partial \dot{q}_j} = 2T \quad \ldots\ldots (6)$

Now for conservative system.

$$p_j = \frac{\partial T}{\partial \dot{q}_j} = \frac{\partial L}{\partial \dot{q}_j}$$

∴ eqⁿ (6) becomes,

$$\sum_j \dot{q}_j p_j = 2T$$

i.e. $\sum_j p_j \dot{q}_j = 2T \quad \ldots\ldots (7)$

Using this eqⁿ (7) in eqⁿ (1); we get,

$$H(p_j, q_j, t) = \sum_j p_j \dot{q}_j - L$$

$$= 2T - L$$

$$= 2T - (T - V)$$

$$= 2T - T + V$$

$$H = T + V$$

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case II) conservative Rheonomic system:

In this case $\sum_j P_j \sum_j \dot{q}_j \frac{\partial T}{\partial \dot{q}_j} = 2T_2 + T_1 \neq 2T$

$\therefore H = \sum_j P_j \dot{q}_j - L$

For conservative system,

$$P_j = \frac{\partial L}{\partial \dot{q}_j} = \frac{\partial T}{\partial \dot{q}_j}$$

$$\therefore \sum_j P_j \dot{q}_j = 2T_2 + T_1$$

Now. consider,

$$H = \sum_j P_j \dot{q}_j - L$$

$$= 2T_2 + T_1 - L$$
$$= 2T_2 + T_1 - (T - V)$$
$$= 2T_2 + T_1 - T + V$$
$$= 2T_2 + T_1 - T_2 - T_1 - T_0 + V$$
$$= T_2 - T_0 + V$$
$$\neq T + V = E$$

$\therefore \quad H \neq E$

Now for rheonomic system, if t is involved in T or V then $\frac{\partial L}{\partial t} \neq 0$

$\therefore \quad \frac{dH}{dt} \neq 0$

$\Rightarrow \quad H \neq$ constant

However there are some rheonomic system where t is not involved explicitly in L.

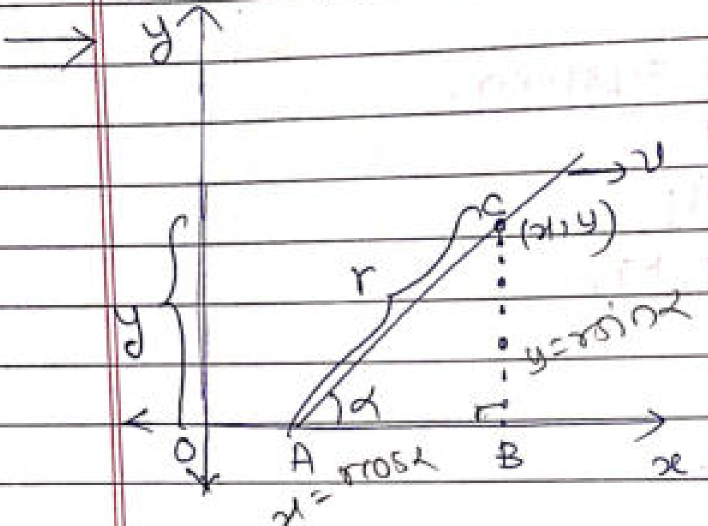
i.e. $\frac{\partial L}{\partial t} = 0$

$\therefore$ From eqn (5); we have,

$$\frac{dH}{dt} = 0.$$

$H =$ const.

Ex. consider a particle moving on an inclined rod with angle of inclination α. Further the inclined rod is moving along x-axis with const. speed v. Discuss the motion.



In this case the rod is moving along x-axis with constant velocity v.

$\therefore$ The position of rod at any time t can be found out by calculating

$OA = vt$ ------- (1)

Fix the particle on rod, we need it's distance from A only.

If $AC = r$ then r is generalised co-ordinate & D.O.F = 1

$y = r\sin\alpha$ ---- (2) (α is constant)

and $x = OA + AB$

$x = vt + r\cos\alpha$ ----- (3)

The transformation relation (3) contain time t explicity.

$\therefore$ Given system is rheonomic

The K.E. of system is.

$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2)$ ... m is mass of particle.

Where $\dot{x} = v + \dot{r}\cos\alpha$

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$\dot{y} = \dot{r}\sin\alpha$

$\therefore \dot{x}^2 + \dot{y}^2 = v^2 + 2v\dot{r}\cos\alpha + \dot{r}^2\cos^2\alpha + \dot{r}^2\sin^2\alpha$

$= v^2 + 2v\dot{r}\cos\alpha + \dot{r}^2$

$\therefore T = \frac{1}{2}m(v^2 + \dot{r}^2 + 2v\dot{r}\cos\alpha)$

Here $\frac{\partial T}{\partial t} = 0$

P.E. is

$V = mgy = mgr\sin\alpha$

$\therefore \frac{\partial V}{\partial t} = 0$

$\therefore \frac{\partial L}{\partial t} = 0$ i.e. Lagrangian does not contain time t explicitly.

$\frac{dH}{dt} = -\frac{\partial L}{dt} = 0 \Rightarrow H = \text{const} \Rightarrow$ H represent const. of motion.

* Routh's procedure :-

If q_j is cyclic co-ordinate then p_j = constant.
i.e. froms Hamilton's canonical eqn of motion is. $\dot{p}_j = \frac{-\partial H}{\partial q_j} = 0$

$\Rightarrow p_j = \text{const.}$

These advantage of Hamiltonian formulation in handling with cyclic ~~of~~ co-ordinate is ~~uait~~ utilised by Routh's and devised a method ~~as~~ by combining with lagragian procedure these new method is knowns as Routh's ~~produres~~ procedure.

Que. Described the Routh's ~~produre~~ procedure to solve the problem involving cyclic & non-cyclic co-ordinate.

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$H(p_j, q_j, t)$ $L(q_j, \dot{q}_j, t)$ s+1 to n → non cyclic

↓ cyclic ↓ non-cyclic.

→ consider a system involving cyclic generalized co-ordinate $q_1, q_2, \ldots, q_s$ and non-cyclic generalized co-ordinate $q_{s+1}, q_{s+2}, \ldots, q_n$.

The Routhian for these system is defined.

$$R(q_1, q_2, \ldots q_n; p_1, p_2, \ldots, p_s; \dot{q}_{s+1}, \dot{q}_{s+2}, \ldots, \dot{q}_n; t) = \sum_{j=1}^{s} p_j \dot{q}_j - L_{(q_j, \dot{q}_j)} \quad \ldots (1)$$

Where L is Lagrangian of these system.

Claim: We find Routhian eq^n of motion.

step. 1) consider $R = R(q_1, \ldots, q_n; p_1, \ldots, p_s; \dot{q}_{s+1}, \ldots \dot{q}_n; t)$

$$\therefore dR = \sum_{j=1}^{n} \frac{\partial R}{\partial q_j} dq_j + \sum_{j=1}^{s} \frac{\partial R}{\partial p_j} dp_j + \sum_{j=s+1}^{n} \frac{\partial R}{\partial \dot{q}_j} d\dot{q}_j + \frac{\partial R}{\partial t} dt \quad \ldots (2)$$

step. 2) $R = \sum_{j=1}^{s} p_j \dot{q}_j - L$

$$\therefore dR = \sum_{j=1}^{s} p_j d\dot{q}_j + \sum_{j=1}^{s} dp_j \cdot \dot{q}_j - \sum_{j=1}^{n} \frac{\partial L}{\partial q_j} dq_j - \sum_{j=1}^{n} \frac{\partial L}{\partial \dot{q}_j} d\dot{q}_j - \frac{\partial L}{\partial t} dt$$

$$\therefore dR = \sum_{j=1}^{s} p_j \cdot d\dot{q}_j + \sum_{j=1}^{s} dp_j \cdot \dot{q}_j - \sum_{j=1}^{s} \frac{\partial L}{\partial q_j} dq_j - \sum_{j=s+1}^{n} \frac{\partial L}{\partial q_j} dq_j - \sum_{j=1}^{s} \frac{\partial L}{\partial \dot{q}_j} d\dot{q}_j - \sum_{j=s+1}^{n} \frac{\partial L}{\partial \dot{q}_j} d\dot{q}_j - \frac{\partial L}{\partial t} \cdot dt$$

We know, $p_j = \frac{\partial L}{\partial \dot{q}_j}$ and $\dot{p}_j = \frac{\partial L}{\partial q_j}$

$$dR = \cancel{\sum_{j=1}^{s} p_j \cdot d\dot{q}_j} + \sum_{j=1}^{s} dp_j \cdot \dot{q}_j - \sum_{j=1}^{s} \dot{p}_j \, dq_j - \sum_{j=s+1}^{n} \dot{p}_j \, dq_j - \cancel{\sum_{j=1}^{s} p_j \cdot d\dot{q}_j} - \sum_{j=s+1}^{n} p_j \cdot d\dot{q}_j - \frac{\partial L}{\partial t} \cdot dt$$

$H = (q_j, p_j, t)$

$L = (q_j, \dot{q}_j, t)$

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$$\therefore dR = \sum_{j=1}^{s} dp_j \cdot \dot{q}_j - \sum_{j=1}^{s} \dot{p}_j dq_j - \sum_{j=s+1}^{n} \dot{p}_j dq_j - \sum_{j=s+1}^{n} p_j d\dot{q}_j - \frac{\partial L}{\partial t} dt \quad \cdots\cdots (3)$$

comparing eqn (2) and (3) we obtain

$$\frac{\partial R}{\partial p_j} = \dot{q}_j \quad ; j = 1, 2, 3, \cdots s \quad \cdots (4)$$

$R = \sum_{j=1}^{s} p_j \dot{q}_j - L$

$$\frac{\partial R}{\partial q_j} = -\dot{p}_j = \frac{-\partial L}{\partial q_j} \quad ; j = 1, 2, \cdots, s \quad \cdots (5)$$

$$\frac{\partial R}{\partial q_j} = -\dot{p}_j = \frac{-\partial L}{\partial q_j} \quad ; j = s+1, s+2, \cdots, n \quad \cdots (6)$$

$$\frac{\partial R}{\partial \dot{q}_j} = -p_j = \frac{-\partial L}{\partial \dot{q}_j} \quad ; j = s+1, s+2, \cdots, n \quad \cdots (7)$$

We see that for cyclic co-ordinates $q_1, q_2, \cdots, q_s$ eqn (4) and eqn (5) represents Hamilton's eqn of motion with R as Hamiltonian.

While eqn (6) and (7) for non-cyclic co-ordinate q_j $(j = s+1, s+2, \cdots, n)$ represents Lagrange's eqn of motion with R as Lagrangian. i.e. from eqn (6) and (7) we obtain

$$\frac{\partial R}{\partial q_j} - \frac{d}{dt}\left(\frac{\partial R}{\partial \dot{q}_j}\right) = 0 \quad ; j = s+1, s+2, \cdots, n$$

Thus by Routh's procedure, the problem involving cyclic and non-cyclic co-ordinates can be solved by solving Lagranges eqn for non-cyclic co-ordinate with R as the Lagrangian and solving Hamiltons eqn for the cyclic co-ordinate with R as the Hamiltonian.

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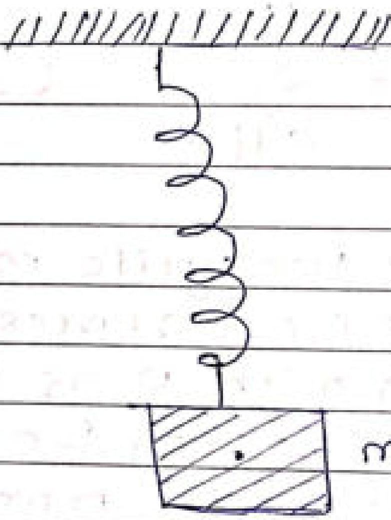
Ex. obtain Hamilton's eqn of motion for one dimens. Harmonic oscillator.

→ The one dimensional harmonic oscillator consist of a mass attached to one end of a spring. The other end of the spring is attached to a fixed support. If the spring is pressed are released then due to elasticity the spring exerts the force F on the body in the opposite direction. These is called restoring force.



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These force is proportional to displacement x of the body i.e. $F \propto x$

$\therefore F = -kx$; k is spring co

(-ve sign indicates that the force is on opposite direction of x).

Since the system is conservative

$$F = -\nabla V$$

$$= -\frac{\partial V}{\partial x}$$

$$dV = -F.dx$$

$$\Rightarrow V = -\int F.dx$$

$$V = \int kx.dx \qquad ; F = -kx$$

$L(q_j, \dot{q}_j, t)$

$H(P_j, q_j, t)$.

$R(\underset{1\text{ to }n}{q_j}, \underset{1\text{ to }s}{\dot{P}_j}, \underset{s+1\text{ to }n}{\dot{q}_j}, t)$

$$V = \frac{K \cdot x^2}{2} + C$$

He can choose reference level properly so that $C = 0$.

$$\therefore V = \frac{Kx^2}{2}$$

Now K.E is,

$$T = \frac{1}{2} m\dot{x}^2$$

$\therefore$ Lagrangian $L = L(x, \dot{x}, t)$

$$= T - V$$

$$L = \frac{1}{2} m\dot{x}^2 - \frac{Kx^2}{2} \quad \text{-----(1)}$$

Now generalized momentum ~~coor~~ corresponding to generalized co-ordinate is,

$$P_x = \frac{\partial L}{\partial \dot{x}} = m\dot{x} \quad \text{----(2)}$$

Now Hamiltonian $H = H(P_x, x, t)$

$$= P_x \cdot \dot{x} - L$$

$$= P_x\left(\frac{P_x}{m}\right) - \frac{1}{2}m\dot{x}^2 + \frac{Kx^2}{2}$$

$$= \frac{P_x^2}{m} - \frac{1}{2} m \frac{P_x^2}{m^2} + \frac{Kx^2}{2}$$

$$= \frac{P_x^2}{m} - \frac{1}{2} \cdot \frac{P_x^2}{m} + \frac{Kx^2}{2}$$

$$\boxed{H = \frac{P_x^2}{2m} + \frac{Kx^2}{2}} \quad \text{---(3)}$$

~~Eq~~ Hamilton's canonical eq^{n}s of motion.

1) $\dot{P}_x = \frac{-\partial H}{\partial x} = -Kx \quad \text{---(4)}$

2) $\dot{x} = \frac{+\partial H}{\partial P_x} = \frac{P_x}{m} \quad \text{---(5)}$

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From (5) $\dot{x} = \frac{p_x}{m}$

$$\ddot{x} = \frac{\dot{p}_x}{m}$$

Use eq^n (4)

$$\ddot{x} = \frac{-kx}{m}$$

$$\Rightarrow \ddot{x} + \frac{kx}{m} = 0$$

Which is 2nd order d.E.

Which represent lagranges equation of motion.

Lagrange's eq^n of motion is

$$\frac{\partial L}{\partial x} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = 0$$

$$-kx - m\ddot{x} = 0$$

$$kx + m\ddot{x} = 0$$

$$\frac{kx}{m} + \ddot{x} = 0$$

IMP

Ex. Find L, H and R for particle (planet) moving under inverse square law of attractive force.

→ The particle will trace a plane curve

∴ The motion of planer motion.

suppose that r and θ are generalized co-ordinates.

Here, $F \propto \frac{1}{r^2}$

$$\therefore F = \frac{-k}{r^2} \quad ; \text{ k is constant.}$$

Negative sign is convension.

since this force is conservative

$$F = -\nabla U = \frac{-\partial U}{\partial r}$$

cyclic - velocity.

non-cyclic - momenta.

$R : (q_1 \dots q_n, p_1 \dots p_s, \dot{q}_{s+1} \dots \dot{q}_n, t)$

$\therefore V = \int \frac{k}{r^2} . dr = -\frac{k}{r}$

$H : H(r, \theta, p_\theta, p_r, t)$

$L : L(r, \theta, \dot{r}, \dot{\theta}, t)$

Now, K.E. is

$T = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2)$ $R : R(r, \theta ; p_\theta ; \dot{r} ; t)$

Where m is mass of particle.

$\therefore$ Lagrangian is

$L(r, \theta, \dot{r}, \dot{\theta}, t) = T - V$

i.e. $\boxed{L = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2) + \frac{k}{r}}$ ...... (1)

It can be observed that θ is cyclic and r is non-cyclic co-ordinate.

Generalised momentum,

$p_r = \frac{\partial L}{\partial \dot{r}} = m\dot{r} \Rightarrow \dot{r} = \frac{p_r}{m}$ ...... (2)

$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta} \Rightarrow \dot{\theta} = \frac{p_\theta}{mr^2}$ ...... (3)

$\therefore$ Hamiltonian is.

$H(r, \theta, p_r, p_\theta, t) = p_r . \dot{r} + p_\theta \dot{\theta} - L$

$= p_r . \dot{r} + p_\theta . \dot{\theta} - \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2) - \frac{k}{r}$

$\boxed{H = \frac{p_r^2}{2m} + \frac{p_\theta^2}{2mr^2} - \frac{k}{r}}$ .... (4)

Now, Routhian

$R(q_j ; p_1 . p_2, \dots, p_s ; q_{s+1} \dots q_n, t) = \sum_{i=1}^{s} p_j \dot{q}_j - L$

$R(\theta ; r ; p_\theta ; \dot{r}, t) = p_\theta . \dot{\theta} - L$

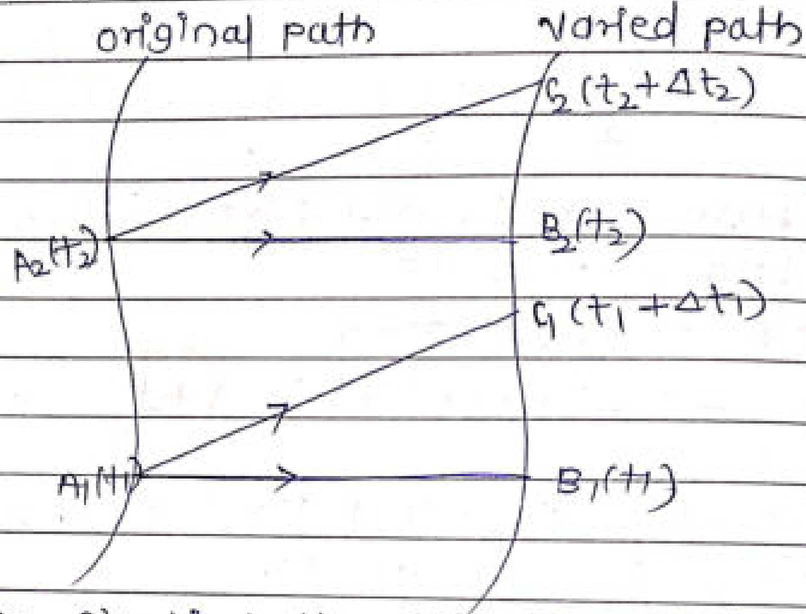
$= p_\theta . \dot{\theta} - \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2) - \frac{k}{r}$

$= \frac{p_\theta^2}{mr^2} - \frac{1}{2} m (\dot{r}^2 + r^2 . \frac{p_\theta^2}{m^2 r^4}) - \frac{k}{r}$

This is Routhian $= \frac{p_\theta^2}{2mr^2} - \frac{p_r^2}{2m} - \frac{k}{r}$

Que. Explain Δ-variation obtain relation betn Δ and δ-variations.

$\Rightarrow$



We have studied Hamilton's principle:

$$\delta \int_{t_0}^{t_1} L\,dt = 0 \quad \text{-------} \quad (1)$$

Where δq_j is variation of co-ordinate q_j such that $\delta q_j = 0$ at t_0 and t_1 i.e. end points were kept fixed.

In (1) we have considering variations where the time was not varied.

i.e. $\delta t = 0$

i.e. we compared the points on original and new orbits at the same time (the points A_n & B_n in figure.)

We shall now consider a different variations of the orbit where we compare q_j at time t with $q_j + \Delta q_j$ at time $t+\Delta t$.

(The points A_n and C_n in figure).

To each pt. A_n of the original orbit, $\exists$ a point B_n at the same time t on the new orbit as well as a point C_n at the varied time $t+\Delta t$

Δ - variations
t is not fixed
$\Delta t \neq 0$

δ-variation
$\delta t = 0$

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End points are fixed $\rightarrow$ $\Delta q = 0$ & $\delta q = 0$

We have,

$$q_{jB} = q_{jA} + \delta q_j \quad \ldots\ldots (1)$$

$$\Rightarrow q_{jA} = q_{jB} - \delta q_j \quad \text{---} (3)$$

(q_{jB} is a co-ordinate q_j at pt. B etc)

$$q_{jC} = q_{jA} + \Delta q_j \quad \ldots\ldots (2)$$

and By (2)

$$\begin{aligned}
\Delta q_j &= q_{jC} - q_{jA} \\
&= q_{jC} + \delta q_j \quad \ldots \text{By (3)} \\
&= q_j(t+\Delta t) - q_j(t) + \delta q_j \\
&= \dot{q}_j \Delta t + \delta q_j \quad \ldots \text{mean value theo.}
\end{aligned}$$

i.e. $\boxed{\Delta q_j = \delta q_j + \dot{q}_j \Delta t.}$ $\leftarrow$ M.C.Q.

Which is required relation betn δ and Δ variation

4th chapter

*. Kinematics of Rigid Body :-

Rigid Body -

Rigid body is regarded as a system of many (at least 3 non-collinear) particles whose position related to one another remain fixed i.e. distance betn any two particles remains constant. The internal forces holding the particles at fixed distances from one-another are known as forces of constraints. These forces obeys the Newton's 3rd law of motion.

(5) mark

VIMP

Generalized co-ordinate of rigid Body :-

Que. Explain how the No. of DOF of rigid body with N particles reduces to 6.

VIMP

Ans. A system of N free particles in space can be have 3N DOF. The constraints involved in the rigid body are of the form

$$r_{ij} = c_{ij} = \text{constant} \quad \ldots\ldots (1)$$

(Where r_{ij} = distance betn ith and jth particles)

$N=8.$ $^8C_2 = \frac{8!}{6!\cdot 2!} = \frac{8\cdot 7}{2} = 28$ $\quad$ $^7C_2 = \frac{7!}{(5!)2!} = \frac{7\cdot 6\cdot 5!}{5!\times 2} = 21.$

$24 - 28 = \underline{-4}$

DOF $= -4$. (These are holonomic and scleronomic)

If these relations are independent then we can find D.O.F as

$N=7.$

$^7C_2 = 21$

$21-21 = 0.$

DOF $= 0.$

$$D.O.F = 3N - N_{C_2}\ {}^NC_2.$$

$N=7$ D.o.f $=0$

But for $N > 7$

$$N_{C_2} > 3N$$

$\therefore$ D.O.F will be $-$ve No.

which is controdiction.

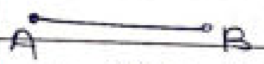
$\therefore$ The relations ① are not independent.

We show that the generalized co-ordinate of rigid body reduces to 6, for this discription.

consider a particle 'A' in a rigid body.

If required 3 generalized co-ordinates to specify the position of A.

Now, consider the another particle B in rigid body. The distance bem A & B is fixed



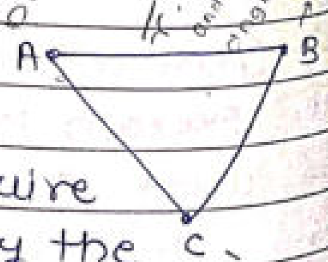
$\therefore$ The co-ordinates of A are known then the motion of B is like a spherical pendulum (with centre A and radius |AB|)

$\therefore$ It requires only two generalized co-ordinates to specify position of B.

Now, consider 3rd particle 'c' which is non-collinear with A and B.

Distance of 'c' from 'A' & from 'B' is always fixed

$\therefore$ In the similar way we require only one co-ordinate to specify the position of 'c'.



If we add any more pts in the system then we does not need any co-ordinate to specify its position if the co-ordinates A, B & C

$\Delta = \dot{q}_j \Delta t + \delta q_j$ $\Delta q_j = \delta q_j + \dot{q}_j \underline{\Delta t}$

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$\Delta = \frac{d}{dt} \cdot \Delta t + \delta$. $\Delta = \delta + \frac{d}{dt} \cdot \underline{\Delta t}$

are known.

Thus, we need only $3+2+1=6$ generalized co-ordinates to fix the position of rigid body

$\therefore$ D.O.F $= 6$.

3rd chapter:

M.C.Q.

Ex. If $f = f(q_j, \dot{q}_j, t)$ then show that $\Delta f = \delta f + \Delta t \frac{df}{dt}$

$\rightarrow$ We have $f = f(q_j, \dot{q}_j, t)$

$$\therefore \Delta f = \sum_j \left[\frac{\partial f}{\partial q_j} \Delta q_j + \frac{\partial f}{\partial \dot{q}_j} \Delta \dot{q}_j \right] + \frac{\partial f}{\partial t} \cdot \Delta t \quad \cdots (1)$$

We know the relation betn Δ and δ is

$$\Delta q_j = \delta q_j + \dot{q}_j \Delta t$$

$$\therefore \Delta \dot{q}_j = \delta \dot{q}_j + \ddot{q}_j \Delta t$$

Using these values in eq (1).

$$\Delta f = \sum_j \left[\frac{\partial f}{\partial q_j} (\delta q_j + \dot{q}_j \Delta t) + \frac{\partial f}{\partial \dot{q}_j} (\delta \dot{q}_j + \ddot{q}_j \Delta t) \right] + \frac{\partial f}{\partial t} \Delta t$$

$$= \cancel{\sum_j} \left[\sum_j \frac{\partial f}{\partial q_j} \delta q_j + \sum_j \frac{\partial f}{\partial \dot{q}_j} \delta \dot{q}_j \right] + \Delta t \left\{ \left[\sum_j \frac{\partial f}{\partial q_j} \dot{q}_j + \sum_j \frac{\partial f}{\partial \dot{q}_j} \ddot{q}_j \right] + \frac{\partial f}{dt} \right\}$$

$f(q_j, \dot{q}_j, t)$

$$\Rightarrow \delta f = \sum_j \left[\frac{\partial f}{\partial q_j} \delta q_j + \frac{\partial f}{\partial \dot{q}_j} \delta \dot{q}_j \right] \text{ and } \frac{df}{dt} = \sum_j \left[\frac{\partial f}{\partial q_j} \dot{q}_j + \frac{\partial f}{\partial \dot{q}_j} \ddot{q}_j \right] + \frac{\partial f}{\partial t}.$$

$$\Delta f = \delta f + \Delta t \cdot \frac{df}{dt}.$$

$$\Delta f = \left(\delta + \Delta t \cdot \frac{d}{dt} \right) f \qquad \Delta = \delta + \frac{d}{dt} \cdot \Delta t$$

M.C.Q. $\therefore \boxed{\Delta = \delta + \Delta t \cdot \frac{d}{dt}}$

$$\Delta = \delta + \frac{d}{dt} \Delta t$$

Que. A system of 2 DOF is described by the Hamiltonian $H = q_1 p_1 - q_2 p_2 - a q_1^2 + b q_2^2$; a, b are const. show that

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1) $\frac{p_1 - aq_1}{q_2}$ 2) $\frac{p_2 - bq_2}{q_1}$ 3) $q_1 \cdot q_2$ 4) H are constants of motion.

→ Here generalized co-ordinates are q_1 and q_2.
The Hamiltonian H is given by,

$$H = q_1 p_1 - q_2 p_2 - aq_1^2 + bq_2^2$$

Now the Hamiltonian canonical eqn of motion corresponding to q_1 and q_2 are given by,

1) $\dot{p}_1 = -\frac{\partial H}{\partial q_1}$ and $\dot{p}_2 = -\frac{\partial H}{\partial q_2}$

2) $\dot{q}_1 = \frac{\partial H}{\partial p_1}$ and $\dot{q}_2 = \frac{\partial H}{\partial p_2}$

$$\therefore \dot{p}_1 = -\frac{\partial H}{\partial q_1} = -(p_1 - 2aq_1) = 2aq_1 - p_1$$

$$\dot{p}_2 = -\frac{\partial H}{\partial q_2} = -(-p_2 + 2bq_2) = p_2 - 2bq_2$$

$$\dot{q}_1 = \frac{\partial H}{\partial p_1} = q_1$$

$$\dot{q}_2 = \frac{\partial H}{\partial p_2} = -q_2$$

1) $$\frac{d}{dt}\left[\frac{p_1 - aq_1}{q_2}\right] = \frac{q_2(\dot{p}_1 - a\dot{q}_1) - \dot{q}_2(p_1 - aq_1)}{q_2^2}$$

$$= \frac{q_2[2aq_1 - p_1 - aq_1] + q_2 p_1 - aq_1 q_2}{q_2^2}$$

$$= \frac{q_2[aq_1 - p_1] + q_2 p_1 - aq_1 q_2}{q_2^2}$$

$$= \frac{aq_1 q_2 - q_2 p_1 + q_2 p_1 - aq_1 q_2}{q_2^2}$$

$$= \frac{0}{q_2^2}$$

$$\frac{d}{dt}\left[\frac{p_1 - aq_1}{q_2}\right] = 0$$

$\Rightarrow \frac{p_1 - aq_1}{q_2}$ constant.

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2) $\frac{d}{dt}\left[\frac{p_2 - bq_2}{q_1}\right] = \frac{q_1(\dot{p}_2 - b\dot{q}_2) - \dot{q}_1(p_2 - bq_2)}{q_1}$

$= q_1(p_2 - 2bq_2 + bq_2) - q_1 p_2 + q_1 b q_2$

$= q_1 p_2 - 2bq_1 q_2 + bq_2 q_1 - q_1 p_2 + q_1 b q_2$

$= 0$

3) $\frac{d}{dt}[q_1 \cdot q_2] = q_1 \dot{q}_2 + q_2 \dot{q}_1$

$= -q_1 q_2 + q_2 q_1$

$= 0$

4) H are const of motion

$\frac{dH}{dt} = q_1 \dot{p}_1 + p_1 \dot{q}_1 - [q_2 \dot{p}_2 + p_2 \dot{q}_2] - 2aq_1\dot{q}_1 + 2bq_2\dot{q}_2$

$= q_1(2aq_1 - p_1) + p_1 q_1 - [q_2(p_2 - 2bq_2) + p_2(-q_2)] - 2aq_1^2 - 2bq_2^2$

$= \cancel{2aq_1^2} - \cancel{q_1 p_1} + \cancel{q_1 p_1} - \cancel{q_2 p_2} + \cancel{2bq_2^2} + \cancel{q_2 p_2} - \cancel{2aq_1^2} - \cancel{2bq_2^2}$

$\frac{dH}{dt} = 0 \Rightarrow H = \text{const}$

H represents const. of motion.

V.IMP. 12 mark.

(12) State and prove principle of least action.

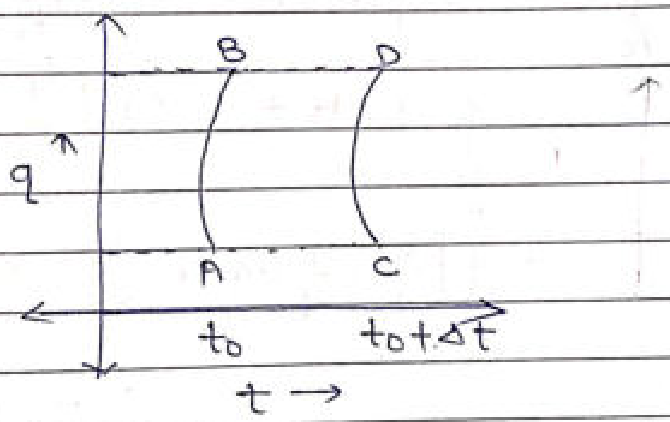
V.IMP

Statement:

For conservative system for which the Hamiltonian H is conserved the principle of least action states that,

$$\Delta \int_{t_0}^{t_1} \sum_j p_j \dot{q}_j \, dt = 0.$$

proof:



We consider a conservative system for which Hamiltonian H is conserved.

Let AB be the actual path and CD be the varied path.

The action integral is given by,

$$A = \int_{t_0}^{t_1} \sum_j p_j \dot{q}_j \cdot dt$$

$$\therefore A = \int_{t_0}^{t_1} (L+H)\, dt$$

$$= \int_{t_0}^{t_1} L \cdot dt + \int_{t_0}^{t_1} H \cdot dt$$

$$A = \int_{t_0}^{t_1} L \cdot dt + Ht \Big|_{t_0}^{t_1} \quad \ldots \text{(1)} \qquad H = \text{const}$$

$$\therefore \Delta A = \Delta \int_{t_0}^{t_1} L \cdot dt + H \Delta t \Big|_{t_0}^{t_1} \quad \ldots \text{(2)}$$

at t_0

$\Delta q_j = \delta q_j + \dot{q}_j \Delta t$

$\Delta L = \delta L + \dot{L}.\Delta t$

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We can not take Δ inside the integral because the limits are also changing during these variation.

*. Let $I = \int_{t_0}^{t_1} L.dt$

$\therefore \dot{I} = L$

$\therefore \Delta I = \delta I + \dot{I}\Delta t$ $\qquad \Delta I = \delta I + \dot{I}.\Delta t$

$\therefore \Delta I = \delta I + \dot{I}.\Delta t$

$$\Delta \int_{t_0}^{t_1} L.dt = \delta \int_{t_0}^{t_1} L.dt + L.\Delta t \Big|_{t_0}^{t_1}$$

$$= \int_{t_0}^{t_1} \delta L.dt + L.\Delta t \Big|_{t_0}^{t_1}$$

$$= \int_{t_0}^{t_1} \left[\sum_j \frac{\partial L}{\partial q_j}\delta q_j + \frac{\partial L}{\partial \dot{q}_j}\delta \dot{q}_j\right] dt + L.\Delta t\Big|_{t_0}^{t_1}$$

; $\delta t = 0$ for any variation.

We know, $p_j = \frac{\partial L}{\partial \dot{q}_j}$

$\therefore$ By using Lagranges eqn of motion,

$\dot{p}_j = \frac{\partial L}{\partial q_j}$

$\therefore$ above eqn becomes,

$$\Delta \int_{t_0}^{t_1} L.dt = \int_{t_0}^{t_1} \left[\sum_j \dot{p}_j \delta q_j + p_j.\delta \dot{q}_j\right] dt + L.\Delta t\Big|_{t_0}^{t_1}$$

$$= \int_{t_0}^{t_1} \left[\frac{d}{dt}\sum_j p_j.\delta q_j\right].dt + L.\Delta t\Big|_{t_0}^{t_1}$$

since, $\Delta = \delta + \Delta t.\frac{d}{dt} \Rightarrow \delta = \Delta - \Delta t\frac{d}{dt}$

$$\Delta \int_{t_0}^{t_1} L dt = \int_{t_0}^{t_1} \cancel{d}\left[\sum_j p_j\left(\Delta - \Delta t.\frac{d}{dt}\right)q_j\right]_{t_0}^{t_1} + L\Delta t\Big|_{t_0}^{t_1}$$

$$= \left[\sum_j p_j \Delta q_j - \sum_j p_j.\Delta t.\frac{dq_j}{dt}\right]_{t_0}^{t_1} + L\Delta t\Big|_{t_0}^{t_1}$$

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$$= \left[\sum_j p_j \Delta q_j - \sum_j p_j \dot{q}_j \Delta t \right]_{t_0}^{t_1} + L \Delta t \Big|_{t_0}^{t_1}$$

$$= -\left[\sum_j p_j \dot{q}_j \Delta t \right]_{t_0}^{t_1} + L \Delta t \Big|_{t_0}^{t_1} \qquad (\Delta q_j)_{t_0}^{t_1} = 0$$

$$= -\left[\sum_j p_j \dot{q}_j \cancel{\Delta t} - L \right] \Delta t \Big|_{t_0}^{t_1}$$

$$= -H \Delta t \Big|_{t_0}^{t_1}$$

$$\therefore \Delta \int_{t_0}^{t_1} L\,dt = -H \Delta t \Big|_{t_0}^{t_1}$$

$\therefore$ eq^n 2 becomes,

$$\Delta A = -H \Delta t \Big|_{t_0}^{t_1} + H \Delta t \Big|_{t_0}^{t_1}$$

$$\Delta A = 0$$

$$\Delta \int_{t_0}^{t_1} \sum_j p_j \dot{q}_j \, dt = 0.$$

Hence, the proof.