★ Cauchy-Riemann Equations :-

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

i.e. $u_x = v_y$ i.e. $u_y = -v_x$

• **Analytic Function :-** A function f of the complex variable z is analytic at a point z_0 if its derivative $f'(z)$ exists not only at z_0 but at every point z in some nhd of z_0. It is analytic in a domain of the z-plane if it is analytic at every pt. in that domain.

The terms "regular" and "holomorphic" are sometimes introduced to denote analyticity in domains of certain classes.

• Theorem :- The necessary condition for $f(z)$ to be analytic.

Statement :- Suppose that $f(z) = u(x,y) + iv(x,y)$ and that $f'(z)$ exists at a point $z_0 = x_0 + iy_0$. Then the first-order partial derivatives of u & v must satisfy the Cauchy-Riemann eq^{ns} - $u_x = v_y$, $u_y = -v_x$

i.e. $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$, $\frac{\partial u}{\partial y} = \frac{-\partial v}{\partial x}$ & Also

$f'(z_0)$ can be written as $f'(z_0) = u_x + iv_x$ where these partial derivatives are to be evaluated at (x_0, y_0).

Proof :- since $f(z) = u(x,y) + iv(x,y)$ ..... ①
is analytic fu^n of z.
i.e. $f'(z_0)$ exists at point $z_0 = x_0 + iy_0$.
Now, write $z_0 = x_0 + iy_0$, $\Delta z = \Delta x + i\Delta y$
and $\Delta f = f(z_0 + \Delta z) - f(z_0)$
i.e. $\Delta w = f(z_0 + \Delta z) - f(z_0)$

$$= [u(x_0 + \Delta x, y_0 + \Delta y) - u(x_0, y_0)] + i[v(x_0 + \Delta x, y_0 + \Delta y) - v(x_0, y_0)]$$

Assuming that the derivative

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z} \text{ exists.} \quad \ldots ②$$

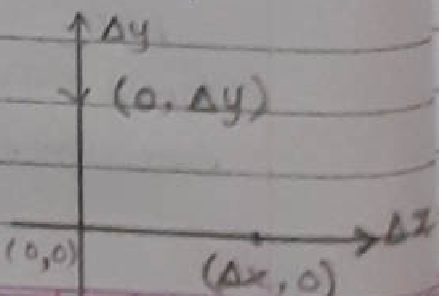
We know that- If $f(z) = u(x,y) + iv(x,y)$, $z = x+iy$
& $z_0 = x_0 + iy_0$, $w_0 = u_0 + iv_0$
then

$$\lim_{z \to z_0} f(z) = w_0 \text{ iff}$$

$$\lim_{(x,y)\to(x_0,y_0)} u(x,y) = u_0 \quad \& \quad \lim_{(x,y)\to(x_0,y_0)} v(x,y) = v_0$$

$$\therefore f'(z_0) = \lim_{(\Delta x,\Delta y)\to(0,0)} \left(\operatorname{Re} \frac{\Delta w}{\Delta z}\right) + i \lim_{(\Delta x,\Delta y)\to(0,0)} \left(\operatorname{Im} \frac{\Delta w}{\Delta z}\right) \quad \ldots ③$$

Expression ③ is valid as $(\Delta x, \Delta y)$ tends to $(0,0)$ horizontally through the points $(\Delta x, 0)$ as indicated in fig. as $\Delta y = 0$ the quotient $\frac{\Delta w}{\Delta z}$ becomes;



$$\frac{\Delta w}{\Delta z} = \frac{u(x_0 + \Delta x, y_0) - u(x_0, y_0)}{\Delta x} + i\,\frac{v(x_0 + \Delta x, y_0) - v(x_0, y_0)}{\Delta x}$$

Thus,

$$\lim_{(\Delta x,\Delta y)\to(0,0)}\left(Re\,\frac{\Delta W}{\Delta Z}\right)=\lim_{\Delta x\to 0}\frac{u(x_0+\Delta x,y_0)-u(x_0,y_0)}{\Delta x}$$

$$=u_x(x_0,y_0)$$

$$\&\ \lim_{(\Delta x,\Delta y)\to(0,0)}\left(Im\,\frac{\Delta W}{\Delta Z}\right)=\lim_{\Delta x\to 0}\frac{v(x_0+\Delta x,y_0)-v(x_0,y_0)}{\Delta x}$$

$$=v_x(x_0,y_0)$$

Where $u_x(x_0,y_0)$ & $v_x(x_0,y_0)$ denote the 1st order partial derivatives w.r.t. x of fu^{ns} u & v resp. at (x_0,y_0). Substituting these limits in eq^n ③

$$f'(z_0)=u_x(x_0,y_0)+i\,v_x(x_0,y_0)\quad\ldots ④$$

Again let Δz tends to zero vertically through the points $(0,\Delta y)$ in this case $\Delta x=0$

$$\boxed{\begin{array}{lll} z=x+iy & \Rightarrow & \Delta z=\Delta x+i\Delta y \\ \Delta x=0 & \Rightarrow & \Delta z=i\Delta y \\ \&\ \Delta y=0 & \Rightarrow & \Delta z=\Delta x. \end{array}}$$

$$\therefore\ \frac{\Delta W}{\Delta Z}=\frac{u(x_0,\cancel{y_0}+\Delta y)-u(x_0,y_0)}{i\Delta y}+i\,\frac{v(x_0,y_0+\Delta y)-v(x_0,y_0)}{i\Delta y}$$

$$=\frac{v(x_0,y_0+\Delta y)-v(x_0,y_0)}{\Delta y}-i\,\frac{u(x_0,y_0+\Delta y)-u(x_0,y_0)}{\Delta y}$$

then

$$\lim_{(\Delta x,\Delta y)\to(0,0)}\left(Re\,\frac{\Delta W}{\Delta Z}\right)=\lim_{\Delta y\to 0}\frac{v(x_0,y+\Delta y)-v(x_0,y_0)}{\Delta y}$$

$$=v_y(x_0,y_0)$$

&

$$\lim_{(\Delta x, \Delta y)\to(0,0)} \left(\operatorname{Im} \frac{\Delta W}{\Delta z}\right) = -\lim_{\Delta y \to 0} \frac{u(x_0, y+\Delta y) - u(x_0, y_0)}{\Delta y}$$

$$= -u_y(x_0, y_0)$$

Hence from eq^n ③,

$$f'(z_0) = v_y(x_0, y_0) - i\, v_y(x_0, y_0) \quad \ldots\ldots ⑤$$

Equating real & imaginary parts of ④ & ⑤, we get

$$u_x = v_y \quad \& \quad v_x = -u_y$$

i.e. $\boxed{u_x = v_y \quad \& \quad u_y = -v_x}$ ..... ⑥

eq^{ns} ⑥ are called Cauchy-Riemann eq^{ns} at (x_0, y_0).

Sufficient condition for differentiability:-

Th^m:-

statement:- Let the fu^n $f(z) = u(x,y) + iv(x,y)$ be defined throughout some ϵ-nhd of a pt. $z_0 = x_0 + iy_0$ & suppose that

ⓐ the first order partial derivatives of the fu^{ns} **u** & **v** w.r.t. x & y exist everywhere in the nhd.

ⓑ those partial derivatives are continuous at (x_0, y_0) & satisfy the C.R. eq^{ns}.

$$u_x = v_y, \quad u_y = -v_x \quad \text{at } (x_0, y_0)$$

then

$f'(z_0)$ exists, its value being

$$f'(z) = u_x + i v_x$$

where r.h.s. is to be evaluated at (x_0, y_0).

Proof:-

We assume that conditions ⓐ & ⓑ in its hypothesis are satisfied & write $\Delta z = \Delta x + i\Delta y$, where $0 < |\Delta z| < \epsilon$, as well as $\Delta w = f(z_0 + \Delta z) - f(z_0)$

Thus

$$\Delta w = \Delta u + i\Delta v \quad \ldots\ldots (1)$$

where

$$\Delta u = u(x_0 + \Delta x,\ y_0 + \Delta y) - u(x_0, y_0)$$

$$\& \quad \Delta v = v(x_0 + \Delta x,\ y_0 + \Delta y) - v(x_0, y_0)$$

The assumption that the 1^{st} order partial derivatives of u & v are continuous at the pt. (x_0, y_0) enables us to write *

$$\Delta u = u_x(x_0, y_0)\Delta x + u_y(x_0, y_0)\Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y \quad \ldots\ldots (2)$$

$$\& \quad \Delta v = v_x(x_0, y_0)\Delta x + v_y(x_0, y_0)\Delta y + \epsilon_3 \Delta x + \epsilon_4 \Delta y \quad \ldots\ldots (3)$$

Where $\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4 \to 0$ as $\Delta x \to 0$, $\Delta y \to 0$

Putting (2) & (3) in eq^n (1) we get

$$\Delta w = u_x(x_0, y_0)\Delta x + u_y(x_0, y_0)\Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y + i[v_x(x_0, y_0)\Delta x + v_y(x_0, y_0)\Delta y + \epsilon_3 \Delta x + \epsilon_4 \Delta y] \quad \ldots\ldots (4)$$

C.R. eq^{ns} are satisfied at (x_0, y_0). We replace $u_y(x_0, y_0)$ by $-v_x(x_0, y_0)$

&

$v_y(x_0, y_0)$ by $u_x(x_0, y_0)$ in eq^n (4)

&

divide by the quantity $\Delta z = \Delta x + i\Delta y$ we get,

$$\frac{\Delta W}{\Delta Z} = \frac{u_x(x_0, y_0)\Delta x}{\Delta Z} - \frac{v_x(x_0, y_0)\Delta y}{\Delta Z} + \epsilon_1 \frac{\Delta x}{\Delta Z} + \epsilon_2 \frac{\Delta y}{\Delta Z}$$

$$+ i\left[\frac{v_x(x_0, y_0)\Delta x}{\Delta Z} + \frac{u_x(x_0, y_0)\Delta y}{\Delta Z} + \epsilon_3 \frac{\Delta x}{\Delta Z} + \epsilon_4 \frac{\Delta y}{\Delta Z}\right]$$

$$= u_x(x_0, y_0)\left[\frac{\Delta x + i\Delta y}{\Delta Z}\right] + i v_x(x_0, y_0)\left[\frac{\Delta x + i\Delta y}{\Delta Z}\right]$$

$$+ (\epsilon_1 + i\epsilon_3)\frac{\Delta x}{\Delta Z} + (\epsilon_2 + i\epsilon_4)\frac{\Delta y}{\Delta Z}$$

i.e.

$$\frac{\Delta W}{\Delta Z} = u_x(x_0, y_0) + i v_x(x_0, y_0) + (\epsilon_1 + i\epsilon_3)\frac{\Delta x}{\Delta Z} + (\epsilon_2 + i\epsilon_4)\frac{\Delta y}{\Delta Z} \quad \text{.......} (5)$$

But $|\Delta x| \le |\Delta z|$ & $|\Delta y| \le |\Delta z|$

i.e.

$\frac{|\Delta x|}{|\Delta z|} \le 1$ i.e. $\frac{|\Delta y|}{|\Delta z|} \le 1$

i.e. $\left|\frac{\Delta x}{\Delta z}\right| \le 1$ & $\left|\frac{\Delta y}{\Delta z}\right| \le 1$

Consequently

$$\left|\epsilon_1 + i\epsilon_3 \cdot \frac{\Delta x}{\Delta z}\right| \le |\epsilon_1 + i\epsilon_3| \le |\epsilon_1| + |\epsilon_3|$$

&

$$\left|\epsilon_2 + i\epsilon_4 \cdot \frac{\Delta y}{\Delta z}\right| \le |\epsilon_2 + i\epsilon_4| \le |\epsilon_2| + |\epsilon_4|$$

i.e. last two terms in eq^n (5) tends to zero as $\Delta z \to 0$

i.e. $\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4 \to 0$ as $\Delta z \to 0$ i.e. $\Delta x \to 0, \Delta y \to 0$

$$\therefore \lim_{\Delta z \to 0} \frac{\Delta W}{\Delta z} = u_x + i v_x$$

$\therefore f'(z) = u_x + iv_x$

$\Rightarrow f'(z)$ exists.

$\therefore f(z)$ is analytic.

Ex:- ① We have to prove that $f(z) = z^2 = x^2 - y^2 + 2ixy$ is differentiable everywhere & $f'(z) = 2z$. To verify that the C-R eq^{ns} are satisfied everywhere.

$\Rightarrow$

$u(x, y) = x^2 - y^2$, $v(x, y) = 2xy$

Thus $u_x = 2x$ $\quad$ $u_y = 2x$

& $u_y = -2y$ $\quad$ $v_x = 2y$

$\therefore u_x = u_y = 2x$ & $u_y = -v_x = -2y$

& $f'(z) = u_x + iv_x$

$\therefore f'(z) = 2x + i2y = 2(x + iy) = \underline{\underline{2z}}$

Ex:- ② When $f(z) = |z|^2$

$\rightarrow$ We have $|z|^2 = x^2 + y^2$

$\therefore u(x, y) = x^2 + y^2$, $v(x, y) = 0$

$\therefore u_x = 2x$, $v_y = 0$

& $u_y = 2y$, $v_x = 0$

$\therefore$ C-R eq^{ns} are not satisfied.

$\therefore u_x = v_y = 0 \Rightarrow 2x = 0 \Rightarrow x = 0$

& $u_y = -v_x \Rightarrow 2y = 0 \Rightarrow y = 0$.

Consequently $f'(z)$ does not exist at any non-zero pts.

Ex:- $f(z) = e^z = e^{x+iy} = e^x e^{iy}$

$\rightarrow$ $f(z) = e^x [\cos y + i \sin y]$

$\therefore f(z) = \underbrace{e^x \cos y}_{u} + i \underbrace{e^x \sin y}_{v}$

Since,

$u_x = e^x \cos y \quad , \quad v_y = e^x \sin y$

$\therefore u_x = v_y$

&

$u_y = -e^x \sin y \quad , \quad v_x = e^x \sin y$

$\therefore u_y = -v_x$

Thus $f'(z)$ exist everywhere

& $f'(z) = u_x + i v_x = e^x \cos y + i e^x \sin y$

$\therefore f'(z) = f(z) \quad \forall z.$

* Polar form of Cauchy-Riemann P.d. eq^n :-

If $f(z) = u + iv$ be analytic fu^n & $z = re^{i\theta}$ where, u, r, v, θ are all real no.s then show that

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \quad , \quad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$$

i.e. $r u_r = v_\theta \quad , \quad u_\theta = -r v_r$

Proof:- Let $x = r\cos\theta \quad , \quad y = r\sin\theta$

$\therefore \frac{\partial x}{\partial r} = \cos\theta \quad \& \quad \frac{\partial y}{\partial r} = \sin\theta$

$\frac{\partial x}{\partial \theta} = -r\sin\theta \quad \& \quad \frac{\partial y}{\partial \theta} = r\cos\theta$

We have, $r = \sqrt{x^2 + y^2} \quad , \quad \theta = \tan^{-1}\left(\frac{y}{x}\right)$

$$\therefore \frac{\partial r}{\partial x} = \frac{1}{2\sqrt{x^2+y^2}} \cdot 2x = \frac{x}{\sqrt{x^2+y^2}} = \frac{x}{r} = \frac{r\cos\theta}{r}$$

$$\therefore \frac{\partial r}{\partial x} = \cos\theta$$

$$\frac{\partial r}{\partial y} = \frac{1}{2\sqrt{x^2+y^2}}(2y) = \frac{y}{\sqrt{x^2+y^2}} = \frac{y}{r} = \frac{r\sin\theta}{r}$$

$$\therefore \frac{\partial r}{\partial y} = \sin\theta$$

Also, $\dfrac{\partial \theta}{\partial x} = \dfrac{1}{1+y^2/x^2}(-y/x^2) = \dfrac{x^2}{x^2+y^2}\left(\dfrac{-y}{x^2}\right)$

$$= \frac{-y}{x^2+y^2} = \frac{-y}{r^2} = \frac{-r\sin\theta}{r^2}$$

$$\therefore \frac{\partial \theta}{\partial x} = \frac{-\sin\theta}{r}$$

$$\& \frac{\partial \theta}{\partial y} = \frac{1}{1+\frac{y^2}{x^2}}\left(\frac{1}{x}\right) = \frac{x^2}{x^2+y^2} \times \frac{1}{x} = \frac{x}{x^2+y^2} = \frac{r\cos\theta}{r^2}$$

$$\therefore \frac{\partial \theta}{\partial y} = \frac{\cos\theta}{r}$$

We have, $\dfrac{\partial u}{\partial x} = \dfrac{\partial u}{\partial r} \cdot \dfrac{\partial r}{\partial x} + \dfrac{\partial u}{\partial \theta} \cdot \dfrac{\partial \theta}{\partial x}$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r} \cdot \cos\theta + \frac{\partial u}{\partial \theta} \cdot \left(\frac{-\sin\theta}{r}\right) \quad \ldots\ldots (1)$$

$$\frac{\partial v}{\partial y} = \frac{\partial v}{\partial r} \cdot \frac{\partial r}{\partial y} + \frac{\partial v}{\partial \theta} \cdot \frac{\partial \theta}{\partial y}$$

$$= \frac{\partial v}{\partial r} \cdot \sin\theta + \frac{\partial v}{\partial \theta} \cdot \left(\frac{\cos\theta}{r}\right) \quad \ldots\ldots (2)$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r} \cdot \frac{\partial r}{\partial y} + \frac{\partial u}{\partial \theta} \cdot \frac{\partial \theta}{\partial y}$$

$$= \frac{\partial u}{\partial r} \cdot \sin\theta + \frac{\partial u}{\partial \theta} \cdot \left(\frac{\cos\theta}{r}\right) \quad \ldots\ldots (3)$$

$$\frac{\partial v}{\partial x} = \frac{\partial v}{\partial r} \cdot \frac{\partial r}{\partial x} + \frac{\partial v}{\partial \theta} \cdot \frac{\partial \theta}{\partial x}$$

$$= \frac{\partial v}{\partial r} \cdot \cos\theta + \frac{\partial v}{\partial \theta}\left(-\frac{\sin\theta}{r}\right) \quad \ldots\ldots (4)$$

since $f(z)$ is analytic.
$\therefore$ u & v are satisfy C-R eq^{ns}.

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \& \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad \ldots\ldots (5)$$

$\therefore$ from eq^{ns} (1), (2) & (5), we get

$$\cos\theta \cdot \frac{\partial u}{\partial r} - \frac{\sin\theta}{r} \cdot \frac{\partial u}{\partial \theta} = \sin\theta \cdot \frac{\partial v}{\partial r} + \frac{\cos\theta}{r} \cdot \frac{\partial v}{\partial \theta} \quad \ldots\ldots (6)$$

$\therefore$ from eq^{ns} (3), (4) & (5), we get

$$\sin\theta \cdot \frac{\partial u}{\partial r} + \frac{\cos\theta}{r} \cdot \frac{\partial u}{\partial \theta} = -\cos\theta \cdot \frac{\partial v}{\partial r} + \frac{\sin\theta}{r} \cdot \frac{\partial v}{\partial \theta} \quad \ldots (7)$$

Multiplying eq^n (6) by $\cos\theta$ & eq^n (7) by $\sin\theta$ & adding them, we get,

$$\cos^2\theta \cdot \frac{\partial u}{\partial r} - \cos\theta \cdot \sin\theta \cdot \frac{1}{r} \cdot \frac{\partial u}{\partial \theta} + \sin^2\theta \cdot \frac{\partial u}{\partial r} + \sin\theta \cdot \cos\theta \cdot \frac{1}{r} \cdot \frac{\partial u}{\partial \theta} = \sin\theta \cdot \cos\theta \cdot \frac{\partial v}{\partial r} + \frac{1}{r}\cos^2\theta \cdot \frac{\partial v}{\partial \theta} - \sin\theta \cdot \cos\theta \cdot \frac{\partial v}{\partial r} + \frac{\sin^2\theta}{r} \cdot \frac{\partial v}{\partial \theta}$$

$$\therefore \frac{\partial u}{\partial r}(\cos^2\theta + \sin^2\theta) = \frac{1}{r}(\cos^2\theta + \sin^2\theta)\frac{\partial v}{\partial \theta}$$

$$\therefore \boxed{\frac{\partial u}{\partial r} = \frac{1}{r} \cdot \frac{\partial v}{\partial \theta}}$$

Multiplying eq^n (6) by $\sin\theta$ & eq^n (7) by $\cos\theta$ & subtracting them, we get,

$$\sin\theta\cdot\cos\theta\cdot\frac{\partial u}{\partial r}-\frac{\sin^2\theta}{r}\cdot\frac{\partial u}{\partial\theta}-\sin\theta\cdot\cos\theta\cdot\frac{\partial u}{\partial r}-\frac{\cos^2\theta}{r}\cdot\frac{\partial u}{\partial\theta}$$

$$=\sin^2\theta\cdot\frac{\partial v}{\partial r}+\frac{\sin\theta\cdot\cos\theta}{r}\cdot\frac{\partial v}{\partial\theta}+\cos^2\theta\cdot\frac{\partial v}{\partial r}-\frac{(\sin\theta\cdot\cos\theta)}{r}\cdot\frac{\partial v}{\partial\theta}$$

$$\therefore -\frac{1}{r}\frac{\partial u}{\partial\theta}(\sin^2\theta+\cos^2\theta)=(\sin^2\theta+\cos^2\theta)\frac{\partial v}{\partial r}$$

i.e. $-\frac{1}{r}\frac{\partial u}{\partial\theta}=\frac{\partial v}{\partial r}$

$$\Rightarrow \boxed{\frac{\partial v}{\partial r}=-\frac{1}{r}\cdot\frac{\partial u}{\partial\theta}} \quad \& \quad \boxed{f'(z)=e^{-i\theta}(u_r+iv_r)}$$

$$\therefore \boxed{r u_r = v_\theta, \quad u_\theta=-r\cdot v_r}$$

Ex:- ① Consider the fu^n $f(z)=\frac{1}{z}$.

Sol^n:- $f(z)=\frac{1}{z}=\frac{1}{re^{i\theta}}=\frac{1}{r}\cdot e^{-i\theta}$

$$=\frac{1}{r}(\cos\theta-i\sin\theta), \quad (z\neq 0)$$

since $u(r,\theta)=\frac{\cos\theta}{r}$ & $v(r,\theta)=\frac{-\sin\theta}{r}$

The C.R. eq^{ns} are satisfied.

$$r u_r=\frac{-\cos\theta}{r}=v_\theta$$

$$\& \quad u_\theta=-r\left(\frac{-\sin\theta}{r}\right)=-r\cdot v_r$$

$$\& \quad f'(z)=e^{-i\theta}\left[\frac{-\cos\theta}{r^2}+i\frac{\sin\theta}{r^2}\right]$$

$$=\frac{-e^{-i\theta}}{r^2}[\cos\theta-i\sin\theta]$$

$$\therefore f'(z)=\frac{-e^{-i\theta}}{r^2}\cdot e^{-i\theta}=-\frac{1}{(re^{i\theta})^2}=\frac{-1}{z^2}$$

Ex:- $f(z) = \sqrt[3]{r}\, e^{i\theta/3}$ $(r>0,\ \alpha<\theta<\alpha+2\pi)$ has derivative everywhere in its domain.

Sol^n:- $f(z) = \sqrt[3]{r}\left[\cos\frac{\theta}{3} + i\sin\frac{\theta}{3}\right]$

$\therefore u(r,\theta) = \sqrt[3]{r}\cos\theta/3$ & $v(r,\theta) = \sqrt[3]{r}\cdot\sin\theta/3$

$u_r = \frac{1}{3}r^{-2/3}\cdot\cos\theta/3$,	$u_\theta = \sqrt[3]{r}\cdot\sin\frac{\theta}{3}\left(\frac{-1}{3}\right)$
$\therefore ru_r = \frac{1}{3}r\cdot r^{-2/3}\cos\theta/3$,	$-rv_r = -r\cdot\frac{1}{3}\cdot r^{-2/3}\sin\frac{\theta}{3}$
$\therefore ru_r = \frac{1}{3}r^{1/3}\cdot\cos\theta/3$	$\therefore -rv_r = -\frac{1}{3}\sqrt[3]{r}\cdot\sin\frac{\theta}{3}$
$\therefore r\cdot u_r = \frac{1}{3}\sqrt[3]{3}\cdot\cos\frac{\theta}{3} = v_\theta$	$\therefore u_\theta = -rv_r$

$\therefore$ C.R. eq^{ns} are satisfied. $f'(z)$ exists.

$\therefore f'(z) = e^{-i\theta}(u_r + iv_r)$

$$\therefore f'(z) = e^{-i\theta}\left[\frac{1}{3r^{2/3}}\cos\frac{\theta}{3} + i\frac{1}{3r^{2/3}}\sin\frac{\theta}{3}\right]$$

$$= e^{-i\theta}\left[\frac{1}{3(\sqrt[3]{r})^2}\cos\frac{\theta}{3} + i\frac{1}{3(\sqrt[3]{r})^2}\cdot\sin\frac{\theta}{3}\right]$$

i.e. $f'(z) = \frac{e^{-i\theta}}{3(\sqrt[3]{r})^2}e^{i\theta/3} = \frac{e^{-i2/3\theta}}{3(\sqrt[3]{r})^2}$

$$\therefore f'(z) = \frac{1}{3(\sqrt[3]{r}\cdot e^{i\theta/3})^2} = \frac{1}{3[f(z)]^2}$$

* Exercise :→ Page 71

Ex:- ① Show that $f'(z)$ does not exist at any point if

a) $f(z) = \bar{z}$

sol^n:- $f(z) = \bar{z} = x - iy$

So, $u = x$, $v = -y$

$u_x = 1 \Rightarrow v_y = -1$ $(\because u_x \neq v_y)$

$\therefore$ C.R. eq^{ns} are not satisfied anywhere.

b) $f(z) = z - \bar{z}$

sol^n:- $f(z) = z - \bar{z} = (x+iy) - (x-iy) = 0 + i2y$

So that $u = 0$, $v = 2y$

since $u_x = 0$ & $v_y = 2$

* $\Rightarrow 0 \neq 2$

$\therefore$ C.R. eq^{ns} are not satisfied.

ⓒ $f(z) = 2x + ixy^2$

Sol^n:- $f(z) = 2x + ixy^2$

Here, $u = 2x$, $v = xy^2$

$u_x = 2$ & $v_y = 2xy$

$\Rightarrow 2 = 2xy$

$\Rightarrow xy = 1$

$u_y = 0$ & $-v_x = -y^2$

$\Rightarrow 0 = -y^2 \Rightarrow y^2 = 0 \Rightarrow y = 0.$

Putting $y = 0$ in $xy = 1$ $\therefore 0 \neq 1$

$\therefore$ C.R. eq^{ns} does not hold anywhere.

ⓓ $f(z) = e^x . e^{-iy}$

Sol^n:- $f(z) = e^x . e^{-iy} = e^x [\cos y - i \sin y]$

$f(z) = e^x . \cos y - ie^x \sin y$

So that $u = e^x \cos y$, $v = -e^x . \sin y$

$u_x = v_y \Rightarrow e^x . \cos y = -e^x . \cos y$

$\Rightarrow 2e^x . \cos y = 0$

$\Rightarrow \cos y = 0$

Thus, $y = \frac{\pi}{2} + n\pi$, $(n = 0, \pm 1, \pm 2, \ldots\ldots)$

$u_y = -v_x \Rightarrow -e^x . \sin y = e^x \sin y$

$\Rightarrow 2 . e^x . \sin y = 0$

$\Rightarrow \sin y = 0$

$\Rightarrow y = n\pi \quad , (n = 0, \pm 1, \pm 2, \pm \ldots\ldots)$

. Since there are two different sets of values of y.

The C.R. eq^{ns} are not satisfied anywhere.

Ex:- ② Using C.R. eq^{ns}. to show that $f'(z)$ & its derivative $f''(z)$ exist everywhere & find $f''(z)$ when

ⓐ $f(z) = iz + 2$

Sol^n:- $f(z) = iz + 2$

$\therefore f(z) = i(x + iy) + 2$

$f(z) = ix - y + 2$

$\therefore f(z) = (2 - y) + ix$

$\therefore u = 2 - y \quad \& \quad v = x$

$\therefore u_x = 0 \quad , \quad v_y = 0 \qquad \therefore u_x = v_y$

$\& \quad u_y = -1 \quad \& \quad v_x = 1 \qquad \therefore u_y = -v_x$

$\therefore$ C.R. eq^{ns} are satisfied.

$\therefore f'(z) = u_x + iv_x$

$\therefore f'(z) = 0 + i(1)$

$\therefore f'(z) = i$

$\& \quad f''(z) = 0$

ⓑ $f(z) = e^{-x} . e^{-iy}$

Sol^n:- $f(z) = e^{-x} . e^{-iy}$

$f(z) = e^{-x} [\cos y - i \sin y]$

$f(z) = e^{-x} \cos y - ie^{-x} . \sin y$

$u(x, y) = e^{-x} . \cos y \quad , \quad v(x, y) = -e^{-x} . \sin y$

$\therefore u_x = -e^{-x}.\cos y \quad , \quad v_y = -e^{-x}.\cos y$

$\therefore \quad u_x = v_y.$

& $u_y = -e^{-x}.\sin y \quad$ & $\quad v_x = e^{-x}.\sin y$

$\therefore u_y = -v_x \quad$ $\therefore$ C.R. eq^{ns} are satisfied.

$\therefore \quad f'(z) = u_x + i v_x$

$$f'(z) = -e^{-x}.\cos y + i\, e^{-x}.\sin y$$

$\therefore \quad f'(z) = -e^{-x}[\cos y - i\sin y]$

$\therefore \boxed{f''(z) = e^{-x}[\cos y - i\sin y] = f(z)}$

© $f(z) = z^3$

Sol^n:- $f(z) = z^3 = (x+iy)^3$

$$= x^3 + 3x^2(iy) + 3x\, i^2 y^2 + (iy)^3$$

$$= x^3 + 3x^2 y\, i - 3xy^2 - iy^3$$

$\therefore \quad f(z) = (x^3 - 3xy^2) + i(3x^2y - y^3)$

$\therefore \; u(x,y) = x^3 - 3xy^2 \quad , \quad v(x,y) = 3x^2y - y^3$

$\therefore \quad u_x = 3x^2 - 3y^2 \quad , \quad v_y = 3x^2 - 3y^2$

$\therefore \quad u_x = v_y \quad$ &

$\therefore \quad u_y = -6xy \quad , \quad v_x = 6xy$

$\therefore \quad u_x = -v_x$

$\therefore$ C.R. eq^{ns} are satisfied.

$\therefore f'(z) = u_x + i v_x$

$f'(z) = 3x^2 - 3y^2 + i6xy$

$f'(z) = 3(x^2 - y^2 + i2xy)$

$f'(z) = 3(x+iy)^2$

$f'(z) = 3z^2$ & $f''(z) = 6z$.

H.W (d) $f(z) = \cosh y \cdot \cos x - i \sin x \cdot \sinh y$

Sol^n:- $f(z) = (\cosh y \cdot \cos x) - i(\sin x \cdot \sinh y)$

$\therefore u = \cosh y \cdot \cos x$ & $v = -\sin x \cdot \sinh y$

$u_x = -\sin x \cdot \cosh y$ & $v_y = -\sin x \cdot \cosh y$

$\therefore \quad u_x = v_y$

And

$u_y = +\cos x \cdot \sinh y$ & $v_x = -\cos x \cdot \sinh y$

$\therefore u_y = -v_x$. Hence C-R eq^{ns} are satisfied.

$\therefore f'(z) = u_x + iv_x$

$= -\sin x \cdot \cosh y + i(-\cos x \cdot \sinh y)$

$\therefore f'(z) = -(\sin x \cdot \cosh y + i \cos x \cdot \sinh y) = -\sin(x+iy)$

$\therefore f'(z) = \underline{-\sin z}$ & $f''(z) = \underline{-\cos z}$.

Ex:- (5) Using C.R. eq^{ns} determine where $f'(z)$ exist & find its value when

a) $f(z) = \dfrac{1}{z}$

Sol^n:- $f(z) = \dfrac{1}{z} = \dfrac{1}{z} \cdot \dfrac{\bar{z}}{\bar{z}} = \dfrac{\bar{z}}{|z|^2} = \dfrac{x}{x^2+y^2} + i\dfrac{(-y)}{x^2+y^2}$

So that

$$u = \frac{x}{x^2+y^2}, \quad v = \frac{-y}{x^2+y^2}$$

Since

$$u_x = \frac{y^2 - x^2}{(x^2+y^2)^2} = v_y \quad \& \quad u_y = \frac{-2xy}{(x^2+y^2)^2} = -v_x$$

$f'(z)$ exist when $z \neq 0$.

$$\therefore f'(z) = u_x + iv_x = \frac{y^2 - x^2}{(x^2+y^2)^2} + i\frac{2xy}{(x^2+y^2)^2}$$

$$= -\left[\frac{x^2 - y^2 - 2ixy}{(x^2+y^2)^2}\right]$$

$$= \frac{-(x-iy)^2}{(x^2+y^2)^2} = \frac{-(\bar{z})^2}{(z\bar{z})^2}$$

$$= \frac{-(\bar{z})^2}{z^2(\bar{z})^2} = \frac{-1}{z^2}$$

(b) $f(z) = x^2 + iy^2$

Sol^n:- Here, $u = x^2$, $v = y^2$

$u_x = 2x$, $v_y = 2y$

$\Rightarrow 2x = 2y \Rightarrow x = y$.

& $u_y = -v_x \Rightarrow 0 = 0$

So $f'(z)$ exists only when $y = x$ & we find that

$f'(x+ix) = u_x(x,x) + iv_x(x,x)$

$= 2x + i \cdot 0 = 2x$

(c) $f(z) = z \, \text{Im}(z)$

Sol^n:- $f(z) = (x+iy) \cdot y = xy + iy^2$.

Here $u = xy$, $v = y^2$

$u_x = v_y \Rightarrow y = 2y \Rightarrow y = 0.$

& $u_y = -v_x \Rightarrow x = 0 \Rightarrow x = 0.$

Hence $f'(z)$ exists only when $z = 0$

i.e. $f'(z) = u_x(0,0) + i\, v_x(0,0)$

$= 0 + i.0$

$= 0$

④ Using polar form of C.R. eq^{ns} show that each of there fu^{ns} is differentiable in the indicated domain of def^{n} & also find $f'(z)$.

ⓐ $f(z) = \frac{1}{z^4}$ $(z \neq 0)$

Sol^n:- $f(z) = \frac{1}{z^4} = \frac{1}{(re^{i\theta})^4}$

$= \frac{1}{r^4}(e^{-i\theta})^4 = \frac{1}{r^4}.[e^{-i4\theta}]$

$= \frac{1}{r^4}[\cos 4\theta - i \sin 4\theta]$

$$= \underbrace{\left(\frac{1}{r^4}\cos 4\theta\right)}_{u} + i\underbrace{\left(-\frac{1}{r^4}.\sin 4\theta\right)}_{v}$$

$\therefore\ r u_r = \frac{-r4}{r^5}\cdot\cos 4\theta = -\frac{4}{r^4}\cdot\cos 4\theta = v_\theta$

& $u_\theta = -\frac{4}{r^4}\sin 4\theta = -r v_r$

f is analytic in its domain.

$\therefore\ f'(z) = e^{-i\theta}\,(u_r + i v_r)$

$= e^{-i\theta}\left(-\frac{4}{r^5}\cdot\cos 4\theta + i\,\frac{4}{r^5}\sin 4\theta\right)$

$= -\frac{4}{r^5}\cdot e^{-i\theta}\,[\cos 4\theta - i\sin 4\theta]$

$= \frac{-4}{r^5}\cdot e^{-i\theta}\cdot e^{-i4\theta}$

$= -\frac{4}{r^5}\cdot e^{-i5\theta}$

$= \frac{-4}{(re^{i\theta})^5} = -\frac{4}{z^5}$

(b) $f(z) = \sqrt{r}\cdot e^{i\theta/2}\quad (r>0,\ \alpha<\theta<\alpha+2\pi)$

Sol^n:- $f(z) = \sqrt{r}\cdot e^{i\theta/2}$

$\therefore\ f(z) = \underbrace{\sqrt{r}\cdot\cos\theta/2}_{u} + \underbrace{i\sqrt{r}\cdot\sin\theta/2}_{v}$

$\therefore\ u_r = \frac{1}{2\sqrt{r}}\cdot\cos\frac{\theta}{2}$

$\therefore\ r\cdot u_r = r\cdot\frac{1}{2\sqrt{r}}\cdot\cos\frac{\theta}{2} = \frac{\sqrt{r}}{2}\cdot\cos\frac{\theta}{2} = v_\theta$

& $u_\theta = \frac{-\sqrt{r}}{2}\cdot\sin\frac{\theta}{2} = -r\cdot v_r = -r\cdot\frac{1}{2\sqrt{r}}\cdot\sin\frac{\theta}{2}$

$$\therefore u_\theta = -r v_r = -\frac{\sqrt{r}}{2} \cdot \sin\frac{\theta}{2}$$

$\therefore$ f is analytic in its domain.

$$\therefore f'(z) = e^{-i\theta}(u_r + i v_r)$$

$$= e^{-i\theta}\left(\frac{1}{2\sqrt{r}} \cdot \cos\frac{\theta}{2} + i\frac{1}{2\sqrt{r}} \cdot \sin\frac{\theta}{2}\right)$$

$$= \frac{1}{2\sqrt{r}} \cdot e^{-i\theta}\left[\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right]$$

$$= \frac{1}{2\sqrt{r}} e^{-i\theta} \cdot e^{i\theta/2}$$

$$= \frac{1}{2\sqrt{r}} \cdot e^{-i\theta/2}$$

$$= \frac{1}{2\sqrt{r} \cdot e^{i\theta/2}}$$

$$\therefore f'(z) = \frac{1}{2 \cdot f(z)}$$

(c) $f(z) = e^{-\theta}\cos(\ln r) + i e^{-i\theta} \cdot \sin(\ln r)$

$(0 < \theta < 2\pi,\ r > 0)$

Sol^n:- $f(z) = \underbrace{e^{-\theta} \cdot \cos(\ln r)}_{u} + i \underbrace{e^{-\theta} \sin(\ln r)}_{v}$

$$\therefore u_r = e^{-\theta} \cdot [-\sin(\ln r)] \cdot \frac{1}{r}$$

$$\therefore r u_r = -e^{-\theta} \cdot \sin(\ln r) = v_\theta$$

& $u_\theta = -e^{-\theta}\cos(\ln r) = -r\, V_r$

$= -\cancel{r}\cdot e^{-\theta}\cdot\sin(\ln r)\cdot\frac{1}{\cancel{r}}$

$= -e^{-\theta}\cdot\sin(\ln r)$

$\therefore u_\theta = -r\cdot V_r$.

f is analytic in its domain.

$\therefore f'(z) = e^{-i\theta}\,[u_r + iv_r]$

$= e^{-i\theta}\left[\frac{-e^{-\theta}\cdot\sin(\ln r)}{r} + i\,\frac{e^{-\theta}\cdot\cos(\ln r)}{r}\right]$

$= \frac{1}{re^{i\theta}}\left[i^2\cdot e^{-\theta}\cdot\sin(\ln r) + i\cdot e^{-\theta}\cdot\cos(\ln r)\right]$

$= \frac{i}{re^{i\theta}}\left[i\cdot e^{-\theta}\cdot\sin(\ln r) + e^{-\theta}\cdot\cos(\ln r)\right]$

$\therefore f'(z) = i\,\frac{f(z)}{z}$

5. Show that when $f(z) = x^3 + i(1-y)^3$ it is legitimate to write $f'(z) = u_x + iv_x = 3x^2$ only when $z = i$.

Sol^n:- When $f(z) = x^3 + i(1-y)^3$

We have $u = x^3$, $v = (1-y)^3$

$\therefore u_x = 3x^2$ & $v_y = 3(1-y)^2(-1) = -3(1-y)^2$

$\therefore u_x = v_x \Rightarrow \cancel{3}x^2 = -\cancel{3}(1-y)^2$

$\Rightarrow x^2 + (1-y)^2 = 0$

& $u_y = 0$, $V_x = 0$

$\therefore u_x = -V_x \Rightarrow 0 = 0$

$\therefore$ The C.R. eq^{ns} are satisfied when $x=0$ & $y=1$.
i.e. $z = 0+i$ i.e. $\Rightarrow z = i$

$f'(z) = u_x + i v_x$

$= 3x^2 + i.0$

$\therefore f'(z) = 3x^2$
is valid when $z = i$ in which case $\boxed{f'(i) = 0}$

- ## Entire function :-

An entire fu^n is a fu^n that is analytic at each non-zero pt. in the finite plane. Since derivative of a polynomial exists. everywhere it follows that every polynomial is an entire function.

Note:- ① If two fu^{ns} are analytic in a domain D their sum & their product are both analytic in D. similarly their quotient is analytic in D provided the fu^n in denominator does not vanish at any pt. in D.

② Composition of two analytic fu^{ns} is analytic.

Th^m:- If $f'(z) = 0$ everywhere in domain D then $f(z)$ must be constant throughout D.

Proof:- $f(z) = u(x,y) + iv(x,y)$

Assuming that $f'(z) = 0$ in D
we denote that $u_x + iv_x = 0$
& in view of C-R. eq^{ns}

$$v_y - iu_y = 0$$

$$\Rightarrow u_x = u_y = 0$$

& $v_x = v_y = 0$ at each pt. in D.

Ex:- ① $f(z) = \dfrac{z^3+4}{(z^2-3)(z^2+1)}$ is analytic throughout the z-plane except for the singular pts.
$z = \pm\sqrt{3}$ & $z = \pm i \Rightarrow z^2+1 \Rightarrow z^2 = -1 \Rightarrow z = \sqrt{-1}$
$z^2 - 3 = 0 \Rightarrow z^2 = 3 \Rightarrow z = \pm\sqrt{3}$ $\Rightarrow z = \pm i$

② If $f(z) = \cosh x \cdot \cos y + i \sinh x \cdot \sin y$
→ Here $u(x,y) = \cosh x \cdot \cos y$, $v(x,y) = \sinh x \cdot \sin y$

$\therefore u_x = \sinh x \cdot \cos y = v_y$

& $u_y = -\cosh x \cdot \sin y = -v_x$ everywhere

$\therefore$ f is entire function.

Exercise :- Page 77

Ex:-① Verify that each of there fu^n is entire.

a) $f(z) = 3x+y+i(3y-x)$

Sol^n:- $f(z) = \underbrace{(3x+y)}_{u} + i\underbrace{(3y-x)}_{v}$

$u_x = 3$, $v_y = 3$ $\therefore u_x = v_y$

&

$u_y = 1$, $v_x = -1$ $\therefore u_y = -v_x$

$\therefore f(z)$ is entire.

b) $f(z) = \sin x.\cosh y + i\cos x.\sinh y$

Sol^n:- $u = \sin x.\cosh y$, $v = \cos x.\sinh y$

$u_x = \cos x.\cosh y = v_y$; $u_y = \sin x.\sinh y = -v_x$

$\therefore f(z)$ is entire.

c) $f(x) = e^{-y}.\sin x - ie^{-y}.\cos x$

Sol^n:- $u = e^{-y}.\sin x$, $v = -e^{-y}.\cos x$

Since,

$u_x = e^{-y}.\cos x = v_y$ &

$u_y = -e^{-y}.\sin x = -v_x$

$\therefore f(x)$ is entire.

d) $f(z) = (z^2-2)\,e^{-x}.e^{-iy}$ is entire it is

Sol^n:→ product of the entire fu^{ns}.

$g(z) = z^2 - 2$ & $h(z) = e^{-x}.e^{-iy}$

$\therefore h(z) = e^{-x}.e^{-iy}$

$= e^{-x}[\cos y - i\sin y]$

$\therefore h(z) = \underbrace{e^{-x} \cdot \cos y}_{u} - i\underbrace{e^{-x} \cdot \sin y}_{v}$

$\therefore u_x = -e^{-x} \cdot \cos y = v_y$ &

$\therefore u_y = -e^{-x} \cdot \sin y = -v_x$

Since g is entire since it is polynomial & h is also entire.

$\therefore f(z)$ is entire.

Ex:- ② Show that each of these fu^{ns}. is nowhere analytic.

a) $f(z) = xy + iy$ c) $f(z) = e^y \cdot e^{ix}$

b) $f(z) = 2xy + i(x^2 - y^2)$

Sol^n:-

a) $f(z) = \underbrace{xy}_{u} + i\underbrace{y}_{v}$

$\therefore u_x = y$ & $v_y = 1$	$u_y = x$ & $-v_x = 0$
$\Rightarrow y = 1$	$\Rightarrow x = 0$

$\therefore f(z)$ is nowhere analytic.
which means that C.R. eq^{ns} hold only at the pt. $z = (0, 1) = \underline{i}$

b) $f(z) = \underbrace{2xy}_{u} + i\underbrace{(x^2 - y^2)}_{v}$

$\therefore u_x = 2y$ & $v_y = -2y$	$u_y = 2x$, $-v_x = -2x$
$\Rightarrow y = 0$	$\Rightarrow x = 0$

$\therefore f(z)$ is nowhere analytic. which means that C.R. eq^{ns} hold only at the pt. $z = \underline{(0, 0)}$ i.e. origin.

ⓒ $f(z) = e^y \cdot e^{ix} = e^y \cdot [\cos x + i \sin x]$

$\therefore f(z) = \underbrace{e^y \cos x}_{u} + i \underbrace{e^y \cdot \sin x}_{v}$

$\therefore u_x = -e^y \cdot \sin x$ and $v_y = e^y \cdot \sin x$ $\Rightarrow$ if $u_x = v_y$

$\Rightarrow -e^y \cdot \sin x = e^y \sin x$

$\Rightarrow 2e^y \sin x = 0$

$\Rightarrow \sin x = 0$

And

$u_y = -v_x \Rightarrow e^y \cos x = -e^y \cdot \cos x$

$\Rightarrow 2e^y \cos x = 0$

$\Rightarrow \cos x = 0$

$\therefore$ roots of $\sin x = 0$ are 0 & $n\pi$ $(n = 0, \pm 1, \pm 2, \ldots$

& $\cos n\pi = (-1)^n \neq 0$

$\therefore$ C-R. eq^{ns} are not satisfied.

$\therefore$ It is nowhere analytic.

④ Determine the singular points of the fu^n:-

a) $f(z) = \dfrac{2z+1}{z(z^2+1)}$ $\therefore z = 0, \pm i$

b) $f(z) = \dfrac{z^3 + i}{z^2 - 3z + 2}$ $\therefore z = 1, 2$

c) $f(z) = \dfrac{z^2+1}{(z+2)(z^2+2z+2)}$ $\therefore z = -2, -1 \pm i$

Ex:- Show that an analytic fu^n with constant modulus in a domain is constant.

Sol^n:- Let $f(z) = u + iv$

So that $|f(z)| = |u + iv|$

$$\therefore \sqrt{u^2 + v^2} = c$$

$$\therefore u^2 + v^2 = c$$

$$\therefore 2u \cdot \frac{\partial u}{\partial x} + 2v \cdot \frac{\partial v}{\partial x} = 0$$

i.e. $u \cdot \frac{\partial u}{\partial x} + v \cdot \frac{\partial v}{\partial x} = 0$

i.e. $u \cdot u_x + v \cdot v_x = 0 \quad \cdots\cdots (1)$

& $2u \cdot \frac{\partial u}{\partial y} + 2v \cdot \frac{\partial v}{\partial y} = 0$

i.e. $u \cdot \frac{\partial u}{\partial y} + v \cdot \frac{\partial v}{\partial y} = 0$

i.e. $u \cdot u_y + v \cdot v_y = 0 \quad \cdots\cdots (2)$

&

By C-R eq^{ns} these eq^{ns} becomes;

$$u \cdot u_x - v \cdot u_y = 0$$

$$u \cdot u_y + v \cdot u_x = 0$$

Eliminating u_y from these eq^{ns}, we get,

$$+ \quad u^2 \cdot u_x - \cancel{uv \cdot u_y} = 0$$

$$\cancel{uv \cdot u_y} + v \cdot v \cdot u_x = 0$$

$$u^2 \cdot u_x + v^2 \cdot u_x = 0$$

$$\therefore u_x (u^2 + v^2) = 0$$

Thus at every pt. z, $w = u + iv \neq 0$

$\therefore u_x = 0$

Similarly, $u_y = 0$ & $v_x = 0$ & $v_y = 0$

Thus at every point of the domain the four partial derivatives of u, v are also zero & hence

$f(z) = u + iv$ is constant.

★ Harmonic Functions :-

A real valued fu^n H of two real variables x & y is said to be harmonic in a given domain of xy-plane if throughout that domain it has continuous partial derivative of 1^{st} & 2^{nd} order & satisfies the p.d.e.

$H_{xx} + H_{yy} = 0$ known as Laplace eq^n

OR

Def^n :- Any fu^n $f(x, y)$ possessing continuous Partial derivatives of the 1^{st} & 2^{nd} order & satisfying Laplace eq^n $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$ is called Harmonic fu^n

Let $f(z) = u + iv$ be an analytic fu^n then u & v are both Harmonic fu^{ns}.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \& \quad \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0.$$

Note :- $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ is known as Laplace eq^n or Potential eq^n.

Th^m:- If a fu^n $f(z) = u+iv$ is analytic in a domain D then its component fu^{ns} u & v are harmonic in D.

Proof:-

Let $f(z) = u+iv$ be an analytic fu^n then we have,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \& \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad \ldots\ldots (1)$$

Also, u & v are real & imaginary parts of analytic fu^{ns}.

$\therefore$ Derivatives of u & v of all orders exist & are continuous fu^n of x & y so that we have,

$$\frac{\partial^2 v}{\partial x \cdot \partial y} = \frac{\partial^2 v}{\partial y \cdot \partial x}$$

$\therefore$ from (1)

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \cdot \partial y} \quad \& \quad \frac{\partial^2 u}{\partial y^2} = \frac{-\partial^2 v}{\partial y \cdot \partial x}$$

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \left[\because \frac{\partial^2 v}{\partial x \cdot \partial y} = \frac{\partial^2 v}{\partial y \cdot \partial x}\right]$$

Also from (1), we have,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

diff. partially w.r.t. 'y' we get,

$$\frac{\partial^2 u}{\partial y \cdot \partial x} = \frac{\partial^2 v}{\partial y^2}$$

&

$$\frac{\partial u}{\partial y} = \frac{-\partial v}{\partial x}$$

diff. partially w.r.t. 'x' we get,

$$\frac{\partial^2 u}{\partial x \cdot \partial y} = -\frac{\partial^2 v}{\partial x^2}$$

$$\rightarrow \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

$\therefore$ u & v are harmonic functions.

Such a fu^ns u & v are called conjugate Harmonic fu^n or simple conjugate fu^n.

★ Th^m :- A fu^n $f(z) = u + iv$ is analytic in a domain **D** **iff** **v** is harmonic conjugate of u.

• Exercise - Page 82

Ex:- If $u = x^2 - 3xy^2$. show that there exist a fu^n $v(x, y)$ s.t. $w = u + iv$ is analytic in a finite region.

Sol^n:- Let $u = x^3 - 3xy^2$

$\therefore u_x = 3x^2 - 3y^2$

$\therefore u_{xx} = 6x$

&

$\therefore u_y = -6xy$

$\therefore u_{yy} = -6x$

$\therefore u_{xx} + u_{yy} = 0$.

Hence, u satisfies Laplace eq^n.

Hence u is Harmonic fun.

Now,

$u_x = v_y \Rightarrow 3x^2 - 3y^2$

$\Rightarrow v_y = \frac{\partial v}{\partial y} = 3x^2 - 3y^2$

Hold x fixed & integrate w.r.t. 'y'.

$\int \frac{\partial v}{\partial y} . dy = \int (3x^2 - 3y^2) dy + \phi(x)$.

$\Rightarrow v = 3x^2 y - \frac{3y^3}{3} + \phi(x)$

$\therefore v = 3x^2 y - y^3 + \phi(x)$

Now, $v_x = 6xy + \phi'(x) \Rightarrow -v_x = -6xy - \phi'(x)$

Then $u_y = -v_x$

$\therefore -6xy = -6xy - \phi'(x)$

$\therefore -\phi'(x) = 0 \Rightarrow \phi'(x) = 0$

$\therefore \phi(x) = c$

$\therefore v = 3x^2 y - y^3 + c$.

$\therefore w = f(z) = u + iv$

$= x^3 - 3xy^2 + i(3x^2 y - y^3 + c)$

$= x^3 - 3xy^2 + i3x^2 y - iy^3 + ic$.

$f(z) = (x+iy)^3 + ic$

$f(z) = z^3 + ic$.

Ex:- (2) Prove that $u = y^3 - 3x^2y$ is harmonic fu^n. Determine its harmonic conjugate & find the corresponding analytic function in-terms of z.

Sol^n:- Let $u = y^3 - 3x^2y$

$\therefore u_x = -6xy$

$\therefore u_{xx} = -6y$

&

$u_y = 3y^2 - 3x^2$

& $\therefore u_{yy} = 6y$

$\therefore u_{xx} + u_{yy} = 0.$

Hence u is Harmonic fu^n.

Now,

$u_x = v_y \Rightarrow -6xy$

$\Rightarrow \frac{\partial v}{\partial y} = -6xy$

Hold x fixed & integrate w.r.t. 'y'

$\Rightarrow \int \frac{\partial v}{\partial y} . dy = \int -6xy . dy + \phi(x)$

$\Rightarrow v = -6x . \frac{y^2}{2} + \phi(x)$

$\Rightarrow v = -3x . y^2 + \phi(x)$

Now $v_x = -3y^2 + \phi'(x)$

$\therefore -v_x = 3y^2 - \phi'(x)$

Then $u_y = -v_x$

$\Rightarrow 3y^2 - 3x^2 = 3y^2 - \phi'(x)$

$\Rightarrow \phi'(x) = 3x^2$

$\Rightarrow \phi(x) = \int 3x^2\,dx + c$

$\Rightarrow \phi(x) = \frac{3x^3}{3} + c$

$\Rightarrow \phi(x) = x^3 + c$

$\therefore V = -3xy^2 + x^3 + c$

$\therefore V = x^3 - 3xy^2 + c$

Now,

$w = f(z) = u + iv$

$= y^3 - 3x^2y + i(x^3 - 3xy^2 + c)$

$= y^3 - 3x^2y + ix^3 - i3xy^2 + ic$

$= ix^3 - 3ixy^2 - 3x^2y + y^3 + ic$

$= (y - ix)^3 + ic$

$= (-i)^3 \left(\frac{y}{-i} + x\right)^3 + ic$

$= -(-i)(iy + x)^3 + ic$

$= i(x + iy)^3 + ic$

$\therefore f(z) = iz^3 + ic$

Ex:- (3) Show that $u(x,y)$ is harmonic in some domain & find a harmonic conjugate $v(x,y)$ when

(a) $u(x,y) = 2x(1-y)$

Sol^n:- Let $u = 2x - 2xy$

$$\frac{\partial u}{\partial x} = 2 - 2y$$

$$\& \quad \frac{\partial^2 u}{\partial x^2} = 0$$

$$\& \quad \frac{\partial u}{\partial y} = -2x$$

$$\& \quad \frac{\partial^2 u}{\partial y^2} = 0$$

$$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Hence u satisfy Laplace eq^n.

To find a harmonic conjugate $v(x,y)$ we start with $u_x = 2 - 2y$

Now,

$$u_x = v_y \Rightarrow v_y = 2 - 2y \Rightarrow \frac{\partial v}{\partial y} = 2 - 2y.$$

Hold x fixed & integrate w.r.t. 'y', we get

$$\therefore v(x,y) = \int (2-2y)\,dy + \phi(x)$$

$$\therefore v(x,y) = 2y - \frac{\cancel{2}y^2}{\cancel{2}} + \phi(x)$$

$$\therefore v(x,y) = 2y - y^2 + \phi(x) \quad \ldots\ldots (1)$$

Then $u_y = -v_x \Rightarrow -2x = -\phi'(x)$

$\Rightarrow \phi'(x) = 2x$

$\Rightarrow \phi(x) = x^2 + c.$

Consequently,

$V(x,y) = 2y - y^2 + x^2 + c$ ....[From ①]

$\therefore \boxed{V(x,y) = x^2 - y^2 + 2y + c}$

Now,

$W = f(z) = u + iv$

$= 2x(1-y) + i(x^2 - y^2 + 2y + c)$

$= 2x - 2yx + ix^2 - iy^2 + i.2y + ic$

$= ix^2 - iy^2 - 2yx + 2x + 2iy + ic$

$= i(x^2 - y^2 + i2xy) + 2(x+iy) + ic$

$= i(x+iy)^2 + 2(x+iy) + ic$

$\therefore W = f(z) = i.z^2 + 2.z + ic$

$\therefore \quad f(z) = iz^2 + 2z + ic$

(b) $u(x,y) = 2x - x^3 + 3xy^2$

$\Rightarrow$ Let $u = 2x - x^3 + 3xy^2$

$u_x = \dfrac{\partial u}{\partial x} = 2 - 3x^2 + 3y^2$

$u_{xx} = \dfrac{\partial^2 u}{\partial x^2} = -6x$

$$u_y = \frac{\partial u}{\partial y} - 6xy$$

&

$$u_{yy} = \frac{\partial^2 u}{\partial y^2} = 6x$$

$$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Hence u satisfy Laplace eq^n.

To find a harmonic conjugate $v(x,y)$

we start with $u_x = 2 - 3x^2 + 3y^2$

Now,

$$u_x = v_y \Rightarrow 2 - 3x^2 + 3y^2$$

Integrate w.r.t. 'y' & hold x fixed.

$$\Rightarrow v(x,y) \Rightarrow 2y - 3x^2 y + \frac{\cancel{3}y^3}{\cancel{3}} + \phi(x)$$

$$\Rightarrow v(x,y) \Rightarrow 2y - 3x^2 y + y^3 + \phi(x)$$

Then $u_y = -v_x$

$$\therefore +6xy = -(-6xy + \phi'(x))$$

$$\therefore 6xy = 6xy + \phi'(x)$$

$$\therefore \phi'(x) = 0$$

$$\therefore \phi(x) = c$$

Consequently,

$$\boxed{v(x,y) = 2y - 3x^2 y + y^3 + c}$$

Now,

$$w = f(z) = u + iv$$

$$= 2x - x^3 + 3xy^2 + i(2y - 3x^2y + y^3 + c)$$

$$= 2x - x^3 + 3xy^2 + i2y - 3ix^2y + i\cdot y^3 + ic$$

$$= -(x^3 + i\,3x^2y - 3xy^2 - iy^3) + 2(x+iy) + ic$$

$$= -(x+iy)^3 + 2(x+iy) + ic$$

$$\therefore f(z) = -z^3 + 2z + ic$$

(c) $u(x,y) = \sinh x \cdot \sin y$

$\rightarrow$ $u(x,y) = \sinh x \cdot \sin y$

$$\therefore u_x = \frac{\partial u}{\partial x} = \cosh x \cdot \sin y$$

$$u_{xx} = \frac{\partial^2 u}{\partial x^2} = \sinh x \cdot \sin y \quad \ldots\ldots (1)$$

$$u_y = \frac{\partial u}{\partial y} = \sinh x \cdot \cos y$$

$$u_{yy} = \frac{\partial^2 u}{\partial y^2} = -\sinh x \cdot \sin y \quad \ldots\ldots (2)$$

$\therefore$ from (1) & (2), we get,

$$u_{xx} + u_{yy} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Hence u satisfies Laplace eq^n.

To find harmonic conjugate $v(x,y)$.

We start with $u_x = \cosh x \cdot \sin y = v_y$

Hold x fixed & integrating each side here w.r.t. 'y'

$\Rightarrow V = -\cosh x \cdot \cos y + \phi(x)$

$\therefore V_x = -\cos y \cdot \sinh x + \phi'(x)$

Then

$u_y = -V_x \Rightarrow \sinh x \cdot \cos y = -[-\sinh x \cdot \cos y + \phi'(x)]$

$\Rightarrow \sinh x \cdot \cos y = \sinh x \cdot \cos y - \phi'(x)$

$\Rightarrow \phi'(x) = 0$

$\Rightarrow \phi(x) = c$

Consequently,

$$\boxed{V = -\cosh x \cdot \cos y + c}$$

Now,

$w = f(z) = u + iv$

$= \sinh x \cdot \sin y + i(-\cosh x \cdot \cos y + c)$

$= \sinh x \cdot \sin y - i \cdot \cosh x \cdot \cos y + ic$

$= (\sinh x \cdot \sin y) - i(\cosh x \cdot \cos y) + ic$

(d) $u(x,y) = \dfrac{y}{x^2+y^2}$

$\Rightarrow$ $u(x,y) = \dfrac{y}{x^2+y^2}$

To show that $u_{xx} + u_{yy} = 0$

$\therefore u_x = \dfrac{(x^2+y^2)\cdot 0 - y(2x+0)}{(x^2+y^2)^2} = \dfrac{-2xy}{(x^2+y^2)^2}$

$$u_{xx} = \frac{(x^2+y^2)^2(-2y) - (-2xy)[2(x^2+y^2)\cdot(2x)]}{(x^2+y^2)^4}$$

$$= \frac{-2y\,\cancel{(x^2+y^2)}\,[(x^2+y^2) - 4x^2]}{(x^2+y^2)^4}$$

$$\therefore u_{xx} = \frac{+2y(-x^2-y^2+4x^2)}{(x^2+y^2)^3} = \frac{2y(3x^2-y^2)}{(x^2+y^2)^3}$$

$$\therefore u_{xx} = \frac{6x^2y - 2y^3}{(x^2+y^2)^3} \quad \ldots\ldots (1)$$

Now,

$$u_y = \frac{(x^2+y^2)\cdot(1) - y(2y)}{(x^2+y^2)^2} = \frac{x^2+y^2-2y^2}{(x^2+y^2)^2}$$

$$\therefore u_y = \frac{x^2-y^2}{(x^2+y^2)^2}$$

$$\therefore u_{yy} = \frac{[(x^2+y^2)^2(-2y)] - \{(x^2-y^2)[2(x^2+y^2)(2y)]\}}{(x^2+y^2)^4}$$

$$= \frac{(-2y)\,\cancel{(x^2+y^2)}\,[x^2+y^2+2x^2-2y^2]}{(x^2+y^2)^4}$$

$$= \frac{(-2y)[3x^2-y^2]}{(x^2+y^2)^3} = \frac{-(6x^2y-2y^3)}{(x^2+y^2)^3} \quad \ldots (2)$$

From (1) & (2), $u_{xx} + u_{yy} = 0$

Hence u satisfies Laplace eq^n.

To find harmonic conjugate $v(x,y)$.

Now, start with $u_x = v_y$.

$\therefore \quad u_x = v_y \Rightarrow v_y = \dfrac{-2xy}{(x^2+y^2)^2}$

$$\Rightarrow \frac{\partial v}{\partial y} = \frac{-2xy}{(x^2+y^2)^2}.$$

Hold x fixed & integrate w.r.t. 'y'

$$\Rightarrow \int \frac{\partial v}{\partial y} \cdot dy = \int \frac{-2xy}{(x^2+y^2)^2} \cdot dy + \phi(x)$$

Put $x^2+y^2 = t$

$\therefore \quad 0 + 2y.dy = dt$

$$\Rightarrow v = -x \int \frac{dt}{t^2} + \phi(x)$$

$$\Rightarrow v = -x \int t^{-2} . dt + \phi(x)$$

$$\Rightarrow v = -x \left[\frac{t^{-1}}{-1}\right] + \phi(x)$$

$$\Rightarrow v = x . t^{-1} + \phi(x)$$

$$\Rightarrow v = \frac{x}{t} + \phi(x)$$

$$\Rightarrow v = \frac{x}{x^2+y^2} + \phi(x)$$

$$\therefore v = \frac{x}{x^2+y^2} + \phi(x)$$

Now,

$$v_x = \frac{(x^2+y^2)\cdot 1 - x(2x)}{(x^2+y^2)^2} + \phi'(x)$$

$$= \frac{x^2+y^2-2x^2}{(x^2+y^2)^2} + \phi'(x)$$

$$\therefore v_x = \frac{y^2-x^2}{(x^2+y^2)^2} + \phi'(x)$$

$$\therefore -v_x = \frac{x^2-y^2}{(x^2+y^2)^2} - \phi'(x)$$

Then

$$u_y = -v_x$$

Now,

$$u_y = \frac{(x^2+y^2)\cdot 1 - y(0+2y)}{(x^2+y^2)^2}$$

$$\therefore \quad u_y = \frac{x^2+y^2-2y^2}{(x^2+y^2)^2}$$

$$\therefore \quad u_y = \frac{x^2-y^2}{(x^2+y^2)^2}$$

$$\Rightarrow \quad u_y = -v_x$$

$$\Rightarrow \frac{x^2-y^2}{(x^2+y^2)^2} = \frac{x^2-y^2}{(x^2+y^2)^2} - \phi'(x)$$

$$\Rightarrow \phi'(x) = 0$$

$$\Rightarrow \phi(x) = c$$

Consequently,

$$v(x, y) = \frac{x}{x^2+y^2} + c$$

Now, $w = f(z) = u + iv$

$$= \frac{y}{(x^2+y^2)} + i\left(\frac{x}{x^2+y^2} + c\right)$$

$$= \frac{y+ix}{(x^2+y^2)} + ic$$

Ex:- Show that $u(x, y)$ is harmonic ~~& find harmonic~~ & find harmonic conjugate $v(x, y)$ when or find conjugate harmonic if real part $u = \ldots$

i) $u(x, y) = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$.

$\Rightarrow u(x, y) = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$

$u_x = 3x^2 - 6y + 6x$

$u_{xx} = 6x + 6 = 6(x+1)$ ...... ①

$u_y = -6xy - 6y$

$u_{yy} = -6x - 6 = -6(x+1)$ .... ②

From ① & ②, we get,

$u_{xx} + u_{yy} = 0$

Hence u satisfies Laplace eq^n.

To find harmonic conjugate $v(x, y)$.

We start with $u_x = 3x^2 - 6y + 6x = v_y$

$\Rightarrow \frac{\partial v}{\partial y} = 3x^2 + 6x - 6y$

Hold x fixed & integrate w.r.t. 'y'.

$$\Rightarrow \int \frac{\partial v}{\partial y}\, dy = \int (3x^2 + 6x - 6y)\cdot dy + \phi(x)$$

$$\Rightarrow v = 3x^2y + 6xy - \frac{6y^2}{2} + \phi(x)$$

$$\Rightarrow v = 3x^2y + 6xy - 3y^2 + \phi(x)$$

Now,

$$v_x = 6xy + 6y + \phi'(x)$$

Then,

$$u_y = -v_x$$

$$\therefore -6xy - 6y = -6xy - 6y - \phi'(x)$$

$$\therefore -(6xy + 6y) = -(6xy + 6y) - \phi'(x)$$

$$\therefore -\phi'(x) = 0$$

$$\therefore \phi'(x) = 0$$

$$\therefore \phi(x) = c$$

Consequently,

$$v(x, y) = 3x^2y + 6xy - 3y^2 + c$$

Now,

$$W = f(z) = u + iv$$

$$= (x^3 - 3xy^2 + 3x^2 - 3y^2 + 1) + i(3x^2y + 6xy - 3y^2 + c)$$

x, vii) $u(x,y) = e^x \cdot \cos y$.

$\Rightarrow$ $u(x,y) = e^x \cdot \cos y$

$\therefore u_x = e^x \cdot \cos y$ ....

$\therefore u_{xx} = e^x \cdot \cos y$ .... ①

&

$u_y = -e^x \cdot \sin y$

$\therefore u_{yy} = -e^x \cdot \cos y$ .... ②

From ① & ②,

$u_{xx} + u_{yy} = 0$

$\therefore$ u satisfies Laplace eq^n.

To find harmonic conjugate $v(x,y)$.

We start with

$u_x = e^x \cdot \cos y = v_y$

$\Rightarrow \frac{\partial v}{\partial y} = e^x \cdot \cos y$

Hold x fixed & integrate w.r.t. 'y'

$\Rightarrow \int \frac{\partial v}{\partial y} = \int e^x \cdot \cos y\, dy + \phi(x)$

$\Rightarrow v = e^x \cdot \sin y + \phi(x)$

Now,

$v_x = e^x \cdot \sin y + \phi'(x)$

Then, $u_y = -v_x$

$\therefore -e^x \cdot \sin y = -e^x \cdot \sin y + \phi'(x)$

$\therefore \phi'(x) = 0$

$\therefore \phi(x) = c$

Consequently,

$$\boxed{v(x,y) = e^{x}.\sin y + c}$$

iii) $u(x,y) = \cos x . \cosh y$

$\Rightarrow u(x,y) = \cos x . \cosh y.$

$\therefore u_x = -\sin x . \cosh y$

$u_{xx} = -\cos x . \cosh y \quad \ldots\ldots (1)$

&

$u_y = \cos x . \sinh y$

$\therefore u_{yy} = \cos x . \cosh y \quad \ldots\ldots (2)$

From (1) & (2), $u_{xx} + u_{yy} = 0$.

$\therefore$ u satisfies Laplace eq^n.

To find harmonic conjugate $v(x,y)$ we start with

$u_x = -\sin x . \cosh y = v_y$

$\Rightarrow \frac{\partial v}{\partial y} = -\sin x . \cosh y.$

Hold x fixed & integrate w.r.t. 'y',

$\Rightarrow \int \frac{\partial v}{\partial y} = \int -\sin x . \cosh y . dy + \phi(x).$

$\Rightarrow v = -\sin x . \sinh y + \phi(x) \quad \ldots (3)$

$\Rightarrow v_x = -\cos x . \sinh y + \phi'(x)$

Then $u_y = -v_x$

$\therefore \cos x . \sinh y = \cos . \sinh y - \phi'(x).$

$\therefore -\phi'(x) = 0$

$\therefore \phi'(x) = 0$

$\therefore \phi(x) = c.$

Consequently,

$$v(x,y) = -\sin x \cdot \sinh y + c \quad \ldots \text{[from (3)]}$$

$$\therefore \boxed{v(x,y) = -\sin x \cdot \sinh y + c}$$

iv) $u(x,y) = \sin x \cdot \cosh y$

$\Rightarrow u_x = \cos x \cdot \cosh y$

$u_{xx} = -\sin x \cdot \cosh y \quad \ldots (1)$

&

$u_y = \sin x \sinh y$

$u_{yy} = \sin x \cdot \cosh y \quad \ldots (2)$

From (1) & (2), $u_{xx} + u_{yy} = 0$

$\therefore$ u satisfies Laplace eq^n.

To find harmonic conjugate $v(x,y)$.

We start with

$$u_x = \cos x \cdot \cosh y = v_y$$

$$\Rightarrow \frac{\partial v}{\partial y} = \cos x \cdot \cosh y.$$

Hold x fixed & integrate w.r.t. 'y'.

$$\Rightarrow \int \frac{\partial v}{\partial y} = \int \cos x \cdot \cosh y \cdot dy + \phi(x).$$

$$\Rightarrow v = \cos x \cdot \sinh y + \phi(x)$$

Now, $v_x = -\sin x \cdot \sinh y + \phi'(x).$

We have, $u_y = -v_x$

$\therefore \sin x \cdot \sinh y = +\sin x \cdot \sinh y + \phi'(x)$

$\therefore \phi'(x) = 0$

$\therefore \phi(x) = c$

Consequently,

$$\boxed{v(x,y) = \cos x \cdot \sinh y + c}$$

v) $u(x,y) = (x-1)^3 - 3xy^2 + 3y^2$

$\Rightarrow \quad u_x = 3(x-1)^2 \cdot (1) - 3y^2 + 0$

$\therefore u_x = 3(x-1)^2 - 3y^2$

$u_{xx} = 6(x-1) = 6x - 6 \quad \ldots (1)$

&

$u_y = -6xy + 6y$

$u_{yy} = -6x + 6 = -(6x-6) \quad \ldots (2)$

From (1) & (2), $u_{xx} + u_{yy} = 0$

$\therefore$ u satisfies Laplace eq^n.

To find harmonic conjugate $v(x,y)$

we start with $u_x = v_y = 3(x-1)^2 - 3y^2$

$\Rightarrow \dfrac{\partial v}{\partial y} = 3(x-1)^2 - 3y^2$

$\therefore \dfrac{\partial v}{\partial y} = 3(x^2 - 2x + 1) - 3y^2 = 3x^2 - 6x + 3 - 3y^2$

$\Rightarrow \dfrac{\partial v}{\partial y} = 3x^2 - 6x + 3 - 3y^2.$

Hold x fixed & integrate with respect to 'y'

$\Rightarrow \int \dfrac{\partial v}{\partial y} = \int (3x^2 - 6x + 3 - 3y^2) \cdot dy + \phi(x)$

$\Rightarrow v = \dfrac{-3y^3}{3} + \phi(x)$

$\Rightarrow v = -y^3 + \phi(x)$

Now, $v_x = \phi'(x)$

Then, we know that

$$u_y = -v_x$$

$$\therefore \; -6xy + 6y = -\phi'(x)$$

$$\therefore \; \phi'(x) = 6xy + 6y$$

$$\therefore \; \phi(x) = \frac{6x^2}{2}y + 6xy + c = \underline{3x^2y + 6xy + c}$$

Consequently,

$$\boxed{v(x, y) = -y^3 + 3x^2y + 6xy + c}$$

vi) $u(x, y) = e^x(x\cdot\cos y - y\sin y)$

$\Rightarrow u(x, y) = xe^x\cos y - e^x\cdot y\sin y$.

$$u_x = (x\cdot e^x + e^x)\cos y - e^x\cdot y\sin y$$

$$u_{xx} = (x\cdot e^x + e^x + e^x)\cos y - e^x\cdot y\sin y$$

$$\therefore u_{xx} = (xe^x + 2e^x)\cos y - e^x\cdot y\cdot\sin y \quad \ldots\ldots (1)$$

&

$$u_y = -xe^x\cdot\sin y - e^x[y\cdot\cos y + \sin y]$$

$$u_{yy} = -x\cdot e^x\cdot\cos y - e^x[-y\cdot\sin y + \cos y + \cos y]$$

$$= -\cos y[xe^x] - e^x[-y\cdot\sin y] - 2e^x\cos y$$

$$= -\cos y(x\cdot e^x + 2e^x) + e^x y\cdot\sin y$$

$$\therefore u_{yy} = -[(xe^x + 2e^x)\cos y - e^x\cdot y\cdot\sin y] \quad \ldots\ldots (2)$$

From (1) & (2), $u_{xx} + u_{yy} = 0$

$\therefore$ u satisfies Laplace eq^n.

To find harmonic conjugate $v(x, y)$

we start with $u_x = v_y = (x\cdot e^x + e^x)\cos y - e^x y\sin y$

$$\Rightarrow \frac{\partial v}{\partial y} = (x\cdot e^x + e^x)\cos y - e^x\cdot y\sin y.$$

Hold x fixed & integrate w.r.t. 'y',

$$\Rightarrow \int \frac{\partial v}{\partial y} = \int [(xe^x + e^x)\cos y - e^x . y\sin y] . dy + \phi(x)$$

$$\Rightarrow v = \sin y(xe^x + e^x) - e^x[y.\cos y + \sin y] + \phi(x)$$

$$\Rightarrow v = \sin y(xe^x + e^x) + e^x y.\cos y - e^x \sin y + \phi(x)$$

$$\Rightarrow v = \sin y[xe^x + e^x - e^x] + e^x y.\cos y + \phi(x)$$

$$\Rightarrow v = (x.e^x)\sin y + e^x y.\cos y + \phi(x).$$

Now, $v_x = (x.e^x + e^x)\sin y + e^x . y.\cos y + \phi'(x)$

Then $u_y = -v_x$

$$\therefore -[(xe^x + e^x)\sin y + ye^x \cos y] = -[(xe^x + e^x)\sin y + e^x y\cos y] + \phi'(x)$$

$$\therefore \phi'(x) = 0$$
$$\therefore \phi(x) = c$$

Consequently,

$$v(x,y) = (xe^x)\sin y + e^x . y.\cos y + c$$
$$\therefore v(x,y) = e^x(x.\sin y + y.\cos y) + c.$$

Ex:- Prove that fu^n $e^x(\cos y + i\sin y)$ is holomorphic & find its derivative.

$\Rightarrow$ $f(z) = e^x\cos y + ie^x . \sin y$

Here, $u = e^x . \cos y$ & $v = e^x . \sin y$.

$u_x = e^x \cdot \cos y \quad , \quad v_y = e^x \cdot \cos y$

$\therefore \; u_x = v_y$

And

$u_y = -e^x \cdot \sin y \quad \& \quad v_x = e^x \cdot \sin y$

$\therefore \quad u_y = -v_x$

Hence C-R eq^{ns} are satisfied.

$\therefore f'(z) = u_x + i v_x$

$= e^x \cdot \cos y + i \, (e^x \cdot \sin y)$

$= e^x (\cos y + i \sin y)$

$= e^x \cdot e^{iy}$

$= e^{x+iy}$

$\therefore \boxed{f'(z) = e^z}$

Note:- ① If two given fu^{ns} u & v are harmonic in a domain D & their 1st order partial derivatives satisfy the Cauchy-Riemann eq^{ns} throughout D then v is said to be harmonic conjugate of u. The meaning of word conjugate here is of coure different from that $\bar{z}$ is defined.

② Two families of curves $u(x,y) = c_1$, $v(x,y) = c_2$ are said to form an orthogonal system if they intersect at right angles at each of their points of intersection.

(3) If $z = u+iv$ is an analytic fu^n in a domain D p.t. the curves $u =$ const., $v =$ const. form two orthogonal system.

(4) An Analytic fu^n with constant real part is constant.

Ex:- H.W. Prove that the fu^n $z.|z|$ is not analytic anywhere.

Proof:- $f(z) = u+iv = z\,|z| = (x+iy)\sqrt{x^2+y^2}$

$$\Rightarrow u = x\sqrt{x^2+y^2} \ \& \ v = y\sqrt{x^2+y^2}$$

changing to polar co-ordinates,

$$u = r^2\cos\theta \quad , \quad v = r^2\sin\theta$$

$$\Rightarrow \frac{\partial u}{\partial r} = 2r.\cos\theta \quad , \quad \frac{1}{r}\frac{\partial u}{\partial \theta} = -r.\sin\theta$$

$$\frac{\partial v}{\partial r} = 2r.\sin\theta \quad , \quad \frac{1}{r}\frac{\partial v}{\partial \theta} = r.\cos\theta$$

$$\Rightarrow \frac{\partial u}{\partial r} \neq \frac{1}{r}\frac{\partial v}{\partial \theta} \text{ and } \frac{1}{r}\frac{\partial u}{\partial \theta} \neq -\frac{\partial v}{\partial r} \quad \forall r.$$

$\Rightarrow f(z) = z.|z|$ is not analytic anywhere.

Ex:- H.W. Find the pt. where the C-R eq^{ns} are satisfied for the fu^n $f(z) = xy^2 + ix^2y$ where does $f'(z)$ exist?

$^n \Rightarrow$ $f(z) = xy^2 + ix^2y$

Here, $u = xy^2$ & $v = x^2y$

$u_x = y^2$, $v_y = x^2$

&

$u_y = 2xy$ & $v_x = 2xy$

As $u_x = v_y \Rightarrow y^2 = x^2$

$\Rightarrow x = y$ ... ①

& $u_y = -v_x \Rightarrow 2xy = -2xy$

$\Rightarrow 4xy = 0$

$\Rightarrow xy = 0$ ... ②

$\Rightarrow x = 0$ or $y = 0$

or

$x = 0$ & $y = 0$ ... ③

But from ① & ③, we get,

$x = 0$ & $y = 0$ [$\because$ As $x = y$]

$\Rightarrow f(z)$ is analytic at the origin.

As at the point $(x, y) \equiv (0, 0)$ the C-R eq^{ns} are satisfied for fu^n $f(z)$.

Now,

$$f'(z) = u_x + iv_x$$

$$= y^2 + i(2xy)$$

$$\therefore f'(z) = y(y + i2x)$$

$$\Rightarrow f'(0) = 0$$

Thus $f(z)$ is analytic only at $z=0$ & there $f'(z) = 0$ for $z = 0$.