Course Code : DSE - F11

Title of Course: Complex Analysis

Unit 1- Analytic functions and Complex Integration:-

Basic algebraic & geometric properties of complex numbers, Function of complex variables, Limits, continuity and differentiation, Cauchy Riemann eq^{ns}, Analytic fu^{ns} & examples of analytic fu^{ns}, Exponential fu^{n}, Logarithmic fu^{n}, Trigonometric fu^{n}, Definite integrals of fu^{ns}, Contours, Contour integrals, and its examples, upper bounds for moduli of contour integrals, Cauchy-Goursat theorem & examples, Cauchy integral formula & examples, Liouville's theorem & the fundamental theorem of algebra.

Unit 2- Sequences, Series and Residue Calculus :-

Convergence of sequences, & series of complex variables, Taylor series & its examples, Laurent series & its examples, absolute & uniform convergence of power series, Isolated singular points, Residue's, Cauchy's residue th^{m}, Residue at infinity, The three types of isolated singularities, Residues at poles & examples, Zeros of analytic functions, Zeros & poles, Application of residue theorem to evaluate real integrals.

Unit 1 - Analytic function and Complex Integration

- Complex Numbers :-

Defⁿ :- The number of the form $z = x+iy$ or $z=(x,y)$ where $i=\sqrt{-1}$ and x & y are real number is called Complex number.

In complex number $z = x+iy$, x is called real part and y is called imaginary part.

- Modulus of Complex number :-

If $z = x+iy$ is complex number then the number $\sqrt{x^2+y^2}$ is called modulus of complex number z and it is denoted by $|z|$. Thus $|z| = \sqrt{x^2+y^2}$.

- Complex Conjugate :-

The complex number $x+iy$ & $x-iy$ are called complex conjugate of each other.

Complex conjugate of complex number z is denoted by $\bar{z}$ i.e. $(\bar{z} = x-iy)$.

- Result :-

We write $z = x+iy$ then

$x = Re(z)$, $y = Im(z)$.

① $z+\bar{z} = x+\cancel{iy}+x-\cancel{iy} = 2x = 2\,Re(z)$

$\therefore x = \frac{z+\bar{z}}{2} = Re(z)$

② $z-\bar{z} = 2iy = 2I(z)$

$\therefore y = \frac{z-\bar{z}}{2i} = I(z)$

(3) $z \cdot \bar{z} = (x+iy)(x-iy)$
$= x^2 + y^2$
$= (|z|)^2$

(4) $|z| = |\bar{z}|$

(5) $|\bar{z}| = \sqrt{x^2+y^2}$.

Que.-1) Prove that $|z_1 \cdot z_2| = |z_1| \cdot |z_2|$

Proof:- Let $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$.

$$z_1 \cdot z_2 = (x_1 + iy_1) \cdot (x_2 + iy_2)$$

$$= x_1 \cdot x_2 + ix_1 y_2 + i x_2 y_1 + i^2 \cdot y_1 y_2$$

$$= x_1 x_2 + ix_1 \cdot y_2 + i x_2 \cdot y_1 - y_1 \cdot y_2$$

$$= (x_1 \cdot x_2 - y_1 \cdot y_2) + i(x_1 \cdot y_2 + x_2 \cdot y_1)$$

$$|z_1 \cdot z_2| = \sqrt{(x_1 \cdot x_2 - y_1 \cdot y_2)^2 + (x_1 \cdot y_2 + x_2 y_1)^2}$$

$$= \sqrt{x_1^2 x_2^2 - \cancel{2x_1 x_2 \cdot y_1 y_2} + y_1^2 y_2^2 + x_1^2 y_2^2 + \cancel{2x_1 x_2 y_1 y_2} + x_2^2 y_1^2}$$

$$= \sqrt{x_1^2 x_2^2 + y_1^2 y_2^2 + x_1^2 y_2^2 + x_2^2 y_1^2}$$

$$= \sqrt{x_1^2 (x_2^2 + y_2^2) + y_1^2 (y_2^2 + x_2^2)}$$

$$= \sqrt{(x_1^2 + y_1^2) \cdot (x_2^2 + y_2^2)}$$

$$= \sqrt{(x_1^2 + y_1^2)} \cdot \sqrt{(x_2^2 + y_2^2)}$$

$$\therefore |z_1 \cdot z_2| = |z_1| \cdot |z_2|$$

Q.2) Prove that $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$

Sol^n:- We have, $|z_1| = \left|\frac{z_1}{z_2} \cdot z_2\right|$

$$= \left|\frac{z_1}{z_2}\right| \cdot |z_2|$$

$$\frac{|z_1|}{|z_2|} = \left|\frac{z_1}{z_2}\right|$$

$$\therefore \left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$$

Q.3) Prove that $\overline{z_1 \cdot z_2} = \overline{z_1} \cdot \overline{z_2}$

Proof:- We have,

$$z_1 = x_1 + iy_1 \quad \text{and} \quad z_2 = x_2 + iy_2$$

$$\therefore z_1 \cdot z_2 = (x_1 + iy_1) \cdot (x_2 + iy_2)$$

$$= x_1 \cdot x_2 + ix_1y_2 + ix_2y_1 + i^2 y_1y_2$$

$$= (x_1 \cdot x_2 - y_1 \cdot y_2) + i(x_1y_2 + x_2y_1) \quad \ldots (i^2 = -1)$$

$$\therefore \overline{z_1 \cdot z_2} = (x_1 \cdot x_2 - y_1 \cdot y_2) - i(x_1y_2 + x_2y_1) \quad \ldots \text{(1)}$$

Now, $\overline{z_1} = (x_1 - iy_1)$ and $\overline{z_2} = (x_2 - iy_2)$

$$\therefore \overline{z_1} \cdot \overline{z_2} = (x_1 - iy_1) \cdot (x_2 - iy_2)$$

$$= x_1 \cdot x_2 - ix_1y_2 - ix_2y_1 + i^2 y_1y_2$$

$$= (x_1 \cdot x_2 - y_1y_2) - i(x_1y_2 + x_2y_1) \quad \ldots \text{(2)}$$

$\therefore$ From ① and ②,

$\therefore \overline{z_1 \cdot z_2} = \overline{z_1} \cdot \overline{z_2}$

Q.4) Prove that $|c \cdot z| = |c| \cdot |z|$.

Proof:- Let $z = x + iy$.

Consider, $c \cdot z = c \cdot (x + iy)$

$$cz = cx + icy$$

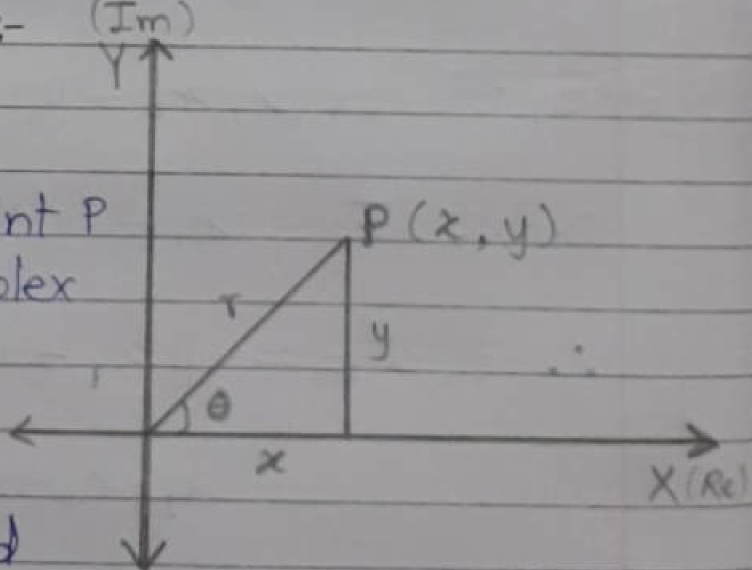
$$|c \cdot z| = \sqrt{c^2x^2 + c^2y^2}$$

$$= \sqrt{c^2(x^2 + y^2)}$$

$$= \sqrt{c^2} \cdot \sqrt{x^2 + y^2}$$

$$\therefore |c \cdot z| = |c| \cdot |z|.$$

- Argand Diagram :-



Let $z = x + iy$.
We say that point P represents the complex number $x + iy$.
Representation of complex number in the plane is called Argand Diagram or this plane is called Complex plane or Gaussian plane.

In XY- plane,
Let $OP = r$. OP makes angle θ with positive direction of X- axis.

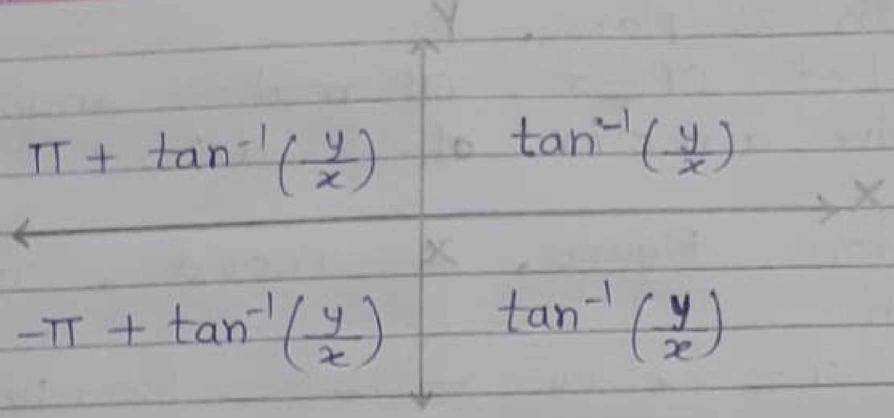
$\therefore$ From figure, $x = r\cos\theta$, $y = r\sin\theta$.
Then
$$r = \sqrt{x^2+y^2} = |z| = |x+iy|$$

and $\theta = \tan^{-1}\left(\frac{y}{x}\right)$

θ is called argument or amplitude of complex number z and
r is called radius vector of complex number z.

It follows that $z = x+iy$
$= r\cos\theta + ir\sin\theta$
$= r(\cos\theta + i\sin\theta)$
$\therefore z = r.e^{i\theta}$ ..... (1)

It is called Polar form of complex number.
'r' is called modulus or absolute value of z and
'θ' is called argument or amplitude of z. i.e. $\theta = \arg(z)$.



① The argument of z is not unique. Eq^n ①, does not alter if we replace 'θ' by '$2\pi + \theta$'.
So, 'θ' can have infinite number of values which differ from each other by 2π.
Thus, general value of argument is given by,

$$\arg(z) = \arg(z) + 2n\pi \quad \forall\, n \in I.$$

② If the value of θ satisfies eq^n ① & lies bet^n $-\pi$ to π.
i.e. $-\pi \le \theta \le \pi$. Then the value of θ is called Principal Value of argument.

③ If $z = 0$ then argument of z i.e. $\arg(z) = \arg(0)$ is not defined and $\arg(z)$ is defined only if $z \neq 0$.

④ If $z = x + iy$ then argument of z
i.e.

1) $$\arg(z) = \begin{cases} \tan^{-1}\left(\frac{y}{x}\right) & ; \text{if } x>0, y\geq 0 \text{ or } y\leq 0. \\ \pi + \tan^{-1}\left(\frac{y}{x}\right) & ; \text{if } x<0, y\geq 0. \\ -\pi + \tan^{-1}\left(\frac{y}{x}\right) & ; \text{if } x<0, y<0. \end{cases}$$

2) $$\arg(z) = \begin{cases} \frac{\pi}{2} & ; \text{if } x=0, y>0. \\ -\frac{\pi}{2} & ; \text{if } x=0, y<0. \end{cases}$$

e.g.- If $z = -\sqrt{3} - i$

We know that $z = r.e^{i\theta}$

$\Rightarrow z = r.e^{i\theta} = -\sqrt{3} - i \Rightarrow -\sqrt{3} - i = r.e^{i\theta}$.

$$\Rightarrow \arg(z) = \tan^{-1}\left(\frac{y}{x}\right)$$
$$= \tan^{-1}\left(\frac{-1}{-\sqrt{3}}\right)$$
$$= \tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$$

As $\tan\theta = \frac{1}{\sqrt{3}}$ then $\theta = \frac{\pi}{6}$.

$\therefore \arg(z) = -\pi + \frac{\pi}{6}$ ... $\left[\begin{array}{l} \because \text{Here, } x<0, y<0 \\ \therefore \arg(z) = -\pi + \tan^{-1}\left(\frac{y}{x}\right) \end{array}\right]$

$\therefore \arg(z) = \frac{-5\pi}{6}$

1. $z = i \Rightarrow \arg(i) = \frac{\pi}{2}$, As $x = 0, y>0$

$\therefore \arg(z) = \frac{\pi}{2}$.

② $z = -i$

$\Rightarrow$ Here, $x = 0$, $y < 0$; hence

$\arg(z) = -\frac{\pi}{2}$.

$\therefore \arg(-i) = \frac{-\pi}{2}$

③ $\arg(1-i) = ?$

$\Rightarrow$

$z = 1-i$.

Here, $x > 0$, $y < 0$. then

$\therefore \arg(z) = \tan^{-1}\left(\frac{y}{x}\right)$

$= \tan^{-1}\left(\frac{-1}{1}\right)$

$= -\tan^{-1}(1)$

$\therefore \arg(1-i) = \frac{-\pi}{4}$.

④ $\arg(1+i) = ?$

$\Rightarrow$ $z = (1+i)$. Here $x > 0$, $y > 0$ then

$\arg(z) = \tan^{-1}\left(\frac{y}{x}\right)$

$= \tan^{-1}\left(\frac{1}{1}\right)$

$\arg(1+i) = \frac{\pi}{4}$.

⑤ $\arg(-1-i) = ?$

$\Rightarrow$

$z = -1-i$. Here $x < 0$, $y < 0$ then

$\arg(z) = -\pi + \tan^{-1}\left(\frac{y}{x}\right)$

$= -\pi + \tan^{-1}\left(\frac{-1}{-1}\right)$

$= -\pi + \tan^{-1}(1)$

$= -\pi + \frac{\pi}{4}$

$\therefore \arg(-1-i) = \frac{-3\pi}{4}$.

⑥ $\arg(-1) = ?$

$\Rightarrow$ $z = -1$. Here, $x<0$, $y=0$ then

$\arg(z) = \pi + \tan^{-1}\left(\frac{y}{x}\right)$

$= \pi + \tan^{-1}\left(\frac{0}{-1}\right)$

$= \pi + \tan^{-1}(0)$

$= \pi + 0$

$\therefore \arg(-1) = \pi$

Que.- Find modulus and argument.

① $z = 1+i$

$\Rightarrow$ $|z| = r = \sqrt{x^2+y^2} = \sqrt{1+1} = \sqrt{2}$.

Here, $x>0$ and $y>0$ then

$\arg(z)$ or $\text{amp}(z) = \tan^{-1}\left(\frac{y}{x}\right)$

$+0$ $= \tan^{-1}\left(\frac{1}{1}\right)$

$= \tan^{-1}(1)$

$\therefore \arg(1+i) = \frac{\pi}{4}$.

② $z = 1 - i$

$\Rightarrow$ $|z| = \sqrt{1+1} = \sqrt{2}$

$\theta = \arg(z) = \tan^{-1}(-1) = \frac{-\pi}{4}$

MCQ ● If the $amp(z) = \theta$ then $amp(iz) = ?$

$\Rightarrow$ $amp(z) = \theta$, $amp(iz) = ?$

We have,

$$amp(z_1 \cdot z_2) = amp(z_1) + amp(z_2)$$

$$\therefore amp(iz) = amp(i) + amp(z)$$

$$\therefore amp(iz) = \frac{\pi}{2} + \theta$$

★ Properties :-

① Prove that $\arg(z_1 \cdot z_2) = \arg(z_1) + \arg(z_2)$.

Proof :- We know that

$$z_1 = r_1(\cos\theta_1 + i\sin\theta_1) = r_1 \cdot e^{i\theta_1}$$

$$z_2 = r_2(\cos\theta_2 + i\sin\theta_2) = r_2 \cdot e^{i\theta_2}$$

And $\arg(z_1) = \theta_1$, $\arg(z_2) = \theta_2$.

Consider, $z_1 \cdot z_2 = r_1 \cdot e^{i\theta_1} \cdot r_2 \cdot e^{i\theta_2}$

$$= r_1 r_2 \, e^{i(\theta_1+\theta_2)}$$

$$= r_1 r_2 [\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)]$$

$$\therefore \arg(z_1 \cdot z_2) = \theta_1 + \theta_2$$

$$\therefore \arg(z_1 \cdot z_2) = \arg(z_1) + \arg(z_2)$$

Hence it is proved.

② P.T. $\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)$

Proof:- $z_1 = r_1 \cdot e^{i\theta_1}$, $z_2 = r_2 \cdot e^{i\theta_2}$

$\arg(z_1) = \theta_1$ and $\arg(z_2) = \theta_2$.

$$\therefore \frac{z_1}{z_2} = \frac{r_1 \cdot e^{i\theta_1}}{r_2 \cdot e^{i\theta_2}}$$

$$\frac{z_1}{z_2} = \frac{r_1}{r_2} \cdot e^{i(\theta_1 - \theta_2)}$$

$$= \frac{r_1}{r_2}\left[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)\right]$$

$$\therefore \arg\left(\frac{z_1}{z_2}\right) = \theta_1 - \theta_2$$

$$\therefore \arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)$$

Hence it is proved.

③ Prove that $|z_1 + z_2| \leq |z_1| + |z_2|$

Proof:- Let, $z_1 = r_1(\cos\theta_1 + i\sin\theta_1) = r_1 e^{i\theta_1}$

$z_2 = r_2(\cos\theta_2 + i\sin\theta_2) = r_2 e^{i\theta_2}$

We have,

$$z_1 + z_2 = r_1(\cos\theta_1 + i\sin\theta_1) + r_2(\cos\theta_2 + i\sin\theta_2)$$

$$= r_1\cos\theta_1 + ir_1(\sin\theta_1) + r_2\cos\theta_2 + ir_2\sin\theta_2$$

$$= (r_1\cos\theta_1 + r_2\cos\theta_2) + i(r_1\sin\theta_1 + r_2\sin\theta_2)$$

$$|z_1 + z_2| = \sqrt{(r_1\cos\theta_1 + r_2\cos\theta_2)^2 + (r_1\sin\theta_1 + r_2\sin\theta_2)^2}$$

$$= \sqrt{\begin{matrix} r_1^2\cos^2\theta_1 + 2r_1r_2\cos\theta_1 \cdot \cos\theta_2 + r_2^2\cos^2\theta_2 + \\ r_1^2\sin^2\theta_1 + 2r_1r_2\sin\theta_1 \cdot \sin\theta_2 + r_2^2\sin^2\theta_2 \end{matrix}}$$

$$= \sqrt{r_1^2(\sin^2\theta_1 + \cos^2\theta_1) + 2r_1 r_2(\cos\theta_1 \cdot \cos\theta_2 + \sin\theta_1 \cdot \sin\theta_2) + r_2^2(\cos^2\theta_2 + \sin^2\theta_2)}$$

$$= \sqrt{r_1^2 + 2r_1r_2\cos(\theta_1 - \theta_2) + r_2^2}$$

$$|z_1+z_2| = \sqrt{r_1^2 + r_2^2 + 2r_1r_2 \cdot \cos(\theta_1 - \theta_2)}$$

But, $\cos(\theta_1 - \theta_2) \leq 1$.

$$\therefore |z_1+z_2| \leq \sqrt{r_1^2 + r_2^2 + 2r_1r_2}$$

$$\leq \sqrt{(r_1 + r_2)^2}$$

$$\leq r_1 + r_2$$

$$\therefore |z_1 + z_2| \leq |z_1| + |z_2|$$

④ Prove that $|z_1 - z_2| \geq |z_1| - |z_2|$

Proof:- We have,

$$z_1 = r_1(\cos\theta_1 + i\sin\theta_1) = r_1 e^{i\theta_1}$$

& $z_2 = r_2(\cos\theta_2 + i\sin\theta_2) = r_2 e^{i\theta_2}$

Consider,

$$z_1 - z_2 = r_1(\cos\theta_1 + i\sin\theta_1) - r_2(\cos\theta_2 + i\sin\theta_2)$$

$$= r_1\cos\theta_1 + ir_1\sin\theta_1 - r_2\cos\theta_2 - ir_2\sin\theta_2$$

$$= (r_1\cos\theta_1 - r_2\cos\theta_2) + i(r_1\sin\theta_1 - r_2\sin\theta_2)$$

$$|z_1 - z_2| = \sqrt{(r_1\cos\theta_1 - r_2\cos\theta_2)^2 + (r_1\sin\theta_1 - r_2\sin\theta_2)^2}$$

$$= \sqrt{r_1^2 \cos^2\theta_1 - 2r_1 r_2 \cos\theta_1 \cdot \cos\theta_2 + r_2^2 \cos^2\theta_2 + r_1^2 \sin^2\theta_1 - 2r_1 r_2 \sin\theta_1 \cdot \sin\theta_2 + r_2^2 \sin^2\theta_2}$$

$$= \sqrt{r_1^2(\cos^2\theta_1 + \sin^2\theta_1) - 2r_1 r_2(\cos\theta_1 \cdot \cos\theta_2 + \sin\theta_1 \cdot \sin\theta_2) + r_2^2(\cos^2\theta_2 + \sin^2\theta_2)}$$

$$|z_1 - z_2| = \sqrt{r_1^2(1) - 2r_1 r_2 \cos(\theta_1 - \theta_2) + r_2^2(1)}$$

But $\quad -\cos(\theta_1 - \theta_2) \geq 1$

$$|z_1 - z_2| \geq \sqrt{r_1^2 - 2r_1 r_2 + r_2^2}$$

$$\geq \sqrt{(r_1 - r_2)^2}$$

$$\geq r_1 - r_2$$

$$\therefore |z_1 - z_2| \geq |z_1| - |z_2|$$

⑤ Prove that $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2|z_1|^2 + 2|z_2|^2$

Proof :- consider,

L.H.S. $= |z_1 + z_2|^2 + |z_1 - z_2|^2$

$$= (z_1 + z_2)(\overline{z_1 + z_2}) + (z_1 - z_2)(\overline{z_1 - z_2})$$

$\qquad \ldots\ldots (\because z \cdot \bar{z} = |z|)$

$$= (z_1 + z_2)(\bar{z}_1 + \bar{z}_2) + (z_1 - z_2)(\bar{z}_1 - \bar{z}_2)$$

$\qquad \ldots\ldots (\because \overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2)$

$$= z_1\bar{z}_1 + \cancel{z_1\bar{z}_2} + \cancel{z_2\bar{z}_1} + z_2\bar{z}_2 + z_1\bar{z}_1 - \cancel{z_1\bar{z}_2} - \cancel{z_2\bar{z}_1} + z_2\bar{z}_2$$

$$= 2z_1\bar{z}_1 + 2z_2\bar{z}_2$$

$$= 2|z_1|^2 + 2|z_2|^2$$

$$= R.H.S$$

$$\therefore |z_1+z_2|^2 + |z_1-z_2|^2 = 2|z_1|^2 + 2|z_2|^2$$

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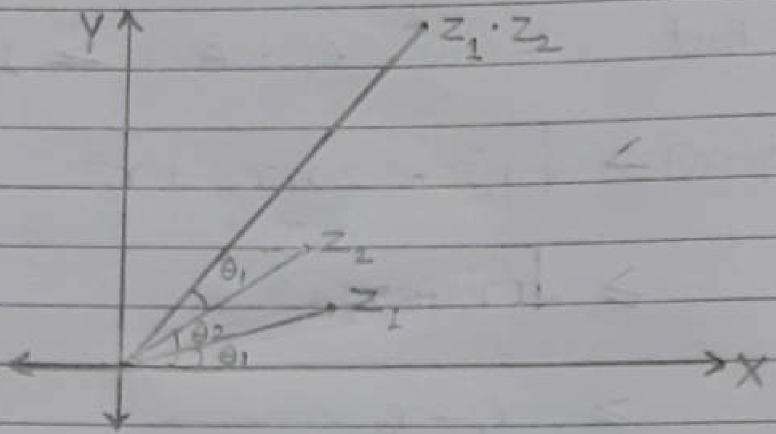


$$\arg(z_1 \cdot z_2) = \arg(z_1) + \arg(z_2)$$

geometrically the length of the vectors $z_1 z_2$ is equal to the product of the lengths z_1 & z_2. The angle of inclination of the vectors $z_1 \cdot z_2$ is the sum of angles θ_1 & θ_2 (see fig.). In particular when a complex no. z is multiplied by i then resulting vector is iz is the one obtained by rotating the vector z through right angle in the positive direction without changing the length of vectors.

Since

$$iz = (\cos \pi/2 + i\sin \pi/2)\, r(\cos\theta + i\sin\theta)$$

$$= r[\cos(\theta + \pi/2) + i\sin(\theta + \pi/2)]$$

We know that

$$z_1 \cdot z_2 = r_1 r_2 [\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)] \quad \ldots (1)$$

It follows from (1) that

$$z_1 z_2 \ldots z_n = r_1 r_2 \ldots r_n [\cos(\theta_1 + \theta_2 + \ldots + \theta_n) + i\sin(\theta_1 + \theta_2 + \ldots + \theta_n)]$$

Consequently if

$z = r(\cos\theta + i\sin\theta)$ & if n is +ve integer.

$$z^n = r^n(\cos n\theta + i\sin n\theta) \quad \ldots (2)$$

When $r = 1$ this formula reduces to Demoivre's th^m for the integral.

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta \quad \ldots (3)$$

The quotient of two complex numbers is given in its polar form by the formula

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)] \quad (r_2 \neq 0).$$

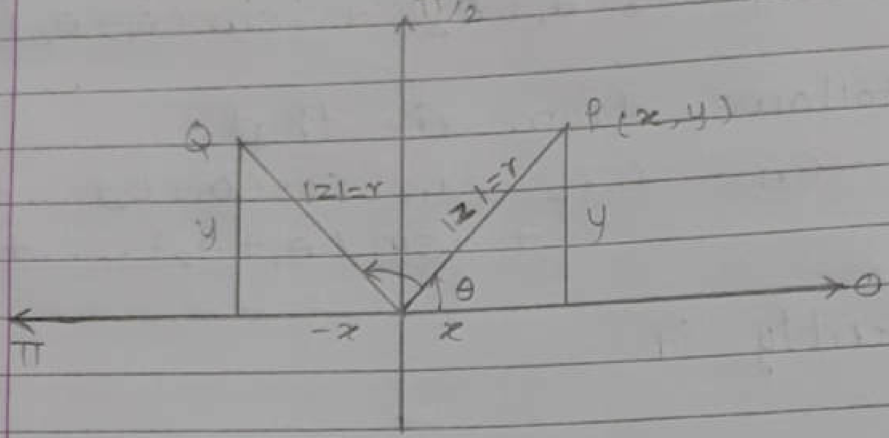
then

$$z^{-n} = \frac{1}{z^n} = \frac{1}{r^n}[\cos(-n\theta) + i\sin(-n\theta)]$$

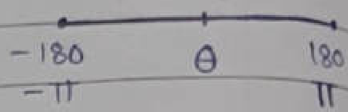
$$= \left(\frac{1}{z}\right)^n$$

Thus formula (2) & Demoivre's th^m (3) are valid when exponent is any -ve integer.

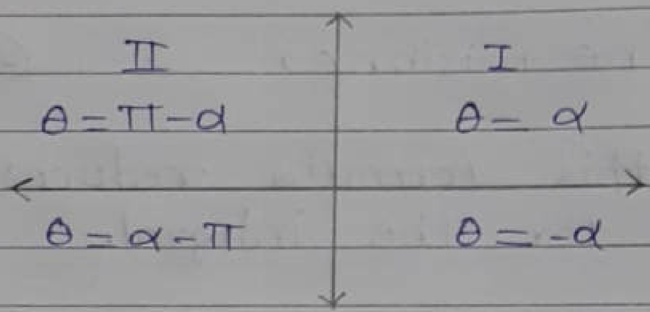
* arg :- θ , $z = x + iy$



$-\pi < \theta \le \pi$



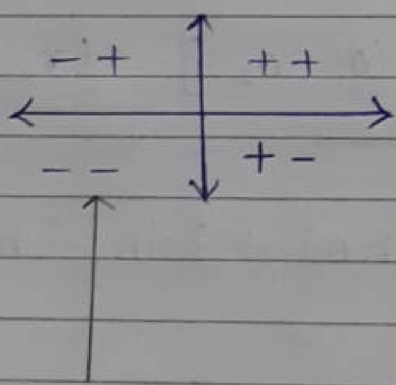
Step :- ① $z = x + iy$, (x, y)



② If $\tan\alpha = \left|\frac{y}{x}\right|$

e.g. – $z = -1 - i\sqrt{3}$ $\qquad \because (x, y) \equiv (-1, -\sqrt{3})$

$\Rightarrow 3^{rd}$ quadrant.



Let $\tan\alpha = \left|\frac{y}{x}\right|$

$= \left|\frac{-\sqrt{3}}{-1}\right|$

$\boxed{\tan\alpha = \sqrt{3}}$

$(-1, -\sqrt{3})$ lies in 3^{rd} quadrant.

$\tan\alpha = \tan\frac{\pi}{3}$

$\therefore \alpha = \frac{\pi}{3}$

$\therefore \theta = \alpha - \pi$
$= \frac{\pi}{3} - \pi$

$\therefore \boxed{\theta = \frac{-2\pi}{3}}$

Ex:- ① If $z = \frac{-2}{1+i\sqrt{3}}$

$\Rightarrow z = -2\left[\frac{1-i\sqrt{3}}{(1+i\sqrt{3})(1-i\sqrt{3})}\right]$

$= -2\left[\frac{(1-i\sqrt{3})}{(1+3)}\right]$

$= \frac{-2}{4}(1-i\sqrt{3})$

$= \frac{-1}{2}(1-i\sqrt{3})$

$= \frac{-1}{2} + i\frac{\sqrt{3}}{2}$

$\therefore \left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ lies in 2^{nd} quad.

Let $\tan\alpha = \left|\left(\frac{y}{x}\right)\right| = \left|\frac{(\sqrt{3}/2)}{(-1/2)}\right| = \sqrt{3} = \tan\frac{\pi}{3}$

Now $\alpha = \frac{\pi}{3}$

$\therefore \theta = \pi - \alpha = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$

& $r = \sqrt{x^2+y^2} = \sqrt{1/4 + 3/4} = \sqrt{1} = 1$

$\therefore$ We get, $\boxed{\theta = \frac{2\pi}{3}}$ & $\boxed{r = 1}$

Ex:- 2 $z = \dfrac{i}{-2-2i}$

$\Rightarrow \therefore z = -\dfrac{1}{2}\left[\dfrac{i}{1+i}\right]$

$= -\dfrac{1}{2}\left[\dfrac{i(1-i)}{(1+i)(1-i)}\right]$

$= \dfrac{-1}{2}\left[\dfrac{i+1}{1+1}\right]$

$= \dfrac{-1}{2}\left(\dfrac{1+i}{2}\right)$

$= \dfrac{-1}{4}(1+i)$

$\therefore z = \dfrac{-1}{4} - \dfrac{1}{4}i$

$\therefore z = \left(-\dfrac{1}{4}, -\dfrac{1}{4}\right)$ lies in 3^{rd} quadrant.

$\therefore \tan\alpha = \left|\dfrac{y}{x}\right| = \left|\dfrac{(-1/4)}{(-1/4)}\right| = 1 = \tan\dfrac{\pi}{4}$

$\therefore \alpha = \dfrac{\pi}{4} = 45^\circ$

Now, $\theta = \alpha - \pi = \dfrac{\pi}{4} - \pi = \underline{\underline{\dfrac{-3\pi}{4}}}$

& $r = \sqrt{x^2+y^2} = \sqrt{\left(\dfrac{-1}{4}\right)^2 + \left(-\dfrac{1}{4}\right)^2} = \sqrt{\dfrac{1}{16}+\dfrac{1}{16}}$

$\therefore r = \sqrt{\dfrac{2}{16}} = \sqrt{\dfrac{1}{8}} = \dfrac{1}{\sqrt{8}} = \underline{\underline{\dfrac{1}{2\sqrt{2}}}}$

Hence we get,

$\boxed{\theta = \dfrac{-3\pi}{4}}$ & $\boxed{r = \dfrac{1}{2\sqrt{2}}}$

③ If $z = (\sqrt{3} - i)^6$ then arg z is

$\Rightarrow$ We have, $\sqrt{3} - i = 2\left(\frac{\sqrt{3}}{2} - i\frac{1}{\sqrt{2}}\right)$

$$\sqrt{3} - i = 2\left(\cos\frac{\pi}{6} - i\sin\frac{\pi}{6}\right)$$

$$\therefore (\sqrt{3} - i)^6 = 2^6\left[\cos\frac{\pi}{6} - i\sin\frac{\pi}{6}\right]^6$$

$$= 2^6\left[\cos\left(-6\left(\frac{\pi}{6}\right)\right) + i\sin\left(-6\left(\frac{\pi}{6}\right)\right)\right]$$

$$= 2^6\left[\cos\pi - i\sin\pi\right]$$

$\therefore$ arg $z = \theta = \underline{\underline{\pi}}$

and also we get, $r = \sqrt{3+1} = \sqrt{4} = \underline{\underline{2}}$.

★ Polar Form :-

Let $z = x + iy$ then

polar form :- $z = r(\cos\theta + i\sin\theta)$

Ex:- ① Express in the polar form $1 - \sqrt{2} + i$

Sol^n:- $z = (1 - \sqrt{2}) + i$

Here, $x = 1 - \sqrt{2}$ & $y = 1$.

$$|z| = r = \sqrt{(1-\sqrt{2})^2 + 1^2} = \sqrt{1 - 2\sqrt{2} + 2 + 1} = \sqrt{4 - 2\sqrt{2}}$$

$$\therefore |z| = r = \sqrt{4 - 2\sqrt{2}}$$

$\cos\theta = \frac{x}{r} = \frac{1-\sqrt{2}}{\sqrt{4-2\sqrt{2}}}$ & $\sin\theta = \frac{y}{r} = \frac{1}{\sqrt{4-2\sqrt{2}}}$

$\therefore$ The polar form is

$$r(\cos\theta + i\sin\theta) = \sqrt{4-2\sqrt{2}}\left[\frac{1-\sqrt{2}}{\sqrt{4-2\sqrt{2}}} + i\left(\frac{1}{\sqrt{4-2\sqrt{2}}}\right)\right]$$

Ex:- ② $z = -\sqrt{3} + i$

$\Rightarrow$ $x = -\sqrt{3}$ & $y = 1$.

$\therefore r = \sqrt{3+1} = \sqrt{4} = 2$

$\cos\theta = \frac{x}{r} = \frac{-\sqrt{3}}{2}$, $\sin\theta = \frac{y}{r} = \frac{1}{2}$.

Since $\cos\theta$ is negative & $\sin\theta$ is +ve in second quadrant.

$\therefore \theta \Rightarrow \tan\alpha = \left|\frac{y}{x}\right| = \left|\frac{1/2}{-\sqrt{3}/2}\right| = \left|\frac{-1}{\sqrt{3}}\right| = \frac{1}{\sqrt{3}} = \tan\frac{\pi}{6}$

$\therefore \alpha = \frac{\pi}{6}$

$\therefore$ Now $\theta = \pi - \alpha = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$

Principal value lying between $-\pi$ to π & is $\frac{5\pi}{6}$

General value is $2n\pi \pm \frac{5\pi}{6}$

$\therefore$ The polar form is

$r(\cos\theta + i\sin\theta) = 2\left(-\frac{\sqrt{3}}{2} + i\,\frac{1}{2}\right)$

③ Find principal values of arguments of the following numbers.

i> x ii> $-x$ iii> iy iv> $-iy$

where $x, y > 0$.

Sol^{ns}:- i> $x = x[\cos 0 + i\sin 0] = x e^{i0}$

$\therefore \arg(x) = \underline{\underline{0}}$

ii> $-x = -x[\cos\pi + i\sin\pi] = x e^{i\pi}$

$\therefore \arg(-x) = \underline{\underline{\pi}}$

iii> $iy = y[\cos\pi/2 + i\sin\pi/2] = y \cdot e^{i\pi/2}$

$\therefore \arg(iy) = \underline{\underline{\frac{\pi}{2}}}$

iv> $-iy = y\left[\cos\frac{\pi}{2} - i\sin\frac{\pi}{2}\right] = y e^{-i\pi/2}$

$\therefore \arg(-iy) = \underline{\underline{\frac{-\pi}{2}}}$

Ex:- If the amplitude of the complex number z be θ then what is amplitude of (iz) is

Sol^n:- $\arg(iz) = \arg(i) + \arg(z)$

$= \frac{\pi}{2} + \theta$

$\therefore \boxed{\arg(iz) = \theta + \frac{\pi}{2}}$

Ex:- If $\sin(x+iy) = p+iq$ where p & q are real then q = ? & p = ?

Sol^n:- $\sin(x+iy) = \sin x \cdot \cos iy + \cos x \cdot \sin iy$

$\sin(x+iy) = \sin x \cdot \cosh y + i\cos x \cdot \sinh y$ ... (1)

Also, Given :- $\sin(x+iy) = p + iq$ ... (2)

From (1) & (2);

Ans:- $q = \cos x \cdot \sinh y$ and $p = \sin x \cdot \cosh y$.

★ Formulae :-

i) $|z| = r = \sqrt{x^2+y^2}$ & $\theta = \tan^{-1}\left(\frac{y}{x}\right)$

ii) $e^{i\theta} = \cos\theta + i\sin\theta$

$e^{-i\theta} = \cos\theta - i\sin\theta$

iii) $\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$

iv) $\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$

v) Polar form :- $Z = r[\cos\theta + i\sin\theta]$
$Z = re^{i\theta}$

This is called exponential form or Euler's form of complex numbers.

$\therefore$ Thus $Z = x+iy$ Cartesian form.
$Z = r(\cos\theta + i\sin\theta)$ Polar form.
$Z = r\cdot e^{i\theta}$ exponential form

& $\bar{Z} = x - iy$
$\bar{Z} = r(\cos\theta - i\sin\theta)$
$\bar{Z} = r\cdot e^{-i\theta}$

vi) $e^{i\pi} + 1 = 0$

L.H.S. $= \cos\pi + i\sin\pi + 1$
$= -1 + 0 + 1$
$= 0$
$=$ R.H.S.

viii) $\arg z = -\arg \bar{z}$

Ex:- Find modulus & principal value of

H.W.

i) $z = \dfrac{(1+i\sqrt{3})^3 (1+i)^{-2} (\sqrt{3}+i)^{-1}}{2}$

Soln:-

$$z = \frac{1 + 3(i\sqrt{3}) + 3(i\sqrt{3})^2 + (i\sqrt{3})^3}{2(1+i)^2(\sqrt{3}+i)}$$

$$\dots [(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3]$$

$$= \frac{1 + i\cdot 3\sqrt{3} + i^2 \cdot 9 + i^3\, 3\sqrt{3}}{2(\cancel{1} + 2i + i^2)(\sqrt{3}+i)}$$

$$= \frac{1 + \cancel{i\,3\sqrt{3}} - 9 - \cancel{i\,3\sqrt{3}}}{4i(\sqrt{3}+i)} \quad \dots (i^2 = -1,\ i^3 = -i)$$

$$= \frac{-8}{4(i\sqrt{3} + i^2)} = \frac{-8}{4(-1+i\sqrt{3})}$$

$$= \frac{-8(-1-i\sqrt{3})}{4(-1+i\sqrt{3})(-1-i\sqrt{3})}$$

$$= -2\left[\frac{(-1-i\sqrt{3})}{(-1)^2 - (i\sqrt{3})^2}\right]$$

$$= -2\left[\frac{(-1-i\sqrt{3})}{1-(i^2\cdot 3)}\right] = 2\left[\frac{(1+i\sqrt{3})}{1+3}\right]$$

$$\therefore z = 2\left[\frac{1+i\sqrt{3}}{4}\right] = \frac{1}{2}(1+i\sqrt{3}) = \frac{1}{2} + \frac{i\sqrt{3}}{2}$$

$\therefore$ Here $x = \frac{1}{2}$ & $y = \frac{\sqrt{3}}{2}$

$\therefore r = |z| = \sqrt{x^2+y^2} = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}$

$= \sqrt{\frac{1}{4}+\frac{3}{4}} = \sqrt{\frac{4}{4}} = \sqrt{\frac{1}{1}}$

$\therefore r = |z| = \underline{\underline{1}}$

$z = (x, y) = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$

$\therefore$ It lies in first quadrant.

$\therefore \theta = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{\sqrt{3}/2}{1/2}\right) = \tan^{-1}(\sqrt{3})$
$= \frac{\pi}{3}$

$\therefore \theta = \underline{\underline{\frac{\pi}{3}}}$

ii) $-1+\sqrt{3}\,i$

Soln:- $z = -1 + i\sqrt{3}$

Here, $x = -1$ & $y = \sqrt{3}$

$\therefore r = |z| = \sqrt{x^2+y^2} = \sqrt{1+3} = \sqrt{4} = \underline{\underline{2}}$

$\therefore z = (x, y) = (-1, \sqrt{3})$ lies in 2^{nd} quadrant.

$\therefore \theta = \pi - \tan^{-1}\left|\frac{y}{x}\right| = \pi - \tan^{-1}\left|\frac{\sqrt{3}}{-1}\right| = \pi - \frac{\pi}{3}$

$= \frac{2\pi}{3}$

$\therefore \theta = \underline{\underline{\frac{2\pi}{3}}}$.

③ $\frac{(2-3i)(5+3i)}{3-2i}$

Sol^n $\Rightarrow$ Let $z = \frac{(2-3i)(5+3i)}{(3-2i)}$

$$= \frac{(10+6i-15i-9i^2)}{(3-2i)} = \frac{(10+9-9i)}{(3-2i)} \quad \ldots (\because i^2=-1)$$

$$= \frac{(19-9i)(3+2i)}{(3-2i)(3+2i)}$$

$$= \frac{(57+38i-27i-18i^2)}{(9-4i^2)}$$

$$= \frac{57+18+11i}{(9+4)} \quad \ldots (\because i^2=-1)$$

$$= \frac{75+11i}{13}$$

$\therefore \quad z = \frac{75}{13} + i\frac{11}{13}$

$\therefore$ Here, $x = \frac{75}{13}$ & $y = \frac{11}{13}$

$|z| = r = \sqrt{x^2+y^2} = \sqrt{\left(\frac{75}{13}\right)^2 + \left(\frac{11}{13}\right)^2} = \underline{\underline{\sqrt{34}}}$

As $(x,y) \equiv \left(\frac{75}{13}, \frac{11}{13}\right)$ lies in I^{st} quadrant.

$\therefore \theta = \tan^{-1}\frac{y}{x} = \tan^{-1}\left(\frac{11/13}{75/13}\right) = \underline{\underline{\tan^{-1}\left(\frac{11}{75}\right)}}$

④ $\frac{(3-i\sqrt{2})^2}{1+2i}$

Solⁿ :- Let $z = \dfrac{(3 - i\sqrt{2})^2}{1 + 2i}$

$$= \frac{9 - i6\sqrt{2} + i^2(2)}{(1+2i)}$$

$$= \frac{(7 - i \cdot 6\sqrt{2})(1-2i)}{(1+2i)(1-2i)}$$

$$= \frac{7 - 14i - i(6\sqrt{2}) + i^2(2)\cdot 6\sqrt{2}}{1 - 4i^2}$$

$$= \frac{7 - 12\sqrt{2} - i(14 + 6\sqrt{2})}{1+4}$$

$$= \frac{7 - 12\sqrt{2} - i(14 + 6\sqrt{2})}{5}$$

$$= \frac{7 - 12\sqrt{2}}{5} - i\left(\frac{14 + 6\sqrt{2}}{5}\right)$$

$\therefore \; z = \dfrac{7 - 12\sqrt{2}}{5} - i\left(\dfrac{14 + 6\sqrt{2}}{5}\right)$

Here, $x = \dfrac{7 - 12\sqrt{2}}{5}$ & $y = \dfrac{-(14 + 6\sqrt{2})}{5}$

$\therefore |z| = r = \sqrt{x^2 + y^2} = \sqrt{\left(\dfrac{7 - 12\sqrt{2}}{5}\right)^2 + \left(\dfrac{-(14 + 6\sqrt{2})}{5}\right)^2}$

$\therefore r = \sqrt{\dfrac{49 - 168\sqrt{2} + 288 + 196 + 168\sqrt{2} + 72}{25}} = \sqrt{\dfrac{\cancel{605}^{121}}{\cancel{25}_{5}}} = \underline{\underline{\dfrac{11}{\sqrt{5}}}}$

Now, $\theta = \tan^{-1}\left|\dfrac{y}{x}\right| = \tan^{-1}\left|\dfrac{-(14 + 6\sqrt{2})}{7 - 12\sqrt{2}}\right| =$

$\therefore \; \theta = \tan^{-1}\left|\dfrac{14 + 6\sqrt{2}}{12\sqrt{2} - 7}\right| = \tan^{-1}\left(\dfrac{14 + 6\sqrt{2}}{12\sqrt{2} - 7}\right)$

$\therefore$ We get, $\underline{\underline{r = \dfrac{11}{\sqrt{5}}}}$ & $\underline{\underline{\theta = \tan^{-1}\left(\dfrac{14 + 6\sqrt{2}}{12\sqrt{2} - 7}\right)}}$

★ Hyperbolic Functions :-

Def^n:- If x is real or complex then $\sinh x = \frac{e^x - e^{-x}}{2}$ is called hyperbolic sine of x & is denoted by $\sinh x$.

Similarly, $\cosh x = \frac{e^x + e^{-x}}{2}$

Other hyperbolic fu^{ns} are defined as,

$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$, $\operatorname{sech} x = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}}$

$\coth x = \frac{e^x + e^{-x}}{e^x - e^{-x}}$ & $\operatorname{cosech} x = \frac{2}{e^x - e^{-x}}$

Again we know that
Circular functions are,

$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$ & $\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$

⊙ Relationship between hyperbolic & Circular fu^{ns}:

1) $\sin(ix) = i\sinh x$ & $\sin(hx) = -i\sin(ix)$

Sol^n:- We have $\sin x = \frac{e^{ix} - e^{-ix}}{2i}$

$$\therefore \sin(ix) = \frac{e^{i^2x} - e^{-i^2x}}{2i} = \frac{e^{-x} - e^{x}}{2i^2}(i)$$

$$= i\left(\frac{e^{-x} - e^{x}}{-2}\right) = i\left(\frac{e^{x} - e^{-x}}{2}\right)$$

$\therefore \sin(ix) = i\sinh x$

& $\sin(hx) = \frac{1}{i}\sin(ix) = \frac{i}{i^2}\sin(ix)$

$\sin(hx) = -i\sin(ix)$

2) $\cos(ix) = \cosh x$ & $\cosh(ix) = \cos x$

Sol^n:- We have,

$\cos x = \frac{e^{ix}+e^{-ix}}{2}$

$\therefore \cos(ix) = \frac{e^{i^2x}+e^{-i^2x}}{2} = \frac{e^{-x}+e^{x}}{2}$

$\cos(ix) = \frac{e^{x}+e^{-x}}{2} = \cosh x$

& $\cosh(ix) = \frac{e^{ix}+e^{-ix}}{2} = \cos x$.

3) $\tanh(ix) = i\tan x$ & $\tan x = -i\tanh(ix)$

- Hyperbolic Identities :-

i) $\sinh(-x) = -\sinh x$ ii) $\cosh(-x) = \cosh x$

Sol^n:- i) since, $\sinh x = \frac{e^{x}-e^{-x}}{2}$

$\therefore \sinh(-x) = \frac{e^{-x}-e^{x}}{2} = -\left[\frac{e^{x}-e^{-x}}{2}\right]$

$\therefore \boxed{\sinh(-x) = -\sinh x}$

ii) $\cosh(-x) = \cosh x$

since, $\cosh x = \frac{e^{x}+e^{-x}}{2}$

$\therefore \cosh(-x) = \frac{e^{-x}+e^{x}}{2} = \frac{e^{x}+e^{-x}}{2}$

$\therefore \boxed{\cosh(-x) = \cosh x}$

iii) $e^{x} = \cosh x + \sinh x$

$\Rightarrow \cosh x + \sinh x = \frac{e^{x}+e^{-x}}{2} + \frac{e^{x}-e^{-x}}{2} = e^{x}$

iv) $e^{-x} = \cosh x - \sinh x$

v) $\cosh^2 x - \sinh^2 x = 1$

vi) $\operatorname{sech}^2 x + \tanh^2 x = 1$

vii) $\coth^2 x - \operatorname{cosech}^2 x = 1$

* Separation of Real & Imaginary Parts:-

i) $\sin(x+iy) = \sin x \cdot \cos(iy) + \cos x \cdot \sin(iy)$

$\therefore \sin(x+iy) = \sin x \cdot \cosh y + i\cos x \cdot \sinh y$

ii) $\cos(x+iy) = \cos x \cdot \cos(iy) - \sin x \cdot \sin(iy)$

$\therefore \cos(x+iy) = \cos x \cdot \cosh y - i \cdot \sin x \cdot \sinh y$

iii) $\sinh(x+iy) = \sinh x \cdot \cosh(iy) + \cosh x \cdot \sinh(iy)$

$= \sinh x \cdot \cos y + \cosh x \, (-i\sin i^2 y)$

$= \sinh x \cdot \cos y + i\cosh x \cdot \sin y$

$\therefore \sinh(x+iy) = \sinh x \cdot \cos y + i\cosh x \cdot \sin y$

iv) $\cosh(x+iy) = \cosh x \cdot \cosh(iy) + \sinh x \cdot \sinh(iy)$
$= \cosh x \cdot \cos y + \sinh x \cdot (-i\sin i^2 y)$
$= \cosh x \cdot \cos y + \sinh x (-i\sin(-y))$
$= \cosh x \cdot \cos y + i\sinh x \cdot \sin y$

$\therefore \cosh(x+iy) = \cosh x \cdot \cos y + i\sinh x \cdot \sin y$

- We use :-

$\sinh(x \pm y) = \sinh x \cdot \cosh y \pm \cosh x \cdot \sinh y$

$\cosh(x \pm y) = \cosh x \cdot \cosh y \pm \sinh x \cdot \sinh y$

★ Algebra of Complex Numbers :-

① Addition :- The sum of two complex no.s is given by

$z_1 + z_2 = (x_1 + iy_1) + (x_2 + iy_2)$

$\therefore z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2)$

to add two complex no.s we add their real parts & imaginary parts separately

② Subtraction :- The difference of two complex no.s is given by

$z_1 - z_2 = (x_1 + iy_1) - (x_2 + iy_2)$

$\therefore z_1 - z_2 = (x_1 - x_2) + i(y_1 - y_2)$

③ Multiplication :- While multiplying one complex no. by another complex no. we multiply them in usual manner & put $i^2 = -1$.

$$z_1 \cdot z_2 = (x_1 + iy_1) \cdot (x_2 + iy_2)$$

$$= x_1(x_2 + iy_2) + iy_1(x_2 + iy_2)$$

$$= x_1x_2 + ix_1y_2 + ix_2y_1 + i^2y_2y_1$$

$$\therefore z_1 \cdot z_2 = x_1x_2 + ix_1y_2 + ix_2y_1 - y_2y_1$$

$$\therefore z_1 \cdot z_2 = (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1)$$

(4) Division :- The quotient $\dfrac{x_1 + iy_1}{x_2 + iy_2}$ is not in standard form $x + iy$. To put the quotient in the std. form we multiply the numerator & denominator by the conjugate of the denominator as follows.

$$\therefore \frac{(x_1 + iy_1)(x_2 - iy_2)}{(x_2 + iy_2)(x_2 - iy_2)} = \frac{x_1x_2 - ix_1y_2 + ix_2y_1 - i^2y_1y_2}{x_2^2 - i^2y_2^2}$$

$$= \frac{(x_1x_2 + y_1y_2) + i(x_2y_1 - x_1y_2)}{x_2^2 + y_2^2}$$

$$\therefore \frac{x_1 + iy_1}{x_2 + iy_2} = \left(\frac{x_1x_2 + y_1y_2}{x_2^2 + y_2^2}\right) + i\left(\frac{x_2y_1 - x_1y_2}{x_2^2 + y_2^2}\right)$$

(5) Equality of complex numbers :- Two complex no.s $z_1 = x_1 + iy_1$ & $z_2 = x_2 + iy_2$ are said to be equal iff their real parts are equal i.e. $x_1 = x_2$ & also their imaginary parts are equal i.e. $y_1 = y_2$.

⑥ Order relation :- If x, y are two real no.s we know that there are three possibilities viz., either $x<y$ or $x=y$ or $x>y$ But if z_1 & z_2 are two complex no.s then there are only two possibilities either $z_1=z_2$ or $z_1 \neq z_2$.

Since complex no. consists of two no.s real & imaginary there is no order relation betn two complex numbers. s.t. $z_1<z_2$ or $z_1>z_2$ e.g. - $3+4i < 5+6i$ or $7+9i > 3+5i$ are meaningless statements.

★ **Power of i** :- In many problems we need various power of i we therefore note that

$i=\sqrt{-1}$, $i^2=-1$, $i^3=i^2 \cdot i=(-1)\cdot i=-i$.

$i^4=(i^2)^2=(-1)^2=1$, $i^5=i^4 \cdot i=1 \cdot i=i$

Similarly,

$$(i)^{47}=(i)^{46} \cdot i=(i^2)^{23} \cdot i=(-1)^{23} \cdot i=-i$$

$$i^{65}=i^{64} \cdot i=(i^2)^{32} \cdot i=(-1)^{32} \cdot i=i$$

$$i^{48}=(i^2)^{24}=(-1)^{24}=1$$

$$i^{66}=(i^2)^{33}=(-1)^{33}=-1$$

Thus the powers of i are $+i$ or $-i$ or $+1$ or -1 depending on the index of i.

Note :- Some authors define,

$z=(x, y)$ be a complex no.

then i) $z_1 + z_2 = (x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$

ii) $z_1 - z_2 = (x_1 - x_2, y_1 - y_2)$

iii) $z_1 \cdot z_2 = (x_1 \cdot x_2 - y_1 y_2, x_1 y_2 + x_2 y_1)$

iv) $\frac{z_1}{z_2} = \left(\frac{x_1 x_2 + y_1 y_2}{x_2^2 + y_2^2}, \frac{x_2 y_1 - x_1 y_2}{x_2^2 + y_2^2} \right)$

* Result :- Suppose z_1, z_2, z_3 belongs to the set S of complex numbers. Then

1) $z_1 + z_2$ & $z_1 \cdot z_2$ belong to S, Closure law.

2) $z_1 + z_2 = z_2 + z_1$, commutative law of addition

3) $z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3$, Associative law of addition

4) $z_1 \cdot z_2 = z_2 \cdot z_1$, commutative law of multiplica^n.

5) $z_1 (z_2 \cdot z_3) = (z_1 \cdot z_2) \cdot z_3$, Associative law of multiplication.

6) $z_1 (z_2 + z_3) = z_1 \cdot z_2 + z_1 \cdot z_3$, Distributive law.

7) $z_1 + 0 = 0 + z_1$, $1 \cdot z_1 = z_1 \cdot 1 = z_1$.

0 is called identity w.r.t. addition.
1 is called identity w.r.t. multiplication.

8) For any complex number z_1 there is unique number z in S such that $z + z_1 = 0$. [z is called the inverse

of z_1 w.r.t. addition & is denoted by $-z_1$].

9) For any $z_1 \neq 0$ there is unique no. z in S s.t. $z_1 \cdot z = z \cdot z_1 = 1$ [z is called inverse of z_1 w.r.t. multiplication] & it is denoted by $\frac{1}{z_1}$ or z_1^{-1}

In general, any set such as S whose members satisfy the above is called field.

Function of Complex Variable :-

Let S be a set of complex numbers. A function f defined on S is a rule that assign to each z in S a complex no. w. The number w is called the value of f at z and it is denoted by $f(z)$. i.e. $w = f(z)$. The set S is called domain of function of complex variable.

i.e. $w = f(z)$

$w = u + iv$

$w = u(x, y) + iv(x, y)$

where u is called real part & v is called imaginary part of the complex fun $f(z)$.

e.g.- 1) $w = z^2 + 1$

$= (x + iy)^2 + 1$

$= x^2 + 2ixy + i^2y^2 + 1$

$= (x^2 - y^2 + 1) + i2xy = u(x, y) + i\,v(x, y)$

w is complex valued funⁿ.

2) $w = \sin z$

$= \sin(x+iy)$

$= \sin x \cdot \cos iy + \cos x \cdot \sin iy$

$= \sin x \cdot \cosh y + i\cos x \cdot \sinh y$

$= u(x,y) + i\, v(x,y).$

3) If f is defined on the set $z \neq 0$ by means of the equation $w = \frac{1}{z}$.

Suppose $w = u+iv$ is the value of fu^n f at $z = x+iy$ so that

$u+iv = f(x+iy).$

$\therefore f(z) = u(x,y) + i\,v(x,y)$ .... ①

If polar co-ors. r & θ instead of x & y are used then

$u+iv = f(re^{i\theta})$

where, $w = u+iv$ & $z = r.e^{i\theta}$

In this case we may write

$f(z) = u(r,\theta) + iv(r,\theta)$ .... ②

4) If $f(z) = z^2$ then

$f(x+iy) = (x+iy)^2 = x^2 - y^2 + i2xy$

Here,

$u(x,y) = x^2 - y^2$, $v(x,y) = 2xy$

When we use polar co-ordinates.

$f(r.e^{i\theta}) = (re^{i\theta})^2 = r^2 e^{i2\theta}$

$= r^2[\cos 2\theta + i\sin 2\theta]$

$= r^2[\cos 2\theta] + i\, r^2[\sin 2\theta]$

$\therefore$ Here $u(r,\theta) = r^2 \cos 2\theta$, $v(r,\theta) = r^2 \sin 2\theta$

If in either eq^{ns} ① & ② the fu^n; v always has value zero. Then the value of f is always real.
i.e. f is real valued fu^n of complex variable.

Ex:-5) $f(z) = |z|^2 = x^2+y^2+i0$
If n is zero or positive integer & if $a_0, a_1, a_2, \ldots\ldots, a_n$ are complex constants where $a_n \neq 0$ the function.

$$P(z) = a_0 + a_1 z + a_2 z^2 + \ldots\ldots + a_n z^n$$

is a polynomial of degree n. Here sum has finite no. of terms & domain is entire z-plane.
Again $\frac{P(z)}{g(z)}$ of polynomials are called rational functions & are defined at each point z where $Q(z) \neq 0$.

Ex:- ① For each of the fu^{ns} below, describe the domain.

ⓐ $f(z) = \frac{1}{z^2+1}$ ⓒ $f(z) = \frac{z}{z+\bar{z}}$

ⓑ $f(z) = Arg(1/z)$ ⓓ $f(z) = \frac{1}{1-|z|^2}$

Sol^n:- ⓐ The function $f(z) = \frac{1}{z^2+1}$ is defined everywhere in the finite plane except at the pts. $z = \pm i$ where $z^2+1 = 0$.

(b) The fu^n $f(z) = Arg(1/z)$ is defined throughout the entire plane except for the point $z=0$.

(c) The fu^n $f(z) = \dfrac{z}{z+\bar{z}}$ is defined everywhere in the finite plane except for the imaginary axis.

This is because $z+\bar{z} = 0 \Rightarrow x+iy+x-iy=0$
$\Rightarrow 2x = 0$
$\Rightarrow x = 0 \Rightarrow Re(z)$

*

(d) The fu^n $f(z) = \dfrac{1}{1-|z|^2}$ is defined everywhere in the finite plane except on $|z|^2 - 1 \Rightarrow |z|=1$ i.e. Circle $|z| = 1$.

Ex:- ② Write the function $f(z) = z^3+z+1$ in the
H.W. form $f(z) = u(x,y) + iV(x,y)$
Ans:- $f(z) = (x^3 - 3xy^2 + x + 1) + i(3x^2y - y^3 + y)$

Sol^n:- $f(z) = z^3 + z + 1$

$= (x+iy)^3 + (x+iy) + 1$

$= x^3 + 3x^2(iy) + 3x(iy)^2 + (i^3y^3) + x + iy + 1$

$= x^3 + i(3x^2y) - 3xy^2 - iy^3 + x + iy + 1$

$\therefore f(z) = (x^3 - 3xy^2 + x + 1) + i(3x^2y - y^3 + y)$

Ex:-③ Suppose that $f(z) = x^2 - y^2 - 2y + i(2x - 2xy)$ where $z = x+iy$. Use the expression $x = \frac{z+\bar{z}}{2}$, $y = \frac{z-\bar{z}}{2i}$ to write $f(z)$ in terms of z & simplify the result.

Sol^n:- We have to simplify given result to write $f(z)$ in terms of z.

$$\therefore f(z) = x^2 - y^2 - 2y + i(2x - 2y)$$

$$= \frac{(z+\bar{z})^2}{4} + \frac{(z-\bar{z})^2}{4} - 2\left(\frac{z-\bar{z}}{2i}\right) + i\left[2\left(\frac{z+\bar{z}}{2}\right)\right] - 2i\left(\frac{z+\bar{z}}{2}\right)\left(\frac{z-\bar{z}}{2i}\right)$$

$$= \frac{(z+\bar{z})^2}{4} + \frac{(z-\bar{z})^2}{4} + i(z-\bar{z}) + i(z+\bar{z}) - \frac{(z+\bar{z})(z-\bar{z})}{2}$$

$$= \frac{z^2}{2} + \frac{\bar{z}^2}{2} + 2iz - \frac{z^2}{2} + \frac{\bar{z}^2}{2}$$

$$= \bar{z}^2 + 2iz$$

$$\therefore \boxed{f(z) = \bar{z}^2 + 2iz}$$

Ex:- Write down function $f(z) = z + \frac{1}{z}$ $(z \neq 0)$
H.W.

in the form $f(z) = u(r,\theta) + iv(r,\theta)$.

$\Rightarrow$ Ans:-

$$f(z) = \left(r + \frac{1}{r}\right)\cos\theta + i\left(r - \frac{1}{r}\right)\sin\theta$$

Sol^n:- $f(z) = z + \frac{1}{z} \quad (z \neq 0)$

$$= r(\cos\theta + i\sin\theta) + \frac{1}{r(\cos\theta + i\sin\theta)}$$

$$= r\cos\theta + ir\sin\theta + \frac{(\cos\theta - i\sin\theta)}{r(\cos\theta + i\sin\theta)(\cos\theta - i\sin\theta)}$$

$$= r\cos\theta + ir\sin\theta + \frac{1}{r} \cdot \frac{(\cos\theta - i\sin\theta)}{(\cos^2\theta - i^2\sin^2\theta)}$$

$$= r\cos\theta + i \cdot r \cdot \sin\theta + \frac{1}{r}(\cos\theta - i\sin\theta)$$

$$\ldots\ldots (\because \cos^2\theta + \sin^2\theta = 1)$$

$$= r\cos\theta + i \cdot r \cdot \sin\theta + \frac{1}{r}\cos\theta - i \cdot \frac{1}{r}\sin\theta$$

$\therefore f(z) = \left(r + \frac{1}{r}\right)\cos\theta + i\left(r - \frac{1}{r}\right)\sin\theta$

Hence we get $f(z)$ in the form of
$f(z) = u(r, \theta) + iv(r, \theta)$.

★ **Limit** :- Let a fu^n f be defined at all points z in some deleted nhd of z_0. The statement that the limit of $f(z)$ as z approaches z_0 in a number w_0.

i.e. $\lim_{z \to z_0} f(z) = w_0 \quad \ldots\ldots (1)$

i.e. for each positive number ϵ there is positive number δ s.t.
$|f(z) - w_0| < \epsilon$, whenever $0 < |z - z_0| < \delta$.

★ **Continuity :-** A function f is continuous at a pt. z_0 if all three conditions are satisfied.

① $\lim\limits_{z \to z_0} f(z)$ exists.

② $f(z_0)$ exists.

③ $\lim\limits_{z \to z_0} f(z) = f(z_0)$

Observe that statement ③ actually contains statements ① & ② statement ③ says that for each positive no. ϵ, there is a positive no. δ s.t.

$$|f(z) - f(z_0)| < \epsilon \quad \text{whenever} \quad |z - z_0| < \delta.$$

★ **Derivative :-** Let f be a function. The derivative of f at z_0 is the limit

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \quad \ldots\ldots ①$$

& the fu^n f is said to be differentiable at z_0 when $f'(z_0)$ exists.

By expressing the variable z in def ① in terms of new complex variable $\Delta z = z - z_0$ $(z \neq z_0)$.

One can write the def^n as,

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} \quad \text{.......} ②$$

Because f is defined throughout a nbd of z_0 the number; $f(z_0 + \Delta z)$ is always defined for $|\Delta z|$ sufficiently small.

When taking form ② of the defn of derivative, we introduce $\Delta w = f(z + \Delta z) - f(z)$ which denote the change in the value $w = f(z)$ of f corresponding to a change Δz in the pt. at which f is evaluated.

Then if we write $\frac{dw}{dz}$ for $f'(z)$.

eqn ② becomes;

$$\frac{dw}{dz} = \lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z} \quad \text{......} ③$$

Ex: ① If $f(z) = z^2$ at any pt. z.

$$\lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z} = \lim_{\Delta z \to 0} \frac{(z + \Delta z)^2 - z^2}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{\cancel{z^2} + 2z\Delta z + \Delta z^2 - \cancel{z^2}}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{\Delta z\,(2z\Delta z) + (\Delta z)\cdot(\Delta z)}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{\cancel{\Delta z}\,(2\cdot z + \Delta z)}{\cancel{\Delta z}}$$

$$= \lim_{\Delta z \to 0} (2z + \Delta z) = 2z$$

$$\therefore \frac{dw}{dz} = 2z \quad \text{or} \quad f'(z) = 2z.$$

Ex:- $f(z) = \bar{z}$

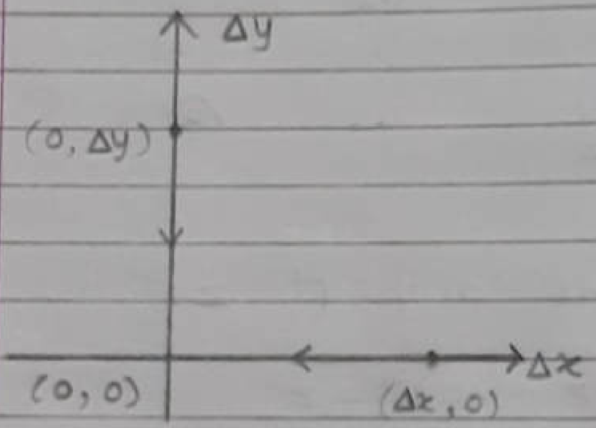
Sol^n:- $\dfrac{\Delta W}{\Delta z} = \dfrac{\overline{(z+\Delta z)} - \bar{z}}{\Delta z} = \dfrac{\bar{z} + \overline{\Delta z} - \bar{z}}{\Delta z}$

$$\frac{\Delta W}{\Delta z} = \frac{\overline{\Delta z}}{\Delta z} \quad \ldots\ldots (1)$$

If the limit of $\dfrac{\Delta W}{\Delta z}$ exists. It can be found by letting the point $\Delta z = (\Delta x, \Delta y)$ approaches the origin $(0,0)$ in the Δz plane.

In particular as Δz approaches $(0,0)$ horizontally through the pts. $(\Delta x, 0)$ on the real axis. (see fig.)

$$\overline{\Delta z} = \overline{\Delta x + i0}$$
$$= \overline{\Delta x} + \overline{i0}$$
$$\overline{\Delta z} = \Delta x + i0 = \Delta z$$



In this case (1) becomes

$$\frac{\Delta W}{\Delta z} = \frac{-\Delta z}{\Delta z} = -1$$

& limit of $\dfrac{\Delta W}{\Delta z}$ exists & equal to -1. Since limits are not unique it follows that $\dfrac{dw}{dz}$ does not exist anywhere

i.e. Thus $f(z) = \bar{z}$ is not differentiable at any point.

Ex:- Prove that the fu^n $f(z) = |z|^2$ is contineous everywhere but no where differentiable except at the origin.

Sol^n:- We have $f(z) = |z|^2$

Now $|z| = \sqrt{x^2+y^2}$

$\therefore |z|^2 = x^2 + y^2$

but x^2+y^2 is continuous, everywhere.

It follows that $|z|^2$ is everywhere continuous.

Now $$\frac{\Delta w}{\Delta z} = \frac{|z+\Delta z|^2 - |z|^2}{\Delta z}$$

$$= \frac{(z+\Delta z)(\bar{z}+\overline{\Delta z}) - z\bar{z}}{\Delta z}$$

$$= \frac{z\bar{z} + \Delta z\,\bar{z} + \overline{\Delta z}\,\Delta z + z\overline{\Delta z} - z\bar{z}}{\Delta z}$$

$$= \bar{z} + \overline{\Delta z} + z\cdot\frac{\overline{\Delta z}}{\Delta z} \quad \ldots\ldots (1)$$

As in Ex. (2) where horizontal & vertical approaches of Δz toward the origin gave as,

$\overline{\Delta z} = \Delta z$ & $\overline{\Delta z} = -\Delta z$ resp.

We have the expressions from (1)

$\frac{\Delta w}{\Delta z} = \bar{z} + \Delta z + z$, when $\Delta z = (\Delta x, 0)$ ...... (2)

& $\frac{\Delta w}{\Delta z} = \bar{z} - \Delta z - z$, when $\Delta z = (0, \Delta y)$ ...... (3)

$\therefore \lim\limits_{\Delta z \to 0} \frac{\Delta w}{\Delta z} = \bar{z} + z$ .... [From (2)]

& from (3),

$\lim\limits_{\Delta z \to 0} \frac{\Delta w}{\Delta z} = \bar{z} - z$

Then $\frac{dw}{dz}$ cannot exist when $z \neq 0$ i.e.

limits are not unique.

When $z=0$; we observe that expression (1) reduces to

$$\frac{\Delta w}{\Delta z} = \overline{\Delta z}$$

When $z=0$; we conclude that $\frac{dw}{dz}$ exists only when $z=0$ it's value being 0.

$\therefore$ function is differentiable at $z=0$.

* Differentiation Formulas :-

(1) $\frac{d}{dz}(c) = 0$, $\frac{d}{dz}(z) = 1$, $\frac{d}{dz}[c.f(z)] = c\,f'(z)$.

(2) If n is +ve integer then

$$\frac{d}{dz} z^n = n z^{n-1}$$

if n is -ve integer then $z \neq 0$.

(3) $\frac{d}{dz}[f(z) + g(z)] = f'(z) + g'(z)$

(4) $\frac{d}{dz}[f(z) \cdot g(z)] = f(z) \cdot g'(z) + g(z)\, f'(z)$

(5) $\frac{d}{dz}\left[\frac{f(z)}{g(z)}\right] = \frac{g(z)\, f'(z) - f(z)\, g'(z)}{[g(z)]^2}$ when $g(z) \neq 0$

(6) If $F(z) = g[f'(z)]$ then

$$F'(z_0) = g'[f(z_0)]\, f'(z_0)$$

⑦ If we write $w = f(z)$ & $W = g(w)$ so that $W = F(z)$

$$\frac{dW}{dz} = \frac{dW}{dw} \cdot \frac{dw}{dz}$$ (page 61)

Ex:- ① Find the derivative of $(2z^2+i)^5$.

Write $w = 2z^2 + i$ & $W = w^5$ then

$$\frac{d}{dz}(2z^2+i)^5 = 5w^4 \, 4z$$
$$= 20w^4 z$$
$$= 20z(2z^2+i)^4$$ using formula⑦

Ex:- Find $f'(z)$ when

a) $f(z) = 3z^2 - 2z + 4$

Sol^n:- $f'(z) = \frac{d}{dz}(3z^2 - 2z + 4) = \underline{\underline{6z - 2}}$.

b) $f(z) = (1-4z^2)^3$

Sol^n:- $f'(z) = 3(1-4z^2)^2 \cdot \frac{d}{dz}(1-4z^2)$

$$= 3(1-4z^2)^2 \cdot (-8z)$$

Ans:- $\therefore f'(z) = -24 \cdot z \cdot (1-4z^2)^2$.

H.W. c) $f(z) = \frac{z-1}{2z+1}$ $\left(z \neq -\frac{1}{2}\right)$

Sol^n:- $f'(z) = \frac{(2z+1)(1) - (z-1)(2)}{(2z+1)^2}$

$$= \frac{\cancel{2z} + 1 - \cancel{2z} + 2}{(2z+1)^2}$$

Ans:- $\therefore f'(z) = \frac{3}{(2z+1)^2}$.

d) $f(z) = \frac{(1+z^2)^4}{z^2}, \quad (z \neq 0)$

Sol^n:- $f'(z) = \frac{(z^2)\left[4(1+z^2)^3 \cdot (2z)\right] - (1+z^2)^4 \cdot (2z)}{(z^2)^2}$

$$= \frac{8z^3 (1+z^2)^3 - (1+z^2)^4 \cdot (2z)}{z^4}$$

$$= \frac{2 \cdot z (1+z^2)^3 \left[4z^2 - 1 + z^2\right]}{z^4}$$

Ans:- $\therefore f'(z) = \frac{2(1+z^2)^3 (3z^2-1)}{z^3}$

Ex:- Use def^n $\frac{dw}{dz} = \lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z}$ of derivative to give a direct proof that $\frac{dw}{dz} = \frac{-1}{z^2}$ when $w = \frac{1}{z}$, $(z \neq 0)$

Proof:- If $f(z) = \frac{1}{z}$ $(z \neq 0)$ then

$$\Delta w = f(z + \Delta z) - f(z)$$

$$= \frac{1}{z + \Delta z} - \frac{1}{z} = \frac{-\Delta z}{z(z + \Delta z)}$$

Hence

$$f'(z) = \lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{-\Delta z}{z \cdot (z + \Delta z) \cdot \Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{-1}{z(z + \Delta z)} = \frac{-1}{z^2}$$

★ Cauchy - Riemann Equations :-

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

i.e. $u_x = v_y$ i.e. $u_y = -v_x$

- **Analytic Function :-** A function f of the complex variable z is analytic at a point z_0 if its derivative $f'(z)$ exists not only at z_0 but at every point z in some nhd of z_0. It is analytic in a domain of the z-plane if it is analytic at every pt. in that domain.
  The terms "regular" and "holomorphic" are sometimes introduced to denote analyticity in domains of certain classes.

- Theorem :- The necessary condition for $f(z)$ to be analytic.

Statement :- Suppose that $f(z) = u(x,y) + iv(x,y)$ and that $f'(z)$ exists at a point $z_0 = x_0 + iy_0$. Then the first-order partial derivatives of u & v must satisfy the Cauchy - Riemann eq^{ns} - $u_x = v_y$, $u_y = -v_x$

i.e. $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$, $\frac{\partial u}{\partial y} = \frac{-\partial v}{\partial x}$ & Also

$f'(z_0)$ can be written as $f'(z_0) = u_x + iv_x$ where these partial derivatives are to be evaluated at (x_0, y_0).