Elementary Functions

1) The exponential function:-

We define $e^z = e^{x+iy} = e^x \cdot e^{iy}$ ... (1)

$e^z = e^x [\cos y + i \sin y]$ ... (2)

Here y is taken in radians.
Now, e^z reduces to usual exponential function in calculus when $y=0$.
The positive n^{th} root $\sqrt[n]{e}$ of e is assigned to e^x when $x = \frac{1}{n}$ $(n = 2, 3, 4, \ldots$

Expression (1) tells us that the complex exponential fu^n e^z is also $\sqrt[n]{e}$ where $z = \frac{1}{n}$ $(n = 2, 3, \ldots\ldots)$

- Properties :-

(1) $e^{z_1} \cdot e^{z_2} = e^{z_1 + z_2}$

(2) $\frac{e^{z_1}}{e^{z_2}} = e^{z_1 - z_2}$

(3) $\frac{d}{dz}(e^z) = e^z$ everywhere in the z-plane

& e^z is entire function, $e^z \neq 0$ for any z.
Now $e^z = e^x [\cos y + i \sin y]$
$\therefore$ $u = e^x \cdot \cos y$ & $v = e^x \cdot \sin y$

& their Partial derivatives are everywhere continuous & satisfy C.R. eq^{ns}.

$u_x = e^x \cos y$ & $v_y = e^x \cos y$

$\therefore u_x = v_y$

& $u_y = -e^x . \sin y$ & $v_x = e^x . \sin y$

$\therefore u_y = -v_x$

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$= e^x \cos y + i e^x . \sin y$

$= e^x [\cos y + i \sin y]$

$= e^x . e^{iy}$

$= e^{x+iy}$

$f'(z) = e^z$

Again writing (1) in the form

i.e. $e^z = e^x . e^{iy} = e^x [\cos y + i \sin y]$

i.e. $e^z = \rho . e^{i\phi}$ where $\rho = e^x$ & $\phi = y$.

$\therefore |e^z| = e^x$ & $\arg(e^z) = y + 2n\pi$

$, (n = 0, \pm 1, \pm 2, \dots)$

$[z = re^{i\theta}, |z| = r$ & $\theta = \phi]$

Again $e^{z+2\pi i} = e^z . e^{2\pi i}$ & $e^{2\pi i} = 1$.

e^z is periodic with a pure imaginary period of $2\pi i$

$\therefore e^{z+2\pi i} = e^z$

Note that e^x is always +ve, e^z can be -ve

$e^{i\pi} = (-1)$

$\therefore e^{i(2n+1)\pi} = e^{i2n\pi} . e^{i\pi} = 1 . (-1) = -1$

$, (n = 0, \pm 1, \dots\dots)$

Ex:- To find numbers $z = x+iy$ s.t. $e^z = 1+i$.

Sol^n $\Rightarrow$ Let $e^z = 1+i$ ....(1)

We write (1) as

$e^x . e^{iy} = 1+i$

$e^x . e^{iy} = \sqrt{2} . e^{i\pi/4}$ .... $\left[\begin{array}{l} |z| = \sqrt{1+1} = \sqrt{2} \\ y = \phi = \tan^{-1}(1) \\ \quad = \pi/4 \end{array} \right]$

Using equality of two non-zero complex numbers in exponential form.

$\therefore e^x = \sqrt{2}$ & $y = \frac{\pi}{4} + 2n\pi$, $(n = 0, \pm 1, \pm 2, ...)$

Taking $\log_e n = \ln$,

$x = \ln\sqrt{2} = \ln 2^{1/2} = \frac{1}{2}\ln 2$

& $y = \left(2n + \frac{1}{4}\right)\pi$, $(n = 0, \pm 1, \pm 2, ...)$

So that

$z = x + iy$

$z = \frac{1}{2}\ln 2 + \left(2n + \frac{1}{4}\right)\pi i$

Ex:- (1) show that

(a) $\exp(2 \pm 3\pi i) = -e^2$

Sol^n :-

$\exp(2 \pm 3\pi i) = e^{2 \pm 3\pi i} = e^2 . e^{\pm 3\pi i}$

$= e^2 [\cos 3\pi \pm i \sin 3\pi]$

$= e^2 [(-1)^3 \pm 0]$ .... $[\because \cos n\pi = (-1)^n$ & $\sin n\pi = 0$, $n = 0, \pm 1, \pm 2, ...]$

$= -e^2$

(b) $\exp\left(\frac{2 + \pi i}{4}\right) = \sqrt{\frac{e}{2}}(1+i)$

Solⁿ:- $e^{\frac{2+\pi i}{4}} = e^{1/2} \cdot e^{i\pi/4} = \sqrt{e}\,[\cos\pi/4 + i\sin\pi/4]$

$$= \sqrt{e}\left[\frac{1}{\sqrt{2}} + \frac{i\,1}{\sqrt{2}}\right]$$

$$= \sqrt{e/2}\ (1+i)$$

$$\therefore e^{\frac{2+\pi i}{4}} = \sqrt{\frac{e}{2}}\,(1+i)$$

© $\exp(z+\pi i) = -\exp(z)$

Solⁿ:-

$$\exp(z+\pi i) = e^{z} \cdot e^{\pi i}$$
$$= e^{z} \cdot [\cos\pi + i\sin\pi]$$
$$= e^{z} \cdot [-1+0] = -e^{z}$$
$$\therefore \exp(z+\pi i) = -\exp(z).$$

② State why funⁿ $f(z) = 2z^2 - 3 - ze^{z} + e^{-z}$ is entire.

Solⁿ:- $f(z) = 2(x+iy)^2 - 3 - (x+iy)e^{(x+iy)} + e^{-(x+iy)}$

$$f(z) = 2(x^2 - y^2 + i2xy) - 3 - x \cdot e^{x} \cdot e^{iy} + iy \cdot e^{x} \cdot e^{iy} + e^{-x} \cdot e^{-iy}$$

$$= 2x^2 - 2y^2 + i4xy - 3 - xe^{x} \cdot e^{iy} + iye^{x} \cdot e^{iy} + e^{-x} \cdot e^{-iy}$$

Satisfies C-R. eqⁿˢ or It is polynomial. Hence it is entire.

③ Use C.R. eqⁿˢ & to show that the fuⁿ $f(z) = e^{\bar{z}} = \exp(\bar{z})$ is not analytic anywhere

:- $\exp(\bar{z}) = \exp(x-iy) = e^x . e^{-iy} = e^x(\cos y - i\sin y)$

where $z = x+iy$

$\therefore \exp(\bar{z}) = u+iv$

Here $u = e^x \cos y$ & $v = -e^x . \sin y$

$\therefore u_x = e^x . \cos y$ & $v_y = -e^x . \cos y$

& $u_y = -e^x . \sin y$ & $v_x = -e^x . \sin y$

$\Rightarrow u_x = v_y \Rightarrow e^x . \cos y = -e^x . \cos y = 2e^x \cos y = 0$

$\Rightarrow \cos y = 0$ ....(1)

&

$u_y = -v_x \Rightarrow -e^x . \sin y = e^x . \sin y \Rightarrow 2e^x . \sin y = 0$

$\Rightarrow \sin y = 0$ ....(2)

But there is no value of y satisfying the pair of eq^{ns} (1) & (2). We may conclude that since the C.R. eq^{ns} fail to be satisfied anywhere.

$\therefore$ The fu^n $\exp(\bar{z})$ is not analytic anywhere.

(4) Show in two ways that the fu^n $f(z) = \exp(z^2)$ is entire. What is its derivative?

Sol^n :- The fu^n $\exp(z^2)$ is entire since it is composition of entire forms z^2 & $\exp(z)$ & the chain rule for derivatives tells us that

$$\frac{d}{dz}\exp(z^2) = \exp(z^2)\,\frac{d}{dz}(z^2) = 2z\cdot\exp(z^2)$$

Alternatively one can show that $\exp(z^2)$ is entire by writing.

$$\exp(z^2) = \exp[x+iy]^2 = \exp[x^2-y^2+2ixy]$$

$$= \exp(x^2-y^2)\cdot\exp(i2xy)$$

$$= e^{x^2-y^2}\cdot[\cos 2xy + i\sin 2xy]$$

$$= \underbrace{\exp(x^2-y^2)\cdot\cos 2xy}_{u} + i\,\underbrace{\exp(x^2-y^2)\cdot\sin 2xy}_{v}.$$

& using C.R. eq^{ns}.

$$u_x = 2x\exp(x^2-y^2)\cos(2xy) - 2y\exp(x^2-y^2)\cdot\sin(2xy) = v_y$$

&

$$u_y = -2y\exp(x^2-y^2)\cdot\cos(2xy) - 2x\exp(x^2-y^2)\sin(2xy) = -v_x$$

Further

$$\frac{d}{dz}(e^{z^2}) = u_x + iv_x = 2(x+iy)[\exp(x^2-y^2)\cdot\cos(2xy) + i\exp(x^2-y^2)\cdot\sin(2xy)]$$

$$= 2z\exp(z^2)$$

Ex:- (5) Write $|\exp(2z+i)|$ & $|\exp(iz^2)|$ in terms of x & y. Then show that $|\exp(2z+i)+\exp(iz^2)| \le e^{2x}+e^{-2yx}$.

Sol^n:- First we write

$$|\exp(2z+i)| = |\exp(2x+i(2y+1))| = e^{2x}$$

& $$|\exp(iz^2)| = |\exp(-2xy+i(x^2-y^2))| = e^{-2xy}$$

Since
$|\exp(2z+i)+\exp(iz^2)| \le |\exp(2z+i)|+|\exp(iz^2)|$

$\Rightarrow |\exp(2z+i)+\exp(iz^2)| \le e^{2x}+e^{-2xy}$

Ex:- ⑥ Show that $|\exp(z^2)| \le \exp(|z|^2)$

Sol^n:- We write,

$|\exp(z^2)| = |\exp(x+iy)^2| = |\exp(x^2-y^2)+i2xy|$
$= \exp(x^2-y^2)$

&

$\exp(|z|^2) = \exp(x^2+y^2)$

Since $x^2-y^2 < x^2+y^2$

$\Rightarrow \exp(x^2-y^2) < \exp(x^2+y^2)$

$\Rightarrow |\exp(z^2)| < \exp|z|^2$

⑦ Find all values of z such that

ⓐ $e^z = -2$ ⓑ $e^z = 1+\sqrt{3}i$ ⓒ $\exp(2z-1)=1$

Sol^n:- ⓐ Write $e^z = -2$

$\Rightarrow e^x \cdot e^{iy} = -2 = (-1)2$
$= 2e^{i\pi}$

$\Rightarrow e^x=2$ & $y=\pi+2n\pi$, $(n=0,\pm1,\pm2,\dots$

i.e.

$x=\ln(2)$ & $y=\pi(2n+1)$

Hence,

$z=\ln(2)+i(2n+1)\pi$, $[n=0,\pm1,\pm2,\dots$

ⓑ $e^z = 1+\sqrt{3}\,i$

$e^x \cdot e^{iy} = 2 \cdot e^{i(\pi/3)}$

$\therefore e^x=2$ & $y=\frac{\pi}{3}+2n\pi$

i.e. $x = \ln(2)$ & $y = \frac{\pi}{3} + 2n\pi = \left(2n + \frac{1}{3}\right)\pi$

$[n = 0, \pm 1, \pm 2, \ldots]$

Hence

$$z = \ln(2) + i\left(2n + \frac{1}{3}\right)\pi \quad , [n = 0, \pm 1, \pm 2, \ldots]$$

(c) We write $\exp(2z - 1) = 1$, as

$$e^{2z-1} = e^{2(x+iy)-1}$$
$$= e^{2x-1} \cdot e^{i2y}$$

$\therefore e^{2x-1} \cdot e^{i2y} = 1$

$\Rightarrow e^{2x-1} \cdot e^{i2y} = 1 \cdot e^{i0}$

$\Rightarrow e^{2x-1}$ & $\cancel{2}y = 0 + \cancel{2}n\pi$ $[n = 0, \pm 1, \pm 2, \ldots]$

$\Rightarrow (2x - 1)\ln(e) = \ln(1)$ & $y = n\pi$

$\Rightarrow (2x - 1) = 0$

$\Rightarrow x = \frac{1}{2}$ & $y = n\pi$

$\therefore z = \frac{1}{2} + i(n\pi)$ $(n = 0, \pm 1, \pm 2, \ldots)$

* The Logarithmic function :-

Let $e^w = z$ ....① is any non-zero complex no.

We write $z = re^{i\theta}$ $(-\pi < \theta \le \pi)$ & $w = u + iv$

eq^n ① becomes;

$e^u \cdot e^{iv} = re^{i\theta}$

$\Rightarrow e^u = r$ & $v = \theta + 2n\pi$

where n is any integer.

since,

$e^u = r$

$\Rightarrow u \cdot \log e = \log r$

$\Rightarrow u = \log r$ it follows that

eq^n ① satisfied iff w has one of the values

$[e^w = z \Rightarrow w = \log z]$

$w = \ln r + i(\theta + 2n\pi)$, $(n = 0, \pm 1, \pm 2, \ldots)$

Thus we write

$\therefore \log z = \ln(r) + i(\theta + 2n\pi)$ .....②

[log z is multivalued fu^n with infinitely many values.]

eq^n ① tells us that

$e^{\log z} = z$, $(z \neq 0)$ .....③

eq^n ② as the def^n of the logarithmic function of non-zero complex variable $z = r.e^{i\theta}$.

Ex:- ① If $z = -1-\sqrt{3}\,i$ then $r=2$ & $\theta = -\frac{2\pi}{3}$.

Hence $\log(-1-\sqrt{3}i) = \ln 2 + i\left(-\frac{2\pi}{3} + 2n\pi\right)$

$$= \ln 2 + 2\left(n - \frac{1}{3}\right)\pi i$$

$(n = 0, \pm 1, \pm 2, \ldots)$.

It should be emphasized that it is not true that the left-hand side of eqn ③ with the order of the exponential & logarithmic funs reversed reduces to just z. More precisely, since eqn ② can be written

$$\log z = \ln|z| + i\arg(z)$$

& since

$|e^z| = e^x$ and $\arg(e^z) = y + 2n\pi$ $(n=0,\pm1,\pm2,\ldots)$

when $z = x+iy$, we know that

$$\log(e^z) = \ln|e^z| + i\arg(e^z) = \ln e^x + i(y+2n\pi)$$

$$= (x+iy) + 2n\pi i$$

$(n = 0, \pm 1, \pm 2, \ldots)$

i.e.

$$\log(e^z) = z + 2n\pi i \quad (n=0,\pm1,\pm2,\ldots) \quad \ldots ④$$

The principal value of $\log z$ is the value obtained from eqn ② when $n=0$ there & is denoted by $\mathrm{Log}\, z$. Thus

$$\mathrm{Log}\, z = \ln r + i\theta \quad \ldots ⑤$$

Note that $\text{Log}\, z$ is well defined & single valued when $z \neq 0$ & that

$$\log z = \text{Log}\, z + 2n\pi i \quad (n = 0, \pm 1, \pm 2, \ldots) \quad \ldots\ldots (6)$$

It reduces to the usual logarithm in calculus when z is +ve real no. $z = r$.

Write $z = re^{i0}$. Hence eq^n (5) becomes;

$$\text{Log}\, z = \ln r.$$

i.e. $\text{Log}\, r = \ln r.$

Ex:- From (2), $\log z = \log(r) + i(\theta + 2n\pi) \ldots$

we find that

$[z = 1 + i0, \quad |z| = r = 1, \quad \theta = 0 \Rightarrow 1^{st}$ quadrant$]$

$$\therefore \log 1 = \ln 1 + i(0 + 2n\pi)$$

$$\therefore \log 1 = \ln 1 + i\underline{2n\pi}, \quad (n = 0, \pm 1, \pm 2, \ldots)$$

$$\therefore \log 1 = 0 + 0$$

$$\therefore \log 1 = 0$$

&

$$\log(-1) = \ln(1) + i(\pi + 2n\pi)$$

$[z = -1 + i0, \quad |z| = r = 1 \quad \& \quad \theta = \pi \ ; \ (-,+) \rightarrow 2^{nd}$ quad.$]$

$$\therefore \log(-1) = 0 + (2n+1)\pi i$$

$$\therefore \log(-1) = \pi i \quad [n = 0, \pm 1, \pm 2, \ldots]$$

★ Branches & Derivative of Logarithms:-

If $z = re^{i\theta}$ is non-zero complex no. & argument θ has any one of the values.

$$\theta = \Theta + 2n\pi \quad (n = 0, \pm 1, \pm 2, \ldots)$$

Where $\Theta = \text{Arg } z$ Hence by defn,

$$\log z = \ln r + i(\Theta + 2n\pi) \quad , (n = 0, \pm 1, \pm 2, \ldots)$$

of multiple-valued logarithmic fun can be written as $\log z = \ln r + i\theta$ ..... (1)

If let α denote any real no. & restrict θ in eqn (1) so that $\alpha < \theta < \alpha + 2\pi$, fun

$$\log z \neq \ln r + i\theta \quad (r > 0, \alpha < \theta < \alpha + 2\pi)$$

..... (2)

with $u(r,\theta) = \ln r$ & $v(r,\theta) = \theta$ is single valued & continuous in stated domain satisfy the polar form,

$$ru_r = v_\theta \quad , \quad u_\theta = -rv_r .$$

Ex:- $\text{Log}(i^3) = \text{Log}(-i) = \ln 1 - i\frac{\pi}{2}$

& $3 \log i = 3\left[\ln 1 + i\frac{\pi}{2}\right] = \frac{3\pi}{2} i$

$\therefore \text{Log}(i^3) \neq 3 \text{ Log } i$

★ Exercises :-

① show that

ⓐ $\text{Log}(-ei) = 1 - \frac{\pi}{2} i$

ⓑ $\text{Log}(1-i) = \frac{1}{2}\ln 2 - \frac{\pi}{4} i$

Solⁿ :- ⓐ $\text{Log}(-ei) = \ln|-ei| + i\,\text{Arg}(-ei)$

$= \ln e - \frac{\pi}{2} i$

$= 1 - \frac{\pi}{2} i$

ⓑ $\text{Log}(1-i) = \ln|1-i| + i\,\text{Arg}(1-i)$

$= \ln\sqrt{2} - \frac{\pi}{4} i$

$= \frac{1}{2}\ln 2 - \frac{\pi}{4} i$

② Show that

ⓐ $\log e = 1 + 2n\pi i$

ⓑ $\log i = \left(2n + \frac{1}{2}\right)\pi i$

ⓒ $\log(-1+\sqrt{3}\,i) = \ln 2 + 2\left(n + \frac{1}{3}\right)\pi i$

$n = 0, \pm 1, \pm 2, \ldots$

Solⁿ :-

ⓐ $\log e = \ln e + i(0 + 2n\pi)$

$\log e = 1 + 2n\pi i$

ⓑ $\log i = \ln 1 + i(\pi/2 + 2n\pi)$

$= 0 + \left(2n + \frac{1}{2}\right)\pi i = \left(2n + \frac{1}{2}\right)\pi i$

ⓒ $\log(-1+\sqrt{3}\,i) = \ln 2 + i\left(\frac{2\pi}{3} + 2n\pi\right)$

$= \ln 2 + 2\left(n + \frac{1}{3}\right)\pi i$

③ Show that

ⓐ $Log(1+i)^2 = 2 Log(1+i)$

ⓑ $Log(-1+i)^2 \neq 2 Log(-1+i)$

Sol^n:- ⓐ $Log(1+i)^2 = Log(1+2i+i^2) = Log(2i)$

$= \ln 2 + \frac{\pi}{2}$

& $2 Log(1+i) = 2\left[\ln\sqrt{2} + i\frac{\pi}{4}\right]$

$= 2\ln\sqrt{2} + 2i\frac{\pi}{4}$

$= \ln(\sqrt{2})^2 + \frac{\pi}{2}i$

$= \ln 2 + \frac{\pi}{2}i$

ⓑ $Log(-1+i)^2 = Log(1-1-2i) = Log(-2i)$

$= \ln 2 - \frac{\pi}{2}i$

& $2 Log(-1+i) = 2\left[\ln\sqrt{2} + i\frac{3\pi}{4}\right]$

$= \ln(\sqrt{2})^2 + 2i\frac{3\pi}{4}$

$= \ln 2 + i\frac{3\pi}{2}$

Hence $Log(-1+i)^2 \neq 2 Log(-1+i)$

* Trigonometric Functions :-

We know that

$e^{ix} = \cos x + i\sin x$ & $e^{-ix} = \cos x - i\sin x$

for every real x.

then

$$\sin x = \frac{e^{ix} - e^{-ix}}{2} \quad \& \quad \cos x = \frac{e^{ix} + e^{-ix}}{2}$$

∴ We define sine & cosine fu^ns of a complex variable z as follows:-

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i} \quad \& \quad \cos z = \frac{e^{iz} + e^{-iz}}{2}$$

- Result :-

(1) $\frac{d}{dz}(e^{iz}) = ie^{iz}$ & $\frac{d}{dz}(e^{-iz}) = -i.e^{-iz}$

(2) $\frac{d}{dz}(\sin z) = \cos z$ & $\frac{d}{dz}(\cos z) = -\sin z$

(3) $\sin(-z) = -\sin z$, $\cos(-z) = \cos z$

(4) $e^{iz} = \cos z + i\sin z$

(5) $\sin(z_1 + z_2) = \sin z_1 \cdot \cos z_2 + \cos z_1 \cdot \sin z_2$

$\cos(z_1 + z_2) = \cos z_1 \cdot \cos z_2 - \sin z_1 \cdot \sin z_2$

$\sin 2z = 2\sin z \cdot \cos z.$

$\cos 2z = \cos^2 z - \sin^2 z$

(6) $\sin\left(z + \frac{\pi}{2}\right) = \cos z$, $\sin\left(z - \frac{\pi}{2}\right) = -\cos z$

(7) $\sin^2 z + \cos^2 z = 1.$

(8) $\sin(z + 2\pi) = \sin z$, $\sin(z+\pi) = -\sin z.$

$\cos(z + 2\pi) = \cos z$, $\cos(z+\pi) = -\cos z.$

★ When y is any real no. then

$\sinh y = \frac{e^{y} - e^{-y}}{2}$ & $\cosh y = \frac{e^{y} + e^{-y}}{2}$

& $\sinh y = i\sinh y$ & $\cos iy = \cosh y$

$\therefore \sin z = \sin x \cdot \cosh y + i\cos x \cdot \sinh y$

$\cos z = \cos x \cdot \cosh y - i\sin x \cdot \sinh y$

& $\sin z$ & $\cos z$ are not bounded on the complex plane.

(9) $\tan z = \frac{\sin z}{\cos z}$ $\cot z = \frac{\cos z}{\sin z}$

$\sec z = \frac{1}{\cos z}$ $\operatorname{cosec} z = \csc z = \frac{1}{\sin z}$

(10) $\frac{d}{dz}(\tan z) = \sec^2 z$, $\frac{d}{dz}(\cot z) = -\operatorname{cosec}^2 z$
$= -\csc^2 z$

$\frac{d}{dz}(\sec z) = \sec z \cdot \tan z$

$\frac{d}{dz}(\operatorname{cosec} z) = -\operatorname{cosec} z \cdot \cot z.$