Integrals

- Definite integrals of Functions $w(t)$:-

If $w(t)$ is complex valued function of real variable t & is written as

$$w(t) = u(t) + i v(t) \quad \ldots\ldots (1)$$

where u & v are real valued fun^{ns}

The definite integral of $w(t)$ over interval $a \le t \le b$ is defined as

$$\int_a^b w(t)\, dt = \int_a^b u(t)\, dt + i \int_a^b v(t)\, dt \quad \ldots\ldots (2)$$

if integrals on the right exist

Thus,

$$\left.\begin{aligned} Re \int_a^b w(t)\, dt &= \int_a^b Re\, [w(t)]\, dt \\ \& \ Im \int_a^b w(t)\, dt &= \int_a^b Im\, [w(t)]\, dt \end{aligned}\right\} \ldots\ldots (3)$$

e.g.- $\displaystyle\int_0^1 (1 + it)^2\, dt = \int_0^1 (1 + 2it - t^2)\, dt$

$$= \int_0^1 (1 - t^2)\, dt + \int_0^1 i2t\, dt$$

$$= \left(t - \frac{t^3}{3}\right)_0^1 + i2\left[\frac{t^2}{2}\right]_0^1$$

$$= \left(1 - \frac{1}{3}\right) + i(t^2)_0^1$$

$$= \frac{2}{3} + i$$

Using fundamental theorem of calculus, suppose that $w(t) = u(t) + iv(t)$ &
$W(t) = U(t) + i\,V(t)$

are continuous on the interval $a \le t \le b$.

If $W(t) = w(t)$ when $a \le t \le b$ then $U'(t) = u(t)$
& $V'(t) = v(t)$.
Hence using ②

$$\int_a^b w(t) = [U(t) + i\,V(t)]_a^b$$

$$= [U(b) + iV(b)] - [U(a) + i(V(a))]$$

i.e. $\int_a^b w(t) = W(b) - W(a) = [W(t)]_a^b$

Ex:- Let $\frac{d}{dt}\left(\frac{e^{it}}{i}\right) = \frac{1}{i}\frac{d}{dt}(e^{it}) = \frac{1}{i}\,i\,e^{it} = e^{it}$

$$\therefore \int_0^{\pi/4} e^{it}\,dt = \left[\frac{e^{it}}{i}\right]_0^{\pi/4} = \frac{e^{i\pi/4}}{i} - \frac{1}{i}$$

$$= \frac{1}{i}\left[e^{i\pi/4} - 1\right]$$

$$= \frac{1}{i}\left[\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) - 1\right]$$

$$= \frac{1}{i}\left[\frac{1}{\sqrt{2}} + \frac{i1}{\sqrt{2}} - 1\right]$$

$$= \frac{1}{\sqrt{2}} + \frac{1}{i}\left(\frac{1}{\sqrt{2}} - 1\right)$$

$$= \frac{1}{\sqrt{2}} - i\left(\frac{1}{\sqrt{2}} - 1\right) \quad \ldots\left[\because \frac{1}{i} = -i\right]$$

$$= \frac{1}{\sqrt{2}} + i\left(1 - \frac{1}{\sqrt{2}}\right)$$

$$\therefore \int_0^{\pi/4} e^{it}\,dt = \left[\frac{1}{\sqrt{2}} + i\left(1 - \frac{1}{\sqrt{2}}\right)\right]$$

Ex:- Evaluate the following integrals.

(a) $\int_1^2 \left(\frac{1}{t} - i\right)^2 dt$ (b) $\int_0^{\pi/6} e^{i2t}\,dt$

(c) $\int_0^{\infty} e^{-zt}\,dt$, $(\mathrm{Re}\, z > 0)$

Soln:- (a) $\int_1^2 \left(\frac{1}{t} - i\right)^2 dt = \int_1^2 \left(\frac{1}{t^2} - \frac{2}{t}i - 1\right) dt$

$$= \int_1^2 \left(\frac{1}{t^2} - 1\right) dt - 2i\int_1^2 \frac{1}{t^2}\,dt$$

$$= \left[-\frac{1}{t} - t\right]_1^2 - 2i\left[\log t\right]_1^2 \ln t$$

$$= -\left(\frac{1}{t} + t\right)_1^2 - 2i\left(\ln 2\right)_1^2$$

$$= -\left(\frac{1}{2} + 2\right) + (1+1) - 2i(\ln 2 - \ln 1)$$

$$= -\frac{5}{2} + 2 - 2i\ln 2 + 0$$

$$= -\frac{1}{2} - i\ln 4$$

(b) $\int_0^{\pi/6} e^{i2t}\,dt = \left[\frac{e^{i2t}}{2i}\right]_0^{\pi/6}$

$$= \frac{1}{2i}\left[e^{i2\pi/6} - 1\right]$$

$$= \frac{1}{2i}\left[e^{i\pi/3} - 1\right] = \frac{1}{2i}\left[\cos\frac{\pi}{3} + i\sin\frac{\pi}{3} - 1\right]$$

$$= \frac{1}{2i}\left[\frac{1}{2} + \frac{i\sqrt{3}}{2} - 1\right]$$

$$= \frac{1}{2i}\left[\frac{i\sqrt{3}}{2} - \frac{1}{2}\right]$$

$$= \frac{\sqrt{3}}{4} - \frac{1}{4i}$$

$$= \frac{\sqrt{3}}{4} - \frac{1 \times i}{4i^2}$$

$$= \frac{\sqrt{3}}{4} + \frac{i}{4}$$

(c) since $|e^{-bz}| = e^{-x}$, $(Re(z) > 0)$

$$\therefore \int_0^\infty e^{-zt}\,dt = \lim_{b\to\infty}\int_0^b e^{-zt}\,dt = \lim_{b\to\infty}\left[\frac{e^{-zt}}{-z}\right]_0^b$$

$$= \frac{1}{z}\lim_{z\to\infty}(1 - e^{-bz}) = \frac{1}{z}(1-0) = \frac{1}{z}\ (Re(z) > 0)$$

Ex:- Show that if m & n are integers.

$$\int_0^{2\pi} e^{im\theta}\cdot e^{-in\theta}\,d\theta = \begin{cases} 0 & \text{when } m \neq n \\ 2\pi & \text{when } m = n \end{cases}$$

Soln:- $I = \int_0^{2\pi} e^{im\theta}\cdot e^{-in\theta}\,d\theta = \int_0^{2\pi} e^{i(m-n)\theta}\,d\theta$

i) when $m \neq n$

$$I = \left[\frac{e^{i(m-n)\theta}}{i(m-n)}\right]_0^{2\pi}$$

$$= \frac{e^{i(m-n)2\pi}}{i(m-n)} - \frac{e^{i(m-n)0}}{i(m-n)}$$

$\ldots\ldots \left[\because e^{i2\pi} = \cos 2\pi + i\sin 2\pi = 1+0=1\right]$

$$= \frac{1}{i(m-n)} - \frac{1}{i(m-n)} = 0$$

ii) when $m=n$ then

$$I = \int_0^{2\pi} e^{im\theta} \cdot e^{-im\theta}\, d\theta$$

$$= \int_0^{2\pi} d\theta$$

$$= 2\pi$$

Ex:- Evaluate $\int_0^{\pi} e^{(1+i)x}\, dx = \int_0^{\pi} e^x \cos x\, dx + i\int_0^{\pi} e^x \sin x\, dx$ the two integrals on right by evaluating the single integral on the left & using real & imaginary parts.

Sol^n:- First of all,

$$\int_0^{\pi} e^{(1+i)x}\, dx = \int_0^{\pi} e^x \cdot e^{ix}\, dx = \int_0^{\pi} e^x(\cos x + i\sin x)\, dx$$

$$= \int_0^{\pi} e^x \cos x \cdot dx + i\int_0^{\pi} e^x \sin x\, dx$$

But also,

$$\int_0^{\pi} e^{(1+i)x}\, dx = \left[\frac{e^{(1+i)x}}{(1+i)}\right]_0^{\pi} = \frac{e^{(1+i)\pi} - e^{(1+i)\cdot 0}}{(1+i)}$$

$$= \frac{e^{\pi} \cdot e^{i\pi} - 1}{(1+i)} = \frac{e^{\pi}(-1) - 1}{(1+i)} \quad \ldots\ldots (e^{i\pi} = -1),$$

$$= \frac{-e^{\pi} - 1}{(1+i)} = \frac{-e^{\pi} - 1}{(1+i)} \times \frac{(1-i)}{(1-i)}$$

$$= \frac{-(e^{\pi}+1)}{(1+i)} \times \frac{(1-i)}{(1-i)} = \frac{-(e^{\pi}+1)}{2} + \frac{i(e^{\pi}+1)}{2}$$

$$= \frac{-(e^{\pi}+1) + i(e^{\pi}+1)}{2}$$

$$= \frac{-(1+e^{\pi})}{2} + \frac{i(e^{\pi}+1)}{2}$$

Equating real & imaginary parts, we get,

$$\int_0^{\pi} e^x \cos x \, dx = -\frac{1+e^{\pi}}{2} \quad \& \quad \int_0^{\pi} e^x \sin x \, dx = \frac{1+e^{\pi}}{2}$$

- Continuous arc :- If the point z on the arc is such that $z = \phi(t) + i\,\psi(t)$ ...... (A)

  Hence we write $x = \phi(t)$ ..... (1)

  $y = \psi(t)$ ..... (2)

  If $\phi(t)$ and $\psi(t)$ are real continuous fun^ns of real variable t defined in the range $\alpha \le t \le \beta$ then arc is called continuous arc.

- Multiple point :- If the eq^n $x = \phi(t)$, $y = \psi(t)$ are satisfied by more than one values of t in the given range the point z or (x, y) is called multiple pt. of the arc.

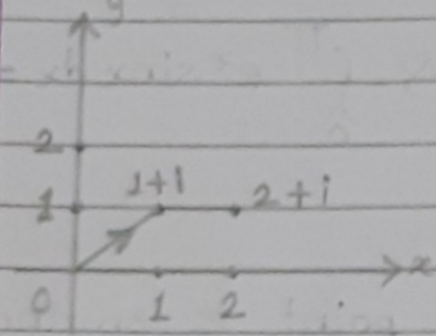
- Jordan arc :- A continuous arc without multiple points is called Jordan arc.

Thus for point $z = x + iy$ on Jordan a curve z is expressed i.e. $z = \phi(t) + i\psi(t)$ is a single valued & $\phi(t)$ & $\psi(t)$ are continuous.

It is called simple arc.

e.g.- The polygonal line defined by means the eq^{ns}

$$z = \begin{cases} x + ix, & \text{when } 0 \le x \le 1 \\ x + i, & 1 \le x \le 2 \end{cases} \quad (*)$$



& consisting line segment from 0 to $1+i$ followed by one from $1+i$ to $2+i$ is a simple arc. (see fig)

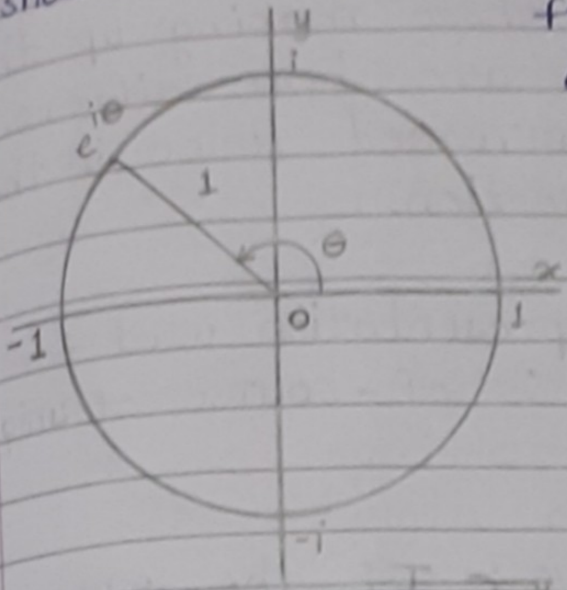
Ex:- ② The unit circle $z = e^{i\theta}$ $(0 \le \theta \le 2\pi)$ ..... ① about the origin is simple closed curve oriented in the clockwise direction.

Again $z = z_0 + Re^{i\theta}$, $(0 \le \theta \le 2\pi)$ centred at point z_0 with radius R.

The same set of points can make up different arc.

Note:- We know that $z = re^{i\theta}$ ..... ①

The expression ① with $r = 1$ tells us that the number $e^{i\theta}$ lie on the circle centred at origin with radius unity as shown in fig. values of $e^{i\theta}$ are then immediately from the fig.

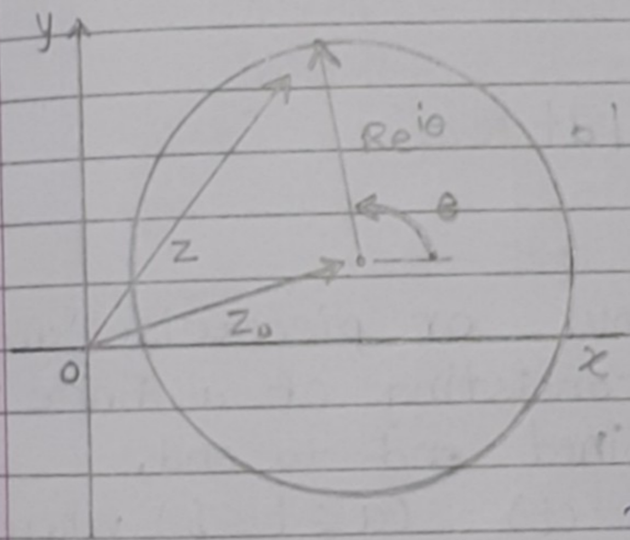


$e^{i\pi} = -1$, $e^{-i\pi/2} = -1$ & $e^{-4\pi i} = 1$

Note that the eq^n
$z = Re^{i\theta}$ $(0 \leq \theta \leq 2\pi)$
is a parametric representa^n of circle $|z| = R$ centred at origin with radius R.
As the parameter θ increases from $\theta = 0$ to 2π the pt. z from the positive real axis & traverses the circle once in the counter-clockwise direction i.e. $|z - z_0| = R$ whose centre is z_0 & Radius is R has the parametric eq^{ns}:
$z = z_0 + Re^{i\theta}$, $(0 \leq \theta \leq 2\pi)$

$e^{i\theta}$ lies on circle $|z| = 1$ & $|e^{i\theta} - 1|$ is the distance bet^n $e^{i\theta}$ & 1. see above fig.

This can be seen vectorially (see fig.)



The pt. z traversing the circle $|z - z_0| = R$ once in counterclockwise direc^n corresponds the sum of the fixed vecter z_0 & a vecter of length R whose angle of inclination θ varies from $\theta = 0$ to $\theta = 2\pi$.

Ex:- 3 The arc $z = e^{-i\theta}$ ..... (2) $(0 \leq \theta \leq 2\pi)$ is not the arc described by eq^n (1). The set of pts. is the same but now the circle is traversed in the clockwise direction.

Ex:- (4) The points on the arc $z = e^{i2\theta}$, $(0 \le \theta \le 2\pi)$ are the same as those making up the arcs (1) & (2). The arc here differs. Here circle is traversed twice in the counterclockwise direction.

The parametric representation used for any given arc C is of-course of unique.

- Smooth arc or curve :- If the fun^s ϕ & ψ in $x = \phi(t)$, $y = \psi(t)$, $(a \le t \le b)$ have continuous derivatives $\phi'(t)$ & $\psi'(t)$ which do not vanish simultaneously for any value of t the arc has a continuous turning tangent. The arc or curve is then smooth. Its length exists & is given by

$$L = \int_a^b \sqrt{\phi'(t) + \psi'(t)}\, dt \qquad (a \le b)$$

$$L = \int_a^b |z'(t)|\, dt$$

★ Contour :- A contour or piecewise smooth arc is an arc consisting of a finite no. of smooth arcs joined end to end. Hence if eq^n $z = z(t)$ $(a \le t \le b)$ where $z(t) = x(t) + iy(t)$ represent a contour $z(t)$ is continuous where its derivative $z'(t)$ is piecewise continuous. The polygonal line (*) is an example of contour.

When only the initial & final values of $z(t)$ are the same; the contour C is called a simple closed contour.

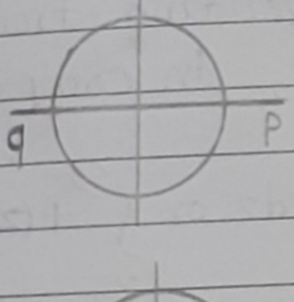
e.g.- The unit circle $z = e^{i\theta}$ $(0 \le \theta \le 2\pi)$
& $z = z_0 + R e^{i\theta}$ $(0 \le \theta \le 2\pi)$

as well as boundary of a triangle or rectangle, taken in specification direction.

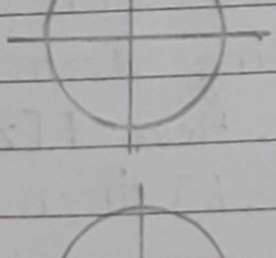
The length of contour or simple closed contour, is the sum of the length of the smooth arcs that make up the contour.

The points of any simple closed curve or simple closed contour C are boundary points of two distinct domains one of which is the interior of C &- is bounded. On the other hand which is exterior of C is unbounded

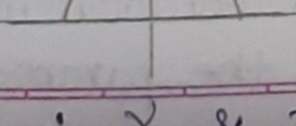
Note:- ① The curve γ is said to be closed curve if $\gamma(a) = \gamma(b)$.
i.e. initial & terminal pts. coinside.
$\gamma(t) = \cos t + i \sin t$, $t \in [0, 3\pi]$



Not closed curve.



$\gamma_1(t) = \cos t + i \sin t$
$t \in [0, 2\pi]$, closed curve.



$\gamma_2(t) = \cos t + i \sin t$
$t \in [0, \pi]$ not closed curve.

$\therefore$ γ_1 & γ_2 are Jordan arc or Simple-curve.

(2) Let $\gamma : [a, b] \to \mathbb{C}$
$\gamma(t) = \cos t + i \sin t$, $t \in [a, b]$

$[a, b]$	Jordan arc	Closed curve	Jordan Curve
$[0, \pi]$	✓	×	×
$[0, 2\pi]$	✓	✓	✓
$[0, 3\pi]$	×	×	×
$[0, 4\pi]$	×	✓	×

- Jordan Curve :- A closed curve is called Jordan Curve if it is Jordan arc as well. i.e. simple closed curve is called Jordan curve.

☆ Contour Integrals :-

Let $f(z)$ be complex valued funⁿ. along a given contour C extending from a pt. $z = z_1$ to a pt. $z = z_2$ in complex plane.

The line of integral $\int_C f(z)\,dz$ or $\int_{z_1}^{z_2} f(z)\,dz$

Suppose that $z = z(t)$, $(a \leq t \leq b)$ represent a contour C, extending from a pt. $z_1 = z(a)$ to a pt. $z_2 = z(b)$. Here $f[z(t)]$ is piecewise continuous on $[a, b]$ then we define the line integral or contour integral of f along C in terms of the parameter t

$$\int_C f(z)\,dz = \int_a^b f[z(t)] \cdot z'(t) \cdot dt$$

Note that since C is contour, $z'(t)$ is also piecewise Continuous on $a \le t \le b$.

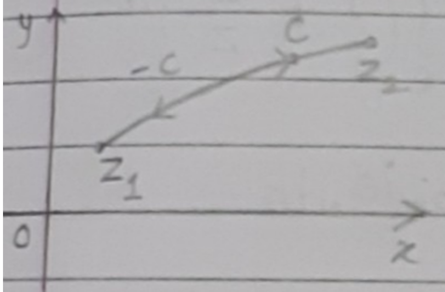
- Properties of Integrals of Complex-Valued fun^{ns}:

① $\int_C z_0 f(z)\,dz = z_0 \int_C f(z)\,dz$, for any complex const. z_0

② $\int_C [f(z) + g(z)]\,dz = \int_C f(z)\,dz + \int_C g(z)\,dz$

③ $\int_{-C} f(z)\,dz = -\int_C f(z)\,dz$.

i.e. $\int_{-C} f(z)\,dz = \int_{-b}^{-a} f[z(-t)] \cdot \frac{d}{dt}[z(-t)]\,dt$



$$= -\int_{-b}^{-a} f[z(-t)]\, z'(-t)\,dt$$

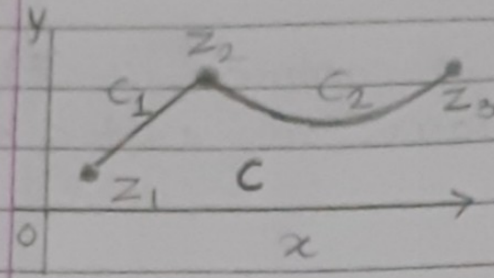
$$= -\int_C f(z)\,dz$$

i.e. $\int_{-C} f(z)\,dz = -\int_C f(z)\,dz$

Here C represents z_1 to z_2

& $-C$ represents z_2 to z_1

④ $\int_c f(z)\,dz = \int_{c_1} f(z)\,dz + \int_{c_2} f(z)\,dz$



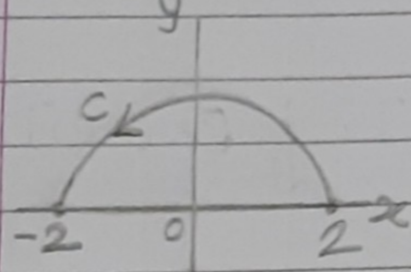
$c = c_1 + c_2$.

Ex :- Evaluate $\int_c f(z)\,dz$

Here $f(z) = \frac{z+2}{z}$ & c is

(i) the Semicircle $z = 2e^{i\theta}$, $(0 \le \theta \le \pi)$
(ii) the semicircle $z = 2e^{i\theta}$, $(\pi \le \theta \le 2\pi)$
(iii) the circle $z = 2e^{i\theta}$, $(0 \le \theta \le 2\pi)$

Solⁿ:- (i) Let c be the semicircle $z = 2e^{i\theta}$, $(0 \le \theta \le \pi)$, show below then



$$\int_c f(z)\,dz = \int_c \frac{z+2}{z}\,dz$$

$$\therefore \int_c f(z)\,dz = \int_c \left(1 + \frac{2}{z}\right) dz$$

$$\therefore \int_c f(z)\,dz = \int_0^{\pi} \left(1 + \frac{2}{2e^{i\theta}}\right) 2ie^{i\theta}.d\theta$$

$$= 2i \int_0^{\pi} \left(e^{i\theta} + \frac{2e^{i\theta}}{2e^{i\theta}}\right) d\theta$$

$$= 2i \int_0^{\pi} (e^{i\theta} + 1)\,d\theta$$

$$= 2i\left[\frac{e^{i\theta}}{i} + \theta\right]_0^{\pi}$$

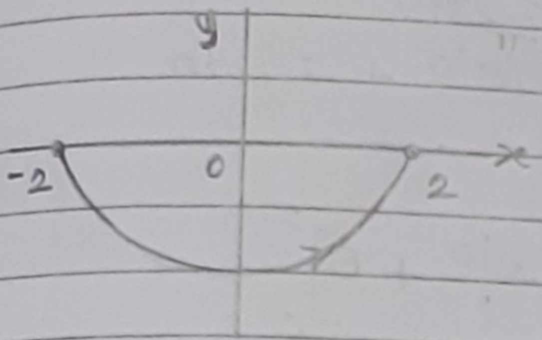
$$= 2i\left[\left(\frac{e^{i\pi}}{i} + \pi\right) - \left(\frac{e^{i\cdot 0}}{i} + 0\right)\right]$$

$$= 2i\left[\frac{-1}{i} + \pi - \frac{1}{i}\right]$$

$$= 2i\left[-\frac{2}{i} + \pi\right]$$

$$\therefore \int_0^{\pi} f(z)\,dz = -4 + 2\pi i$$

(ii) Now let c be semicircle $z = 2e^{i\theta}$, $(\pi \le \theta \le 2\pi)$



then

$$\int_c f(z)\cdot dz = \int_c \frac{z+2}{z}\,dz$$

$$\therefore \int_c f(z)\,dz = \int_c \left(1 + \frac{2}{z}\right) dz$$

$$\therefore \int_c f(z)\,dz = \int_{\pi}^{2\pi} \left(1 + \frac{2}{2e^{i\theta}}\right) 2ie^{i\theta}\,d\theta$$

$$= 2i\int_{\pi}^{2\pi} \left(e^{i\theta} + \frac{2e^{i\theta}}{2e^{i\theta}}\right) d\theta$$

$$= 2i\int_{\pi}^{2\pi} (e^{i\theta} + 1)\,d\theta$$

$$= 2i\left[\frac{e^{i\theta}}{i} + \theta\right]_{\pi}^{2\pi}$$

$$= 2i\left[\left(\frac{e^{i2\pi}}{i} + 2\pi\right) - \left(\frac{e^{i\pi}}{i} + \pi\right)\right]$$

$$= 2i\left[\frac{e^{i2\pi}}{i} + 2\pi - \frac{e^{i\pi}}{i} - \pi\right]$$

$$= 2i\left[\frac{e^{i2\pi}}{i} - \frac{(-1)}{i} + \pi\right] \ldots \left[\because e^{i\pi} = -1\right]$$

$$= 2i\left[\frac{1+1}{i} + \pi\right] \ldots \left[\because e^{i2\pi} = 1\right]$$

$$= 2i\left[\frac{2}{i} + \pi\right]$$

$$\int_{\pi}^{2\pi} f(z)\,dz = \underline{\underline{4 + 2\pi i}}$$

(iii) Let c denote the entire circle $z = 2e^{i\theta}$ $(0 \le \theta \le 2\pi)$

$$\therefore \int_C \left(z + \frac{2}{z}\right) dz = 2i\int_0^{2\pi} (e^{i\theta} + 1)\, d\theta$$

$$= 2i\left[\frac{e^{i\theta}}{i} + \theta\right]_0^{2\pi}$$

$$= 2i\left[\left(\frac{e^{i2\pi}}{i} + 2\pi\right) - \left(\frac{1}{i} + 0\right)\right]$$

$$= 2i\left[\cancel{\frac{1}{i}} + 2\pi - \cancel{\frac{1}{i}}\right]$$

$$= 2i\,(2\pi)$$

$$\therefore \int_0^{2\pi} f(z)\,dz = \underline{\underline{4\pi i}}$$

Ex:- (2) $f(z) = z - 1$ & C is the arc from $z = 0$ to $z = 2$ consisting of

① the semicircle $z = 1 + e^{i\theta}$, $(\pi \le \theta \le 2\pi)$
② the segment $z = x$, $(0 \le x \le 2)$ of the real axis.

Sol^n:- (i) The arc $C: z = 1 + e^{i\theta}$, $(\pi \le \theta \le 2\pi)$

then $\int_C (z-1)\,dz = \int_{\pi}^{2\pi} [1 + e^{i\theta} - 1](ie^{i\theta})\,d\theta$

$$= i\int_{\pi}^{2\pi} e^{i\theta}(e^{i\theta}).d\theta$$

$$= i\int_{\pi}^{2\pi} e^{2i\theta}\,d\theta$$

$$= i\left[\frac{e^{i2\theta}}{2i}\right]_{\pi}^{2\pi}$$

$$= \frac{1}{2}\left[\frac{e^{i4\pi}}{1} - \frac{e^{i2\pi}}{1}\right]$$

$$= \frac{1}{2}\left[e^{i4\pi} - e^{i2\pi}\right]$$

$$\therefore \int_C (z-1)\,dz = \frac{1}{2}[1-1] = \underline{\underline{0}}.$$

② Here $C: z = x$, $(0 \le x \le 2)$; then

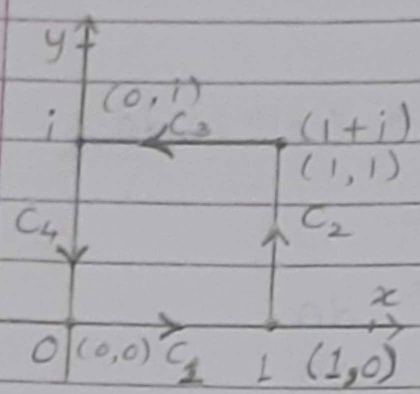
$$\int_C (z-1)\,dz = \int_0^2 (x-1)\,dx = \left[\frac{x^2}{2} - x\right]_0^2$$

$$= [(2-2)-(0-0)] = (0-0)$$

$$\therefore \int_C (z-1)\,dz = \underline{\underline{0}}$$

(3) If $f(z) = \pi \exp(\pi \bar{z})$ & c is the boundary of the square with vertices at the pts $0, 1, 1+i, i$, the orientation of c being the counterclockwise direcn.

soln :- The path c is the sum of paths c_1, c_2, c_3, c_4 that are shown below.



The fun to be integrated around the closed path c is $f(z) = \pi e^{\pi \bar{z}}$.

We observe that $c = c_1 + c_2 + c_3 + c_4$ & find the values of the integrals along the individual legs of the square c.

i) Since c_1 is $z = x$, $(0 \le x \le 1)$

$$\therefore \int_{c_1} \pi e^{\pi \bar{z}} dz = \pi \int_0^1 e^{\pi x} dx \qquad \left[\because z = x,\ dz = dx \ \&\ \bar{z} = \bar{x} = x \right.$$

$$= \pi \left[\frac{e^{\pi x}}{\pi} \right]_0^1$$

$$= \pi \left[\frac{e^{\pi}}{\pi} - \frac{e^0}{\pi} \right] = e^{\pi} - e^0$$

$$= e^{\pi} - 1$$

ii) Along c_2, $x = 1$

i.e. $z = 1 + iy$, $(0 \le y \le 1)$

Now $z = x + iy$
$z = 1 + iy$
$dz = i\,dy$ &
$\bar{z} = 1 - iy$

$$\therefore \int_{c_2} \pi . e^{\pi \bar{z}} dz = \pi \int_0^1 e^{\pi (1 - iy)} (i\,dy)$$

$$= (\pi i) e^{\pi} \int_0^1 e^{-\pi i y} . dy = (\pi i) e^{\pi} \left[\frac{e^{-\pi i y}}{-\pi i}\right]_0^1$$

$$= \pi i e^{\pi} \left[\frac{e^{-i\pi}}{-\pi i} - \frac{e^{-\pi i (0)}}{-\pi i}\right]$$

$$= \frac{\cancel{\pi i} e^{\pi}}{\cancel{\pi i}} [-e^{-i\pi} + 1]$$

$$= e^{\pi} [-(\cos\pi - i\sin\pi) + 1]$$

$$= e^{\pi} [-[(-1) - 0] + 1]$$

$$= e^{\pi} [1 + 1]$$

$$= \underline{\underline{2e^{\pi}}}$$

iii) Along C_3, $y = 1$

$\therefore z = x + iy$ & $\bar{z} = x - iy$

$\therefore z = x + i$ & $\bar{z} = x - i$

$\therefore dz = dx$

& x varies from 1 to 0.

$$\therefore \int_{C_3} \pi e^{\pi \bar{z}} dz = \int_1^0 \pi . e^{\pi(x-i)} dx$$

$$= \pi \int_1^0 e^{\pi x} e^{-\pi i} dx$$

$$= (e^{-i\pi}) \pi \int_1^0 e^{\pi x} . dx$$

$$= \pi\,[\cos\pi - i\sin\pi]\left[\frac{e^{\pi x}}{\pi}\right]_1^0$$

$$= \pi\,[(-1) - 0]\,\frac{1}{\pi}\,[e^{\pi x}]_1^0$$

$$= (-1)(e^0 - e^{\pi})$$

$$= -1 + e^{\pi}$$

$$= e^{\pi} - 1$$

or

since C_3 is $z = (1-x) + i$, $(0 < x < 1)$

$$\therefore \int_{C_3} \pi e^{\pi \bar{z}}\, dz = \pi \int_0^1 e^{\pi[(1-x)-i]}(-dx)$$

$[\because$ Here $z = (1-x) + i$

$\therefore \bar{z} = (1-x) - i$

& $dz = -dx$

$$\therefore \int_{C_3} \pi e^{\pi \bar{z}} dz = -\pi \int_0^1 e^{\pi} . e^{-\pi x} . e^{-\pi i} . dx$$

$$= -\pi\, e^{\pi}\,[e^{-i\pi}] \int_0^1 e^{-\pi x} dx$$

$$= -\pi . e^{\pi} . e^{i\pi} \left[\frac{e^{-\pi x}}{-\pi}\right]_0^1$$

$$= e^{\pi}(-1)[e^{-\pi} - e^0]$$

$$= -e^{\pi}\,[e^{-\pi} - 1]$$

$$= -e^0 + e^{\pi} = e^{\pi} - 1$$

iv) Since C_4 is $x = 0$,

$\therefore z = x + iy$

Here, $z = iy$

$\therefore \bar{z} = -iy$ & $dz = i\,dy$.

& y varies from 1 to 0.

$$\therefore \int_{C_4} \pi e^{\pi \bar{z}}\, dz = \pi \int_1^0 e^{\pi(-iy)}(i\,dy)$$

$$= i\pi \int_1^0 e^{-i\pi y}\, .dy$$

$$= \cancel{i\pi} \left[\frac{e^{-i\pi y}}{-i\pi} \right]_1^0$$

$$= -e^{-i\pi(0)} + e^{-i\pi}$$

$$= -e^{0} + (-1) \quad \ldots (e^{-i\pi} = 1)$$

$$= -1 - 1$$

$$= -2$$

Finally then $\int_C \pi e^{\pi \bar{z}}\, dz = \int_{C_1} \pi e^{\pi \bar{z}}\, .dz + \int_{C_2} \pi e^{\pi \bar{z}}\, dz$

$$+ \int_{C_3} \pi . e^{\pi \bar{z}}\, dz + \int_{C_4} \pi e^{\pi \bar{z}}\, dz$$

$$= (e^{\pi} - 1) + 2e^{\pi} + e^{\pi} - 1 + (-2)$$

$$= 4e^{\pi} - 4$$

$$\therefore \int_C \pi e^{\pi \bar{z}}\, .dz = 4(e^{\pi} - 1)$$

(4) $f(z)$ is defined by means of eqns

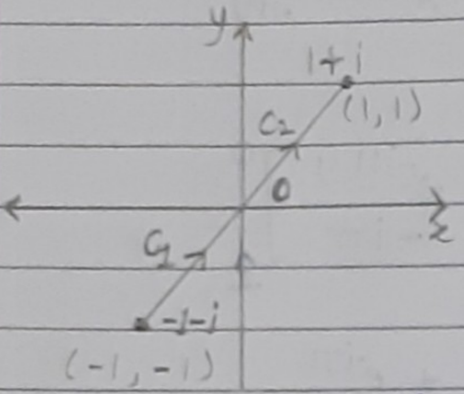
$$f(z) = \begin{cases} 1 & \text{when } y<0 \\ 4y & \text{when } y>0 \end{cases}$$

& C is the arc from $z = -1-i$ to $z = 1+i$ along the curve $y = x^3$.

Soln:- The path c is the sum of the paths

$C_1: z = x + ix^3 \quad (-1 \le x \le 0)$

& $C_2: z = x + ix^3 \quad (0 \le x \le 1)$



Using $f(z) = 1$ on C_1

&

$f(z) = 4y = 4x^3$ on C_2

∴ We have,

$$\int_C f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz$$

$$= \int_{-1}^{0} 1(1+i3x^2)\,dx + \int_0^1 4x^3(1+i3x^2)\,dx$$

$$= \int_{-1}^{0} 1\,dx + 3i\int_{-1}^{0} x^2\,dx + 4\int_0^1 x^3\,dx + 12i\int_0^1 x^5\,dx$$

$$= [x]_{-1}^{0} + 3i\left[\frac{x^3}{3}\right]_{-1}^{0} + 4\left[\frac{x^4}{4}\right]_0^1 + 12i\left[\frac{x^6}{6}\right]_0^1$$

$$= [0-(-1)] + i[0-(-1)] + [1-0] + 2i[1-0]$$

$$= 1 + i + 1 + 2i$$

$$= \underline{2+3i}$$

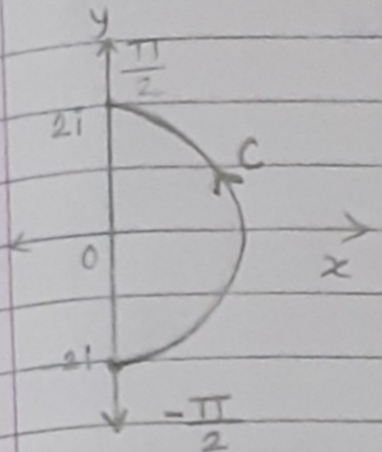
5) Find the value of the integral

$I = \int_C \bar{z}\, dz$ when C is right hand half

$z = 2e^{i\theta}$, $(-\pi/2 \le \theta \le \pi/2)$

of the circle $|z| = 2$ from $z = -2i$ to $2i$.

Solⁿ:-
$$I = \int_{-\pi/2}^{\pi/2} \overline{2e^{i\theta}}\, (2e^{i\theta})'\, d\theta$$



Since $\overline{e^{i\theta}} = e^{-i\theta}$ &

$(e^{i\theta})' = i e^{i\theta}$

$$\therefore I = 4 \int_{-\pi/2}^{\pi/2} (e^{-i\theta}) \cdot i e^{i\theta} \cdot d\theta$$

$$= 4i \int_{-\pi/2}^{\pi/2} 1 \cdot d\theta = 4i\, [\theta]_{-\pi/2}^{\pi/2}$$

$$= 4i\, [\pi/2 + \pi/2] = \underline{\underline{4\pi i}}$$

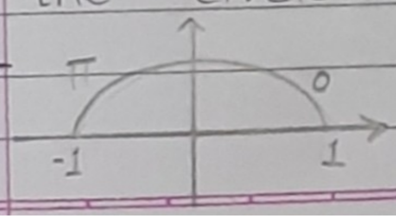
Ex:- The value of $\int_C \frac{1}{z}\, dz$, where z is circle $z = e^{i\theta}$, $0 \le \theta \le \pi$

Solⁿ:- $z = e^{i\theta}$ $\quad \therefore dz = i e^{i\theta} \cdot d\theta$

$$\therefore \int_C \frac{1}{z}\, dz = \int_0^{\pi} \frac{1}{\cancel{e^{i\theta}}} \cdot i \cancel{e^{i\theta}} \cdot d\theta = i\, [\theta]_0^{\pi} = \pi i$$

Ex:- Evaluate $\int_C |z|\, dz$, where C is upper half of the circle $|z| = 1$.

Solⁿ:-



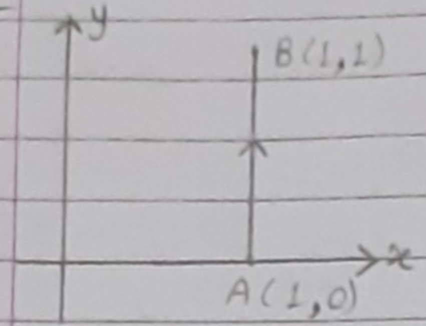
Here $|z| = 1$, $z = e^{i\theta}$ $\therefore dz = i e^{i\theta} d\theta$

$$\int_C |z|\, dz = \int_0^{2\pi} 1 \cdot e^{i\theta} \cdot i\, d\theta = i \left[\frac{e^{i\theta}}{i}\right]_0^{\pi} = [e^{i\pi} - e^{0}] = -1 - 1$$

$$= \underline{\underline{-2}}$$

Ex:- Evaluate the integral $\int_C \bar{z}\, dz$ where C is straight line from (1, 0) to (1, 1)

soln:-



Eqn of line AB is $x = 1$

$\therefore dx = 0$

Now, $z = x + iy$

$\therefore z = 1 + iy$ & $\bar{z} = 1 - iy$

$\therefore dz = i\,dy$

$$I = \int_{AB} \bar{z}\, dz = \int_{AB} (x - iy)\, i\,dy$$

$$= \int_{AB} (1 - iy) \cdot i\,dy$$

$$= i \int_0^1 (1 - iy) \cdot dy$$

$$= i \left\{ [y]_0^1 - i \left[\frac{y^2}{2}\right]_0^1 \right\}$$

$$= i[1 - 0] - i^2\left[\frac{1}{2} - 0\right]$$

$$= i - \frac{i^2}{2}$$

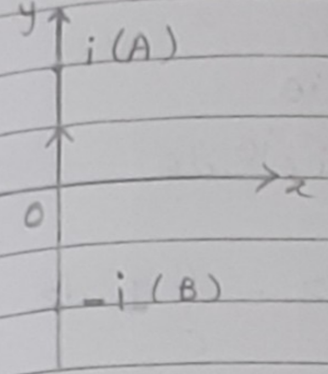
$$\therefore I = \int_{AB} \bar{z}\, dz = \underline{\underline{\frac{1}{2} + i}}$$

Ex:- Evaluate :- $\int_0^{1+i} z^2 . dz$.

soln:- $\int_0^{1+i} z^2 dz = \left[\frac{z^3}{3}\right]_0^{1+i} = \underline{\underline{\frac{1}{3}(1+i)^3}}$.

Ex:- Evaluate :- $\int_C |z|\,dz$, where C is st. line from $z=-i$ to $z=i$.

Sol^n:-



Let $I = \int_C |z|\,dz$

Here C is the path line AB.

Whose eq^n is $x=0$.

Now, $z = x+iy$

$\therefore z = 0+iy$

$\therefore dz = i\,dy$

$|z| = |iy| = |y|$.

$$\therefore \int_C |z|\,dz = \int_C |y|\cdot i\,dy = \int_{BO} |y|\,i\,dy + \int_{OA} |y|\,i\,dy$$

$$= \int_{-1}^{0} |y|\cdot i\,dy + \int_{0}^{1} |y|\cdot i\,dy$$

$$= i\left[\int_{-1}^{0} -y\cdot dy + \int_{0}^{1} y\cdot dy\right]$$

$$= i\left\{\left[-\frac{y^2}{2}\right]_{-1}^{0} + \left[\frac{y^2}{2}\right]_{0}^{1}\right\}$$

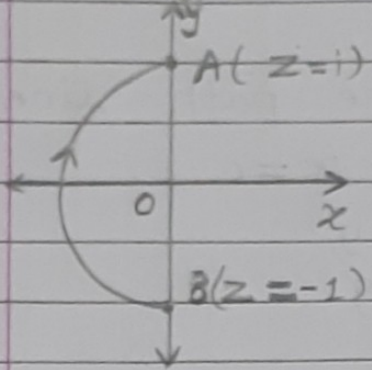
$$= i\left[\left(0-\frac{(-1)}{2}\right) + \left(\frac{1}{2}-0\right)\right]$$

$$= i\left[\frac{1}{2}+\frac{1}{2}\right]$$

$$\therefore \int_C |z|\cdot dz = \underline{\underline{i}}.$$

Ex:- Evaluate $\int_C |z|\, dz$ where c is the left of the unit circle $|z|=1$ from $z=-1$ to $z=i$.

Soln:- At A, $\theta = \frac{\pi}{2}$ & at B, $\theta = \frac{3\pi}{2}$



$|z|=1 \Rightarrow z = e^{i\theta}$

$dz = i e^{i\theta} d\theta$

$\therefore I = \int_C |z|\, dz = \int_C 1 . dz$

$= \int_{3\pi/2}^{\pi/2} e^{i\theta} . i d\theta = i \left[\frac{e^{i\theta}}{i} \right]_{3\pi/2}^{\pi/2}$

$= e^{i\pi/2} - e^{i3\pi/2}$

$= e^{i\pi/2} - e^{i\pi} . e^{i\pi/2}$

$= i - (-1).(i)$

$= i + i$

$\therefore \int_C |z|\, dz = \underline{\underline{2i}}$

$\because e^{i\pi/2} = \cos\pi/2 + i\sin\pi/2 = 0 + i(1) = i$

& $e^{i\pi} = \cos\pi + i\sin\pi = (-1) + 0 = -1$

Ex:- Evaluate $\int_C \log z\, dz$ where C is Unit circle $|z| = 1$.

Sol^n:- $I = \int_C \log z \, dz \quad \ldots\ldots (1)$

Now $z = e^{i\theta} \quad \therefore dz = i e^{i\theta} . d\theta$

& $\log z = \log e^{i\theta} = i\theta$

Now (1) $\Rightarrow I = \int_0^{2\pi} i\theta . (i e^{i\theta}) . d\theta$

$= i^2 \int_0^{2\pi} (\theta . e^{i\theta}) d\theta \ldots$ (Integrate by parts).

$= i^2 \left[\left(\theta . \frac{e^{i\theta}}{i} \right)_{\theta=0}^{2\pi} - \int_0^{2\pi} (1) . \frac{e^{i\theta}}{i} d\theta \right]$

$= -1 \left[\frac{2\pi e^{i2\pi}}{i} - \frac{1}{i} . \frac{1}{i} (e^{i\theta})_0^{2\pi} \right]$

$= \frac{-2\pi}{i} [\cos 2\pi + i \sin 2\pi] - [e^{i2\pi} - e^0]$

$= \frac{-2\pi}{i} [1 + 0] - [1 - 1]$

$= \frac{-2\pi}{i} - 0$

$= \frac{-2\pi \times i}{i \times i}$

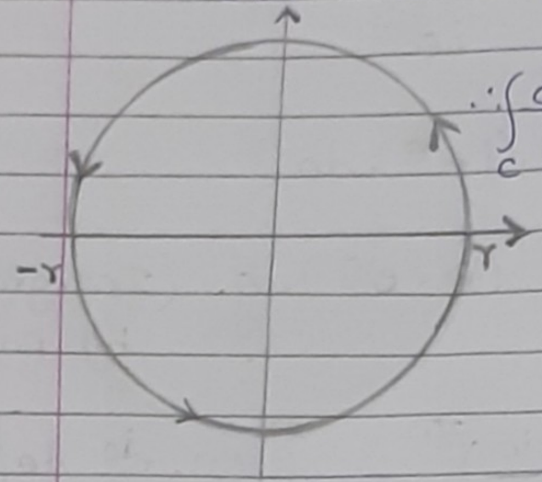
$= \frac{-2\pi i}{i^2}$

$= \frac{-2\pi i}{-1} \quad \ldots\ldots [i^2 = -1]$

$\therefore \int_C \log z . dz = \underline{\underline{2\pi i}}$

Ex:- $\int_C \frac{dz}{z}$. Here, c denotes the circle $|z| = r$ described in positive sense.

Soln:- Here $|z| = r$.
Let $z = re^{i\theta}$



$\therefore \int_C \frac{dz}{z} = \int_0^{2\pi} \frac{\cancel{r}.i\cancel{e^{i\theta}}}{\cancel{r}.\cancel{e^{i\theta}}} d\theta$

$= i\int_0^{2\pi} 1 \cdot d\theta = i[\theta]_0^{2\pi}$

$= i[2\pi - 0]$

$\therefore \int_C \frac{dz}{z} = \underline{2\pi i}$.

Ex:- Evaluate $\int_C \frac{dz}{z-a}$ where c is given by

$|z-a| = R$.

$\Longrightarrow$ Since $|z-a| = R$

$\therefore z-a = R \cdot e^{i\theta}$

$\therefore |z-a| = R$ & θ varies from 0 to 2π.

As $z-a = R \cdot e^{i\theta}$

$\therefore dz = R \cdot ie^{i\theta} \cdot d\theta$

$\therefore \int_C \frac{dz}{z-a} = \int_0^{2\pi} \frac{\cancel{R} \cdot i\cancel{e^{i\theta}}}{\cancel{R} \cdot \cancel{e^{i\theta}}} \cdot d\theta$

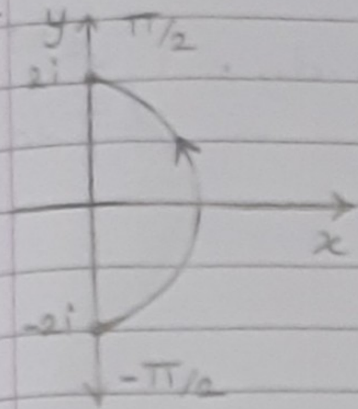
$= i\int_0^{2\pi} 1 \cdot d\theta$

$= i[\theta]_0^{2\pi} = i[2\pi - 0]$

$\therefore \int_C \frac{dz}{z-a} = \underline{2\pi i}$

Ex:- Find the value of integral $I = \int_C \frac{dz}{z}$ when the right of hand half of $z = 2e^{i\theta}$, $(-\pi/2 \le \theta \le \pi/2)$ of the circle $|z| = 2$ from $z = -2i$ to $2i$

Soln:-



$$\therefore I = \int_{-\pi/2}^{\pi/2} \frac{1}{2e^{i\theta}} (2e^{i\theta} \cdot i\,d\theta)$$

$$= i \int_{-\pi/2}^{\pi/2} 1 \cdot d\theta = i [\theta]_{-\pi/2}^{\pi/2}$$

$$= i [\pi/2 - (-\pi/2)]$$

$$= i [\pi]$$

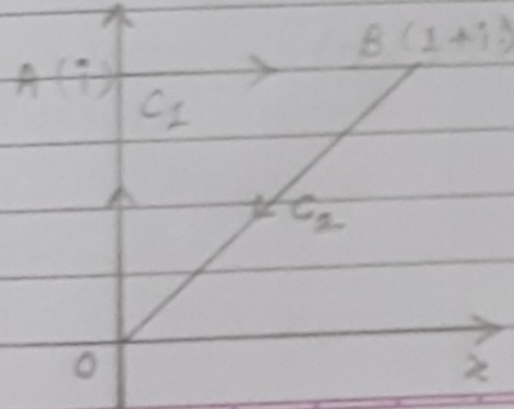
$$\therefore \int_C \frac{dz}{z} = \underline{\underline{\pi i}}$$

Ex:- P.T. $\int_{z_1}^{z_2} z\,dz = \frac{z_2^2 - z_1^2}{2}$

Ex:- Here C_1 denote the polygonal line OAB shown in fig. & evaluate the integral.

$$I_1 = \int_{C_1} f(z)dz = \int_{OA} f(z)dz + \int_{AB} f(z)dz \quad \ldots (1)$$

where $f(z) = y - x - i3x^2$, $(z = x + iy)$



Soln:- The leg OA may be represented as $x=0$. i.e. $z = 0+iy \quad (0 \le y \le 1)$

$\therefore z = iy$ & $dz = idy$ $\Big\}$ $\therefore f(z) = y-0-0$
$\therefore f(z) = y$

$$\therefore \int_{OA} f(z)\,dz = \int_0^1 y.i\,dy = i\int_0^1 y.dy = i\left[\frac{y^2}{2}\right]_0^1$$

$$\therefore \int_{OA} f(z)\,dz = \frac{i}{2}$$

On the leg AB $\Rightarrow y=1$.

$\therefore z = x+iy \Rightarrow x+i(1)$

$\therefore z = x+i \quad (0 \le x \le 1)$

$\therefore dz = dx$

$f(z) = y-x-i3x^2$

$\therefore f(z) = 1-x-i3x^2$

$$\therefore \int_{AB} f(z)\,dz = \int_{AB} (1-x-i3x^2)\,dx$$

$$= \int_0^1 (1-x)\,dx - 3i\int_0^1 x^2\,dx$$

$$= \left[x-\frac{x^2}{2}\right]_0^1 - \cancel{3}i\left[\frac{x^3}{\cancel{3}}\right]_0^1$$

$$= \left(1-\frac{1}{2}\right) - i(1-0)$$

$$\therefore \int_{AB} f(z)\,dz = \frac{1}{2} - i$$

In view of eq^n ①, $I_1 = \frac{i}{2} - \frac{1}{2} - i = \frac{1-i}{2}$

& C_2 denotes the segment OB i.e. $y = x$.

$\therefore \quad z = x + iy$

$\therefore \quad z = x + ix \quad , (0 \le x \le 1)$

$\therefore \quad z = x(1+i)$

$\Rightarrow \quad dz = (1+i)\,dx$

$\therefore \quad f(z) = y - x - i3x^2$

$\therefore \quad f(z) = x - x - i3x^2$

$\therefore \quad f(z) = -i3x^2$

$$I_2 = \int_{OB} f(z)\,dz = \int_0^1 -i3x^2\,(1+i)\,dx$$

$$= -3i(1+i)\int_0^1 x^2\,dx$$

$$= \cancel{3}(1-i)\left[\frac{x^3}{\cancel{3}}\right]_0^1$$

$$= (1-i)\cdot(1-0)$$

$$\therefore \int_{OB} f(z)\cdot dz = \underline{\underline{1-i}}$$

Evidently then these integrals of $f(z)$ along the two paths C_1 & C_2 have ~~the~~ different values even though those paths have the same initial & final pts.

Observe that the integrals of $f(z)$ over the simple closed contour OABO, have the non-zero Value

$$I_1 - I_2 = \frac{1-i}{2} - (1-i) = \frac{1}{2} - \frac{i}{2} - 1 + i = -\frac{1}{2} + \frac{i}{2} = \underline{\underline{\frac{-1+i}{2}}}$$