* Upper bounds for moduli of contour integrals:-

- **Lemma :-** If $w(t)$ is a piecewise continuous complex-valued function defined on interval $a \le t \le b$ then

$$\left| \int_a^b w(t)\, dt \right| \le \int_a^b |w(t)|\, dt \quad \text{(i)}$$

Thm :- Let C denote a contour of length L. & suppose that a fun $f(z)$ is piecewise continuous on C. If M is a non-negative constant. such that $|f(z)| \le M$ for all points z on C at which $f(z)$ is defined, Then

$$\left| \int_C f(z)\, dz \right| \le M \cdot L \quad \ldots\ldots (1)$$

Proof:- Let $z = z(t)$, $(a \le t \le b)$ be a parametric representation on C.

According to above lemma

$$\left| \int_C f(z)\, dz \right| = \left| \int_a^b f[z(t)]\, z'(t)\, dt \right| \le \int_a^b \left| f[z(t)]\, z'(t) \right| dt$$

As $\left| f[z(t)]\, z'(t) \right| = \left| f[z(t)] \right| |z'(t)| \le M |z'(t)|$

When $a \le t \le b$ it follows that

$$\left| \int_C f(z)\, dz \right| \le M \int_a^b |z'(t)|\, dt$$

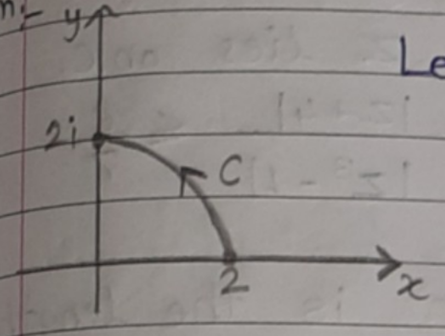
Since the integral of the right hand side represents the length L of C.

i.e. $\left| \int_C f(z)\, dz \right| \le M \cdot L$

Ex:- Without evaluating the integral [using Inequality ①]. Show that $\left|\int_C \frac{dz}{z^2-1}\right| \le \frac{\pi}{3}$.

Where c is the arc of the circle $|z|=2$ from $z=2$ to $z=2i$ lies in the 1^{st} quad.

Soln:-



Let C be the arc of the circle $|z|=2$. Shown below (fig.) without evaluating the integral, let us find an upper bound for $\left|\int_C \frac{dz}{z^2-1}\right|$.

To do this, we note that if z is a pt. on C.

$$|z^2-1| \ge \big||z^2|-1\big| = \big||z^2|-1\big| = |4-1| = 3.$$

Thus $$\left|\frac{1}{z^2-1}\right| = \frac{1}{|z^2-1|} \le \frac{1}{3}.$$

Also the length of C is $\frac{1}{4} \times (4\pi) = \pi$

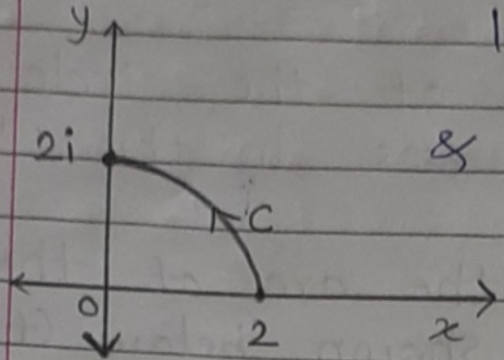
[$\because$ Perimeter of circle $= 2\pi r \Rightarrow 2\pi 2 = 4\pi$]

So taking $M = \frac{1}{3}$ & $L = \pi$

We find that
$$\left|\int_C \frac{dz}{z^2-1}\right| \le M.L \le \frac{\pi}{3}$$

Ex:- ② Let C be the arc of the circle $|z|=2$ from $z=2$ to $z=2i$ that lies in 1^{st} quad. Using inequality ①. Show that $\left|\int_C \frac{z+4}{z^3-1}\,dz\right| \le \frac{6\pi}{7}$

Sol^n:- We denote that z is a point on C so that $|z| = 2$ then



$$|z+4| \le |z| + 4 = 2 + 4 = 6$$

& $$|z^3 - 1| \ge ||z^3| - 1| = |2^3 - 1| = 7$$

Thus when z lies on C.

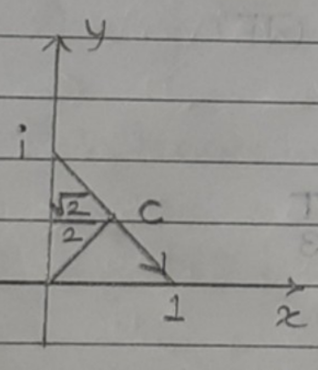
$$\left|\frac{z+4}{z^3-1}\right| = \frac{|z+4|}{|z^3-1|} \le \frac{6}{7}$$

Here $M = \frac{6}{7}$ & $L = \pi$ is the length of C.

$$\therefore \left| \int_C \frac{z+4}{z^3-1}\, dz \right| \le M \cdot L \le \frac{6\pi}{7}$$

Ex:- (3) If C denotes the line segment from $z = i$ to $z = 1$. By observing that of all the points on the line segment, the midpoint is the closest to the origin. Show that $\left| \int_C \frac{dz}{z^4} \right| \le 4\sqrt{2}$ without evaluating integral.

Sol^n:-



The path C as shown in the fig. below.

The midpoint of C is clearly the closest pt. on C to the origin.

The distance of the midpt. from the origin is clearly $\frac{\sqrt{2}}{2}$, the length of C being $\sqrt{2}$.

Hence if z is any pt. on C, then

$$|z| \geq \frac{\sqrt{2}}{2}$$

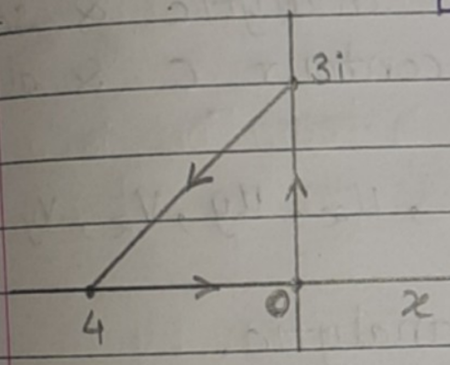
i.e. $$\left|\frac{1}{z^4}\right| = \frac{1}{|z|^4} \leq \frac{1}{\left(\frac{\sqrt{2}}{2}\right)^4} \leq \frac{1}{\frac{4}{16}} \leq 4$$

$\therefore$ Here $M = 4$ & $L = \sqrt{2}$

$$\therefore \left|\int_C \frac{dz}{z^4}\right| \leq M \cdot L \leq 4\sqrt{2}.$$

Ex:- Show that if c is boundary of the triangle with vertices at the pts. $0, 3i, \& -4$ oriented in counterclockwise direction. then $\left|\int_C (e^z - \bar{z})\,dz\right| \leq 60$

Soln:-



Let z be a pt. on the C.

$$|e^z - \bar{z}| \leq |e^z| + |\bar{z}|$$
$$\leq e^x + \sqrt{x^2+y^2}$$

But $e^x \leq 1$. since $x \leq 0$.
& the distance $\sqrt{x^2+y^2}$ of the pt. z from the origin is always less than or equal to 4.

Thus $|e^z - \bar{z}| \leq 1+4 \leq 5$. Then z is on the c.
The length of C is $4+3+5 = 12$.
$\therefore$ Here $M = 5$ & $L = 12$.

$$\therefore \left|\int_C (e^z - \bar{z})\,dz\right| \leq M \cdot L = 5 \times 12 = 60.$$

* Cauchy's theorem using Green's Theorem:-

- Green's theorem :- If $P(x,y)$, $Q(x,y)$, $\frac{\partial Q}{\partial x}$, $\frac{\partial P}{\partial y}$ all are continuous. fu^{ns} of x & y in a closed contour c, then $\int_C P\,dx + Q\,dy =$

$$\iint_C \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx \, . dy.$$

* Cauchy's theorem :-

If $f(z)$ is analytic function of z & if $f'(z)$ is continuous at each point within & on a closed contour C $\int_C f(z)dz = 0$

Proof :- If $f(z) = u + iv$ is analytic & so it is continuous on contour C & also $f'(z)$ exists.

It means that u, u_x, u_y, v_x, v_y all are continuous in C.

$f(z) = u + iv$ is analytic.

Inview of C.R. eq^{ns}.

$\Rightarrow u_x = v_y$ & $u_y = -v_x$

$$\Rightarrow \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} = 0 \quad \ldots\ldots (1)$$

$$, \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = 0 \quad \ldots\ldots (2)$$

$$\int_C f(z)\,dz = \int_C (u+iv)(dx + i\,dy)$$

$$= \int_C (u\,dx + iu\,dy + iv\,dx + i^2 v\,dy)$$

$$= \int_C [(u \cdot dx - v\,dy) + i(u \cdot dy + v \cdot dx)]$$

$$= \int_C (u \cdot dx - v \cdot dy) + i\int_C (v\,dx + u\,dy).$$

$\therefore$ By using Green's theorem,

$$\int_C f(z)\,dz = \iint_C \left(\frac{\partial(-v)}{\partial x} - \frac{\partial u}{\partial y}\right) dx \cdot dy + i\iint_C \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dx \cdot dy.$$

..... [using 1 & 2]

$$= \iint_C \left(\frac{\partial u}{\partial y} - \frac{\partial u}{\partial y}\right) dx \cdot dy + i\iint_C \left(\frac{\partial v}{\partial y} - \frac{\partial v}{\partial y}\right) dx \cdot dy$$

$$= 0 + i0 = 0$$

$$\therefore \int_C f(z) \cdot dz = 0$$

Note:- Goursat showed that for the condition of continuity on $f'(z)$ can be omitted. i.e. continuity of $f'(z)$ is not necessary & so Cauchy's theorem holds iff $f(z)$ is analytic within & on C. We now state the revised form of Cauchy's result known as Cauchy-Goursat theorem.

* **Cauchy - Goursat theorem :-**

If a function f is analytic at all points interior to & on a simple closed contour c then $\int_C f(z)\, dz = 0$.

Note:- $\int_C f(z)\, dz = -\int_{-C} f(z)\, dz = 0$

c is taken in the clockwise direction.

Ex:- ① If $f(z) = \frac{z^2+5z+6}{z-2}$, does Cauchy's theorem apply.

i) when path of integration is a circle c of radius 3 & centre at origin.

ii) When the path of integration is a circle c of radius 1 & centre at origin.

Sol^n:- i) When path is circle c given by $|z-0|=3$. then $z=2$ lies inside c & so $f(z) = \frac{z^2+5z+6}{z-2}$ is not analytic inside c.

Hence Cauchy's theorem is not applicable. & so $\int_C f(z) \neq 0$.

ii) When c is circle $|z-0| = 1$ then $z=2$ lies outside c.

$\therefore f(z) = \frac{z^2+5z+6}{z-2}$ is analytic inside c

& hence $\int_C f(z)\, dz = 0.$

Ex:- ② If C is any simple closed contour C then $\int_C e^{z^3}\, dz = 0.$

Soln:- $f(z) = \exp(z^3)$ is analytic everywhere, & $f'(z) = 3z^2 \exp(z^2)$ is continuous everywhere.

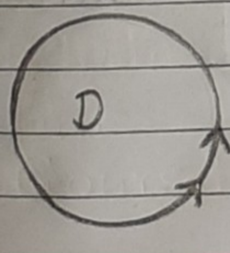
$\therefore \int_C \exp(z^3) = 0.$

* Connected region :-

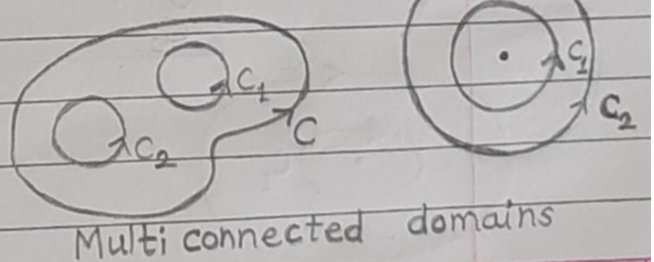
A region is said to be connected region if any two points of region D can be connected by curve which lies entirely within the region.

* Simply Connected domains :- A simply connected domain D is an open connected region (a domain) such that every closed contour within it encloses only points of D. The interior of a closed contour is an example, but the exterior is not simply connected, nor is the annular region between two concentric circles.

A domain that is not simply connected is said to be multi connected.



simply connected domains

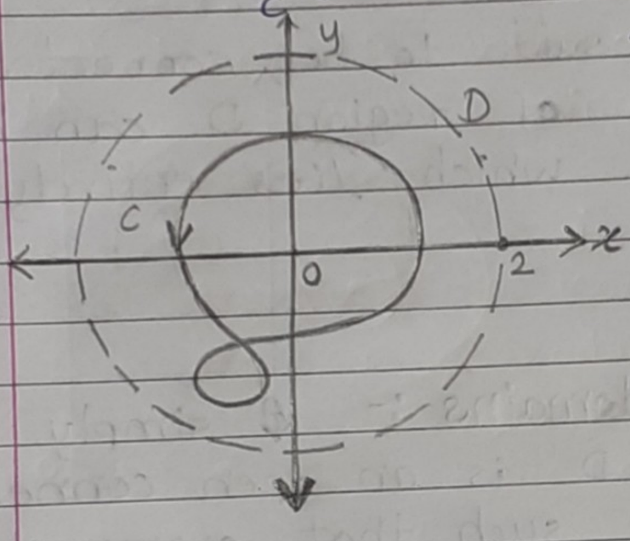


Multi connected domains

The Cauchy - Goursat theorem can be stated in the following alternate form.

* If $f(z)$ is analytic throughout a simply connected domain D then for every closed contour C within D $\int_C f(z)\,dz = 0$.

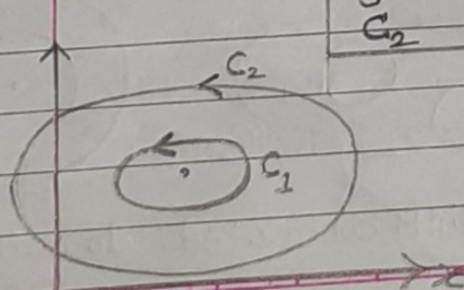
e.g. :- If C denotes any closed contour lying in the open disk $|z| < 2$ (see Fig.) then $\int_C \frac{z e^z}{(z^2+9)^5}\,dz = 0$.



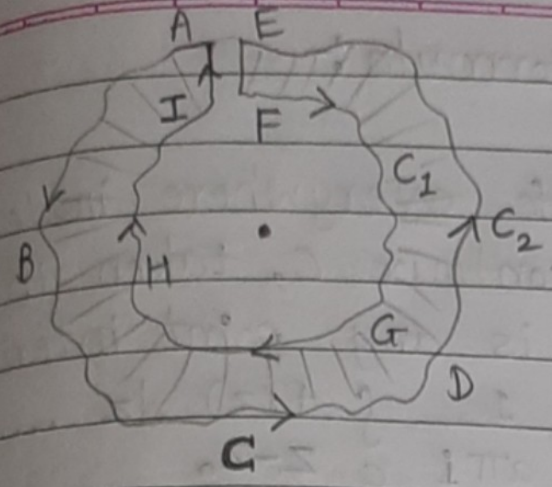
This is because the disk is simply connected domain & the two singularities $z = \pm 3i$ of the integrand are exterior of the disk.

Corr:- If C_1 & C_2 denote positively oriented simple closed contours where C_1 is interior to C_2 (fig.). If a fun f is analytic in the closed region consisting those contours & all pts. betn them.

Then $\boxed{\int_{C_2} f(z)\,dz = \int_{C_1} f(z)\,dz}$ [Cauchy's Extension thm]



e.g.:-



Cauchy's th^m,

$$\int_C f(z)\,dz = 0$$

$$\int_{ABCDEFGHIA} f(z)\,dz = 0$$

$$\Rightarrow \int_{ABCDE} f(z)\,dz + \cancel{\int_{EF} f(z)\,dz} + \int_{FGHI} f(z)\,dz + \cancel{\int_{IA} f(z)\,dz} = 0$$

$$\Rightarrow \int_{ABCDE} f(z)\,dz + \int_{FGHI} f(z)\,dz = 0$$

$$\Rightarrow \int_{ABCDE} f(z)\,dz = -\int_{FGHI} f(z)\,dz$$

$$\Rightarrow \int_{ABCDE} f(z)\,dz = \int_{IHGF} f(z)\,dz$$

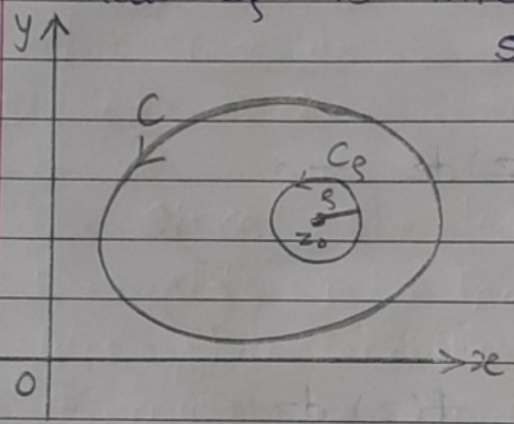
$$\Rightarrow \int_{C_2} f(z)\,dz = \int_{C_1} f(z)\,dz$$

* Cauchy's Integral formula:-

Theorem :- Let f be analytic everywhere inside and on a simple closed contour C, taken in the positive sense. If z_0 is any point interior to C, then $f(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z-z_0} dz$ ..... ①

formula ① is called the Cauchy integral formula.

Proof :- Here C_ρ denote a positively oriented circle $|z-z_0| = \rho$, where ρ is small enough that C_ρ is interior to C (see fig.)



since the quotient $\frac{f(z)}{z-z_0}$ is analytic betn & on the contours C_ρ & C, It follows from the principle of deformation of paths. [or corr. to Cauchy's thm]

$$\int_C \frac{f(z)}{z-z_0} dz = \int_{C_\rho} \frac{f(z)}{z-z_0} dz$$

i.e. $$\int_C \frac{f(z)}{z-z_0} dz = \int_{C_\rho} \frac{f(z)-f(z_0)}{z-z_0} dz + \int_{C_\rho} \frac{f(z_0)}{z-z_0} dz$$

i.e. $$\int_C \frac{f(z)}{z-z_0} dz - f(z_0) \int_{C_\rho} \frac{dz}{z-z_0} = \int_{C_\rho} \frac{f(z)-f(z_0)}{z-z_0} dz$$ ..... ②

since for $\int_{C_\rho} \frac{dz}{z-z_0}$, Here, $|z - z_0| = \rho$

$\therefore \quad z - z_0 = \rho e^{i\theta}$

$\therefore \quad dz = \rho i e^{i\theta} d\theta$

& θ varies from 0 to 2π

$$\therefore \int_{C_\rho} \frac{\rho i e^{i\theta} d\theta}{\rho e^{i\theta}} = i\int_0^{2\pi} d\theta = i[\theta]_0^{2\pi} = 2\pi i$$

$$\therefore \int_{C_\rho} \frac{dz}{z - z_0} = 2\pi i$$

& so eq^n (2) becomes;

$$\int_C \frac{f(z)}{z - z_0} dz - 2\pi i f(z_0) = \int_{C_\rho} \frac{f(z) - f(z_0)}{z - z_0} dz \quad \ldots\ldots (3)$$

Now the fact that f is analytic & therefore continuous at z_0 ensures that corresponding to each positive number ϵ, however small there is positive number δ s.t.

$$|f(z) - f(z_0)| < \epsilon \quad \text{whenever } |z - z_0| < \delta \quad \ldots\ldots (4)$$

Let the radius ρ of the circle C_ρ be smaller than the number δ in the second of these inequalities. Since $|z - z_0| = \rho < \delta$, when z is on C_ρ it follows that the 1st of inequalities (4) hold when z is such a point & theorem giving upper bounds for the moduli of contour integral tells us that

$$\left|\int_{C_\rho} \frac{f(z)-f(z_0)}{z-z_0}\,dz\right| = \int_{C_\rho} \frac{|f(z)-f(z_0)|}{|z-z_0|}\,|dz|$$

$$< \epsilon \int_{C_\rho} \frac{|dz|}{\rho}$$

$$< \frac{\epsilon}{\rho}\, 2\pi\rho \quad \left(\because \int_{C_\rho} |dz| = \text{arc of } C_\rho\right)$$

$$= 2\pi\epsilon$$

Inview of eq^n (3) then

$$\left|\int_C \frac{f(z)\,dz}{z-z_0} - 2\pi i f(z_0)\right| < 2\pi\epsilon$$

Since the left hand side of this inequality is a non-negative constant i.e. less than arbitaryly small positive number, it must be equal to zero. Hence above eq^n becomes;

$$\int_C \frac{f(z)}{z-z_0}\,dz = 2\pi i\, f(z_0)$$

i.e. $$f(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-z_0}\,dz.$$

- An Extension of the Cauchy-Integral Formula :-

* Higher order derivative of an analytic fu^n :-

If $f(z)$ is analytic in a domain D then $f(z)$ has at any pt. $z = z_0$ of D, derivatives

of all orders are again analytic funs in D
there values are given by

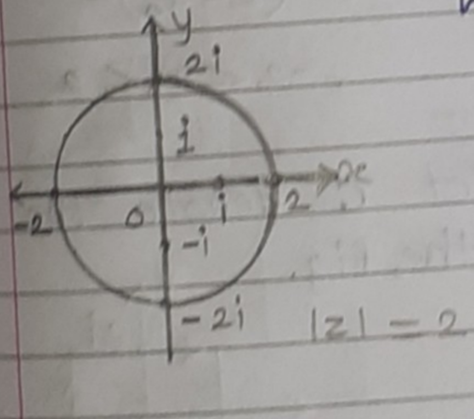
$$f^n(z_0) = \frac{n!}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz \,, (n=0,1,2,\ldots\ldots)$$

where c is any closed contour in D
surrounding the pt. $z = z_0$.

Note:- Cauchy integral formula expresses the value of an analytic function at any pt. within the contour C in terms of the value of the function at the boundary function.

Ex:- ① Evaluate $\int_C \frac{z}{(9-z^2)(z+i)}\,dz$ where c is the circle $|z| = 2$.

Solⁿ:- Here $|z|=2$ represent a circle with centre (0, 0) & radius 2.



We have $(9-z^2)(z+i) = 0$

$z^2 = 9 \,,\quad z = -i$

$z = \pm 3$

$\therefore z = -i$ lies inside the circle

& $z = \pm 3$ lies outside the circle

$\therefore$ Take $\frac{z}{9-z^2} = f(z)$

$$\therefore I = \int_C \frac{f(z)}{[z-(-i)]}\,dz$$

$= 2\pi i\, f(-i)$ using thm (C.I.F.)

$= 2\pi i \left[\frac{-i}{9-(-i)^2}\right]$

$$= 2\pi i \frac{(-i)}{9+1}$$

$$= \frac{2\pi}{10}$$

$$\therefore I = \frac{\pi}{5}$$

Ex :- (2) Evaluate $\int_C \frac{dz}{z-z_0}$ where C is any simply closed curve & $z = z_0$ is (i) outside C (ii) inside C then

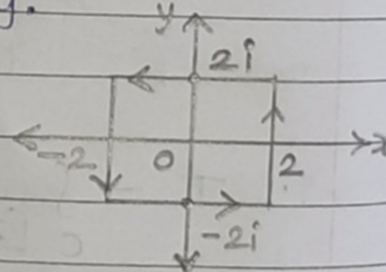
$\rightarrow$ (i) zero (ii) $2\pi i$

Ex :- (3) Let c denote the positively oriented boundary of the square whose sides lie along the lines $x = \pm 2$ & $y = \pm 2$. Evaluate each of these integrals.

(a) $\int_C \frac{e^{-z}}{z-(\pi i/2)} dz$ (b) $\int_C \frac{\cos z}{z(z^2+8)} dz$

(c) $\int_C \frac{z\, dz}{2z+1}$ (d) $\int_C \frac{\cosh z}{z^4} dz$ (e) $\int \frac{\tan(z/2)}{(z-x_0)^2} dz$, $-2 < x_0 < 2$

Sol^n :- In this problem, we let C denote the square Contour shown in the fig.



(a) $I = \int_C \frac{e^{-z}\, dz}{z-(\pi i/2)}$

Take $f(z) = e^{-z}$

$$= 2\pi i\, f\left(\frac{\pi i}{2}\right) = 2\pi i\left[e^{-z}\right]_{z=\pi i/2}$$

$$= 2\pi i\left[e^{-\pi i/2}\right]$$

$$= 2\pi i\left[\cos\frac{\pi}{2} - i\sin\frac{\pi}{2}\right]$$

$$= 2\pi i\,[0 - i]$$

$$= +2\pi$$

$$\therefore\ I = \underline{2\pi}.$$

(b) $$\int_C \frac{\cos z}{z(z^2+8)} = \int_C \frac{\cos z / z^2+8}{(z-0)}\,dz$$

$$= 2\pi i\left[\frac{\cos z}{z^2+8}\right]_{z=0} \qquad \left[\because \text{Take } f(z) = \frac{\cos z}{z^2+8}\right]$$

$$= 2\pi i\left[\frac{\cos 0}{0^2+8}\right]$$

$$= 2\pi i\left[\frac{1}{8}\right]$$

$$= \frac{2\pi i}{8}$$

$$\therefore \int_C \frac{\cos z}{z(z^2+8)} = \frac{\pi i}{4}$$

(c) $$\int_C \frac{z\,dz}{2z+1} = \int_C \frac{z/2}{z+\frac{1}{2}}\,dz = \int_C \frac{z/2}{z-(-\frac{1}{2})}\,dz$$

$$= 2\pi i\left[\frac{z}{2}\right]_{z=-\frac{1}{2}} = 2\pi i\left[-\frac{1}{4}\right]$$

$$\therefore \int_C \frac{z\,dz}{2z+1} = \underline{\underline{-\frac{\pi i}{2}}}$$

(d) $\int_C \frac{\cosh z}{z^4} dz = \int_C \frac{\cosh z}{(z-0)^4} dz = \frac{\cosh z}{(z-0)^{3+1}} dz$

$$= \frac{2\pi i}{3!}\left[\frac{d^3}{dz^3}\cosh z\right]_{z=0}$$

$\therefore f(z) = \cosh z$

$\therefore \frac{df}{dz} = \sinh z$

$\therefore \frac{d^2f}{dz^2} = \cosh z$

$\therefore \frac{d^3f}{dz^2} = \sinh z$

$\therefore \left[\frac{d^3f}{dz^2}\right]_{z=0} = [\sinh z]_{z=0} = \underline{\underline{0}}$

$\therefore \int_C \frac{\cosh z}{z^4} dz = \frac{2\pi i}{3 \times 2!}[0] = \frac{\pi i}{3}(0) = \underline{\underline{0}}$

(e) $\int_C \frac{\tan(z/2)}{(z-x_0)^2} dz = \int_C \frac{\tan(z/2)}{(z-x_0)^{1+1}} dz \quad , -2 < x_0 < 2$

$$= \frac{2\pi i}{1!}\left[\frac{d}{dz}\tan\left(\frac{z}{2}\right)\right]_{z=x_0} \text{.....(as } -2<x_0<2)$$

$$= \frac{2\pi i}{1!}\left[\frac{1}{2}\sec^2\frac{z}{2}\right]_{z=x_0}$$

$$= \frac{2\pi i}{2}\sec^2\frac{x_0}{2}$$

$\therefore \int_C \frac{\tan(z/2)}{(z-x_0)^2} dz = \pi i\left[\sec^2\left(\frac{x_0}{2}\right)\right]$, when $-2<x_0<2$

Ex:- ③ If c is positively oriented circle $|z| = 1$ & $f(z) = e^{2z}$ then $\int_C \frac{e^{2z}}{z^4} dz = ?$

Sol^n:- Let $I = \int_C \frac{e^{2z}}{z^4} dz = \int_C \frac{e^{2z}}{(z-0)^{3+1}} dz$

We know that $\int_C \frac{f(z)}{(z-z_0)^{n+1}} dz = \frac{2\pi i}{n!} [f^n(z_0)]$

Now, $f(z) = e^{2z}$

$\therefore \frac{df}{dz} = 2e^{2z}$

$\therefore \frac{d^2f}{dz^2} = 4e^{2z}$

$\therefore \frac{d^3f}{dz^3} = 8e^{2z} \Rightarrow \left[\frac{d^3f}{dz^3}\right]_{z=0} = 8e^{2(0)} = \underline{\underline{8}}.$

$\therefore \int_C \frac{e^{2z}}{z^4} dz = \frac{2\pi i}{3!}(8) = \frac{2\pi i}{3\times 2!}(8)$

$\therefore \int_C \frac{e^{2z}}{z^4} dz = \underline{\underline{\frac{8\pi i}{3}}}.$

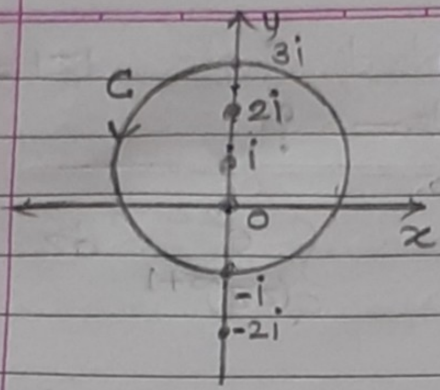
Ex:- ④ Find the value of the integral $g(z)$ around the circle $|z-i| = 2$ in the positive sense when

ⓐ $g(z) = \frac{1}{z^2+4}$ ⓑ $g(z) = \frac{1}{(z^2+4)^2}$

Sol^n:- Let c denote the positively oriented circle $|z-i| = 2$ shown below.

i.e. $|x+iy-i| = 2$

$|(x-0)+i(y-1)| = 2$



(a) $$\int_C \frac{dz}{z^2+4} = \int_C \frac{dz}{(z-2i)(z+2i)}$$

$$= \int_C \frac{1/(z+2i)}{(z-2i)}\, dz$$

$$= 2\pi i \left[\frac{1}{z+2i}\right]_{z=2i}$$

$$= 2\pi i \left(\frac{1}{4i}\right)$$

$$\therefore \int_C \frac{dz}{z^2+4} = \frac{\pi}{2}$$

(b) $$\int_C \frac{dz}{(z^2+4)^2} = \int_C \frac{dz}{(z-2i)^2(z+2i)^2}$$

$$= \int_C \frac{1/(z+2i)^2}{(z-2i)^2}\, dz$$

$$= \int_C \frac{1/(z+2i)^2}{(z-2i)^{1+1}}\, dz$$

$$= \frac{2\pi i}{1!}\left[\frac{d}{dz}\frac{1}{(z+2i)^2}\right]_{z=2i}$$

$$= 2\pi i \left[\frac{-2}{(z+2i)^3}\right]_{z=2i}$$

$$= 2\pi i \left[\frac{-2}{(4i)^3}\right]$$

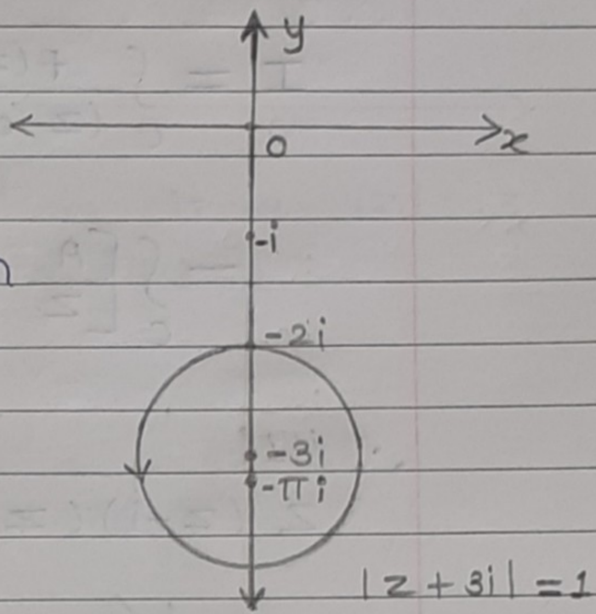
$$= \frac{-4\pi i}{-(4)^3 i} \qquad (\because i^3 = -i)$$

$$= \frac{4\pi}{64} \; 16$$

$$\therefore \int_C \frac{dz}{(z^2+4)^2} = \frac{\pi}{16}$$

Ex:- Using Cauchy Integral Formula Calculate the following integrals.

(1) $\int_C \frac{dz}{z(z+\pi i)}$ where C is $|z+3i| = 1$



Sol^n:- $I = \int_C \frac{dz}{z(z+\pi i)}$

Take $f(z) = \frac{1}{z}$ then

$$I = \int_C \frac{f(z)}{z-(-\pi i)}$$

$$= 2\pi i \, f(-\pi i)$$

Centre $(0,-3)$ & $r = 1$

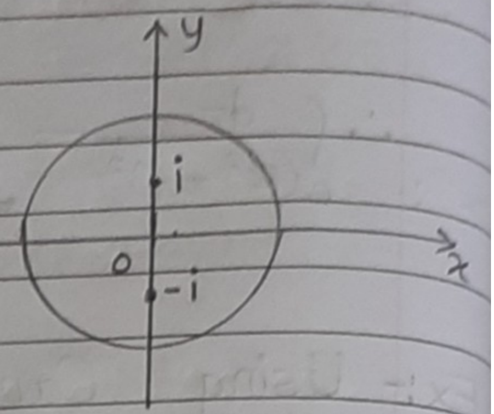
$$= 2\pi i \left(\frac{1}{-\pi i}\right)$$

$$\therefore I = -2$$

Here $f(z) = -\pi i$ lies inside C & $f(z)$ is analytic within C.

(2) $\int_C \frac{\cosh(\pi z)\, dz}{z(z^2+1)}$ where C is circle $|z| = 2$

Sol^n:- Let $I = \int_C \frac{\cosh(\pi z)}{z(z^2+1)} dz$



Take $f(z) = \cosh(\pi z)$
$= \cos(i\pi z)$

&

Circle c is $|z| = 2$

centre $(0,0)$

& $r = 2$

Then

$$I = \int_C \frac{f(z)}{z(z^2+1)} dz$$

$$I = \int_C \frac{f(z)}{(z-0)(z+i)(z-i)} dz$$

$$= \int_C \left[\frac{A}{z} + \frac{B}{z-i} + \frac{C}{z+i}\right] f(z)\, dz \quad \ldots\ldots (2)$$

$$\therefore \frac{1}{z(z-i)(z+i)} = \frac{A}{z} + \frac{B}{z-i} + \frac{C}{z+i}$$

$\therefore A = 1/(z-i)(z+i) = 1$, at $z = 0$

$\therefore B = \frac{1}{z(z+i)} = \frac{-1}{2}$, at $z = i$

$\therefore C = \frac{1}{z(z-i)} = -\frac{1}{2}$, at $z = -i$

Here $z = 0, i, -i$ are points inside c

$$\therefore \int_C \frac{f(z)}{z-z_0} dz = 2\pi i\,[f(z_0)]$$

$\therefore I = 2\pi i \,[A\, f(0) + B\, f(-i) + C f(i)]$

$= 2\pi i \,[1 \cdot \cos(0) - \frac{1}{2}\cos(-i^2\pi) - \frac{1}{2}\cos(i^2\pi)]$

$= 2\pi i \,[1 + \frac{1}{2} + \frac{1}{2}]$

$\therefore I = \underline{\underline{4\pi i}}$

③ $\int_C \frac{dz}{z-2}$ where c is the circle $|z| = 3$.

Sol^n:- $I = \int_C \frac{dz}{z-2}$

Here, $f(z) = 1$ & C is circle $|z| = 3$
centre $z = 0$ & radius $= 3$.

$\therefore z_0 = 2$ lies inside C.

$\therefore I = \int_C \frac{dz}{z-2} = 2\pi i \, f(z_0)$

$= 2\pi i \, f(2)$

$= 2\pi i \,(1)$

$= 2\pi i$ ...... $[\because f(z) = 1 \Rightarrow f(2) = 1]$

④ $\int_C \frac{dz}{z-5}$ where c is the circle $|z| = 3$.

Sol^n:- The pt. $z = 5$ lies outside the c as $|z| = 3$
Here fu^n $\frac{1}{z-5}$ is analytic inside & on c.

$\therefore$ By Cauchy's th^m, $\int_C \frac{dz}{z-5} = 0$.

⑤ $\int_C \frac{dz}{z(z-2)}$ where c is the circle $|z|=3$

Sol^n:- $I = \int_C \frac{dz}{z(z-1)} = \int_C \frac{dz}{(z-0)(z-1)}$

$C : |z| = 3$, Circle's centre at $(0,0)$, $r=3$

$\therefore z=0$, $z=1$ both lies inside c.

$\therefore I = \int_C \left[\frac{1}{z-1} - \frac{1}{z}\right] dz$

$= \int_C \frac{dz}{z-1} - \int_C \frac{1}{z} dz$

Here $f(z) = 1$.

$= 2\pi i\,[f(z_0)] - 2\pi i\,[f(z_0)]$

$= 2\pi i\,(1) - 2\pi i\,(1)$

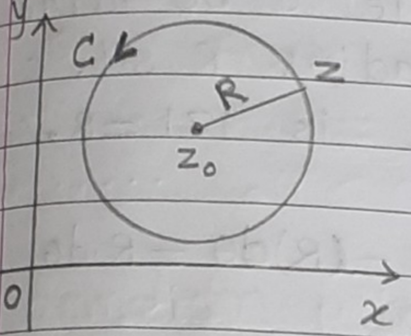
$\therefore I = \underline{\underline{0}}$.

★ Liouville's Theorem :-

If a fu^n f is entire & bounded in the complex plane, then $f(z)$ is constant throughout the plane.

Proof:- We use Cauchy's inequality
Suppose that a fu^n f is analytic

inside and on a positively oriented circle C_R centred at z_0 & with radius R (see fig). If M_R denotes the maximum value of $|f(z)|$ on C_R then $|f^n(z_0)| \le \frac{n! M_R}{R^n}$, $(n=1, 2, \ldots\ldots)$



$\ldots\ldots$ ①

We assume that f is as stated and note that since f is entire. We apply ① with any choice of z_0 & R. In particular we put $n=1$ in eq^n ①,

$$\therefore \quad |f^n(z_0)| \le \frac{M_R}{R} \quad \ldots\ldots ②$$

Moreover the boundedness condition on f tells us that a non-negative const. M exists s.t. $|f(z)| \le M \quad \forall z$ because the const. M_R in inequality ② is always less than or equal to M.

It follows that $|f'(z_0)| \le \frac{M}{R} \quad \ldots\ldots ③$

Where R can be arbitrarily large. The no. M in inequality ③ is independent of the value of R that is taken. Hence inequality holds for arbitrarily large values of R only if $f'(z_0) = 0$.

If $f'(z) = 0$ every-where in complex plane consequently f is constant.

OR

* Liouville's Theorem :-

If an entire function is bounded for all values of z then it is constant.

Proof:-

Let a & b be arbitrary distinct points in z-plane and let c be large circle with centre z = 0 & radius R s.t. c encloses a & b. Eq^n of c is $|z| = R$, so that

$$dz = iRe^{i\theta} d\theta, \quad |dz| = (R) d\theta = R \cdot d\theta$$

f is bounded $\forall z \Rightarrow |f(z)| \le M \ \forall z, M > 0$

By Cauchy's integral formula,

$$f(a) = \frac{1}{2\pi i} \int_c \frac{f(z)}{z-a} dz, \quad f(b) = \frac{1}{2\pi i} \int_c \frac{f(z)}{z-b} dz$$

$$f(a) - f(b) = \frac{1}{2\pi i} \int_c \left[\frac{1}{z-a} - \frac{1}{z-b} \right] f(z)\, dz$$

$$= \frac{1}{2\pi i} \int_c \left[\frac{z - b - z + a}{(z-a)(z-b)} \right] f(z)\, dz$$

$$= \frac{(a-b)}{2\pi i} \int_c \frac{f(z)\, dz}{(z-a)(z-b)}$$

$$|f(a) - f(b)| \le \frac{|a-b|}{|2\pi i|} \int_c \frac{|f(z)| \cdot |dz|}{(|z|-|a|)(|z|-|b|)}$$

$$\le \frac{|a-b|\, M \; 2\pi R}{2\pi (R-|a|)(R-|b|)} \quad \left[\because \int_c |dz| = 2\pi R \Downarrow \text{circumference of Circle } c \right]$$

$$\left[\because \int_c |dz| = \int_0^{2\pi} R\, d\theta = R \int_0^{2\pi} d\theta = R[2\pi - 0] = 2\pi R \right.$$

= circumference of circle.

$$|f(a)-f(b)| \leq \frac{MR\,|a-b|}{(R-|a|)(R-|b|)} \to 0 \text{ as } R\to\infty$$

$\therefore f(b)-f(a)=0$ or $f(a)=f(b)$

i.e. $f(z)$ is constant.

* Fundamental Theorem of Algebra :-

Any polynomial $P(z)=a_0+a_1z+a_2z^2+\dots+a_nz^n$, $(a_n\neq 0)$

of degree n $(n\geq 1)$ has at least one zero. That is there exists at least one point z_0 s.t. $P(z_0)=0$.