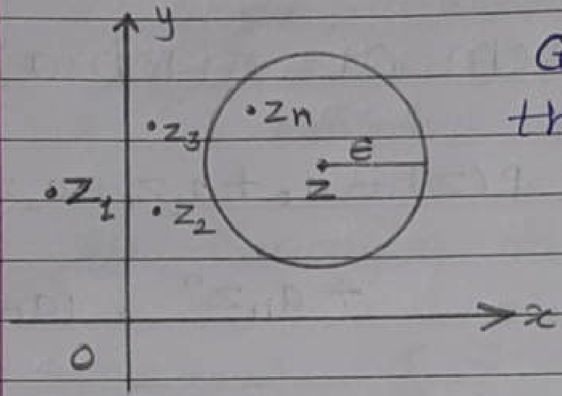
Unit- 2 : Sequences, Series and Residue Calculus

* Power Series :-

* Convergence of Sequences :-

An infinite seq^n $z_1, z_2, \ldots\ldots z_n \ldots\ldots$ ① of complex numbers has a limit z for each positive number ϵ, there exist a positive integer n_0 s.t.

$$|z_n - z| < \epsilon \quad \text{whenever} \quad n > n_0 \quad \ldots\ldots ②$$



Geometrically this means that for sufficiently large values of n, the points z_n lie in any given ϵ nhd of z (see fig) Since we choose ϵ as small as we please.

It follows that the points z_n become arbitrarily close to z as their subscripts increase.

The seq^n ① can have at most one limit. i.e. limit z is unique if it exists.

When limit exists, the sequence is said to be converges to z,

& we write $\lim\limits_{n\to\infty} z_n = z$

If the seq^n has no limit it diverges.

* Theorem :- Suppose that $z_n = x_n + iy_n$, $(n = 1, 2, \ldots)$ & $z = x + iy$ then $\lim\limits_{n\to\infty} z_n = z$ ..... ①

iff $\lim\limits_{n\to\infty} x_n = x$ & $\lim\limits_{n\to\infty} y_n = y$ ..... ②

Proof:- We 1st assume that condition ② hold & obtain condition ① from it.

According to condition ② there exists for each positive number ϵ, positive integer n_1 & n_2 such that

$$|x_n - x| < \epsilon/2 \quad , \text{whenever } n > n_1$$

& $$|y_n - y| < \epsilon/2 \quad , \text{whenever } n > n_2 .$$

Hence if n_0 is larger of two integers n_1 & n_2.

$$\left.\begin{array}{l} |x_n - x| < \epsilon/2 \\ \& \\ |y_n - y| < \epsilon/2 \end{array}\right\} \text{when } n > n_0$$

Since

$$|(x_n + iy_n) - (x + iy)| = |(x_n - x) + i(y_n - y)|$$

$$\leq |x_n - x| + |y_n - y|$$

then $|z_n - z| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$ whenever $n > n_0$.

Condition ① thus holds.

Conversely,

if we start with condition ①.

We know that for each positive number ϵ, there exists a positive integer n_0 s.t.

$$|(x_n + iy_n) - (x + iy)| < \epsilon , \text{ whenever } n > n_0$$

But $|x_n - x| \leq |(x_n - x) + i(y_n - y)| = |(x_n + iy_n) - (x + iy)|$

& $|y_n - y| \leq |(x_n - x) + i(y_n - y)| = |(x_n + iy_n) - (x + iy)|$

& this means that $|x_n - x| < \epsilon$ &
$|y_n - y| < \epsilon$ when $n > n_0$.
i.e. condition ② is satisfied. & we write
$$\lim_{n\to\infty} (x_n + i y_n) = \lim_{n\to\infty} x_n + i \lim_{n\to\infty} y_n.$$

Ex:- The seq^n $z_n = \frac{1}{n^3} + i \quad (n = 1, 2, \ldots\ldots)$
converges to i.

Sol^n:- since $\lim_{n\to\infty} \left(\frac{1}{n^3} + i\right)$
$$= \lim_{n\to\infty} \frac{1}{n^3} + i \lim_{n\to\infty} 1$$
$$= 0 + i \cdot 1 = i$$

or
$$|z_n - i| = \frac{1}{n^3} < \epsilon \quad \text{whenever } n > \frac{1}{\sqrt[3]{\epsilon}}$$

* Convergence of series :-

An infinite series
$$\sum_{n=1}^{\infty} z_n = z_1 + z_2 + \ldots\ldots + z_n + \ldots \quad \ldots\ldots ①$$

of complex numbers converges to the sum S if the seq^n
$$S_N = \sum_{n=1}^{N} z_n = z_1 + z_2 + \ldots\ldots + z_N \quad (N = 1, 2, \ldots\ldots) \quad \ldots\ldots ②$$

of partial sums converges to S, then we write $\sum_{n=1}^{\infty} z_n = S$.

Note that since a seq^n can have atmost one limit, a series can have atmost one sum when a series does not converge we say that it diverges.

Theorem :- Suppose that $z_n = x_n + iy_n$, $(n = 1, 2, \ldots)$ and

$$S = X + iY \text{ then } \sum_{n=1}^{\infty} z_n = S \quad \ldots\ldots (1)$$

iff

$$\sum_{n=1}^{\infty} x_n = X \quad \& \quad \sum_{n=1}^{\infty} y_n = Y \quad \ldots\ldots (2)$$

i.e.

$$\sum_{n=1}^{\infty} (x_n + iy_n) = \sum_{n=1}^{\infty} x_n + i \sum_{n=1}^{\infty} y_n.$$

Proof :- We first write the Partial sums (2) as

$$S_N = X_N + iY_N \quad \ldots\ldots (3)$$

Where $X_N = \sum_{n=1}^{N} x_n$ & $Y_N = \sum_{n=1}^{N} y_n$

Now statement (1) is true iff

$$\lim_{N \to \infty} S_N = S \quad \ldots\ldots (4)$$

In view of relation (3) & the th^m of seq^n (above th^m)

limit (4) holds iff

$$\lim_{N \to \infty} X_N = X \quad \& \quad \lim_{N \to \infty} Y_N = Y \quad \ldots\ldots (5)$$

$\therefore$ limits (5) therefore imply statement (1) & conversely.

Corr:→ ① If a series of complex numbers converges then n^{th} term converges to zero as n tends to infinity.

② The absolute convergence of a series of complex numbers implies the convergence of that series.

★ Taylor's Theorem :-

Let $f(z)$ be analytic at all points within a circle C_0 with centre at z_0 & radius r_0. Then for every pt. z within C_0, we have

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$$

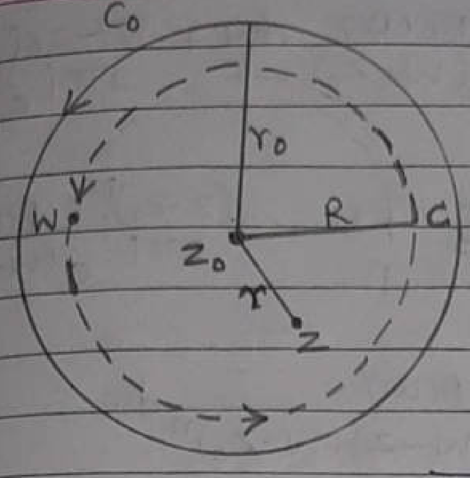
where $a_n = \dfrac{f^n(z_0)}{n!}$ $(n = 0, 1, 2, \ldots\ldots)$

i.e.

$$f(z) = f(z_0) + (z-z_0)\frac{f'(z_0)}{1!} + (z-z_0)^2\frac{f''(z_0)}{2!} + \ldots\ldots + (z-z_0)^n\frac{f^n(z_0)}{n!} + \ldots\ldots \quad \ldots\ldots ①$$

i.e. series ① converges to $f(z)$ when z lies in the stated open disk.

Proof:- Let a circle with centre z_0 & radius r_0. We have defined $|z - z_0| = r$ & another circle c within centre z_0 & Radius R, defined by $|W - z_0| = R$.



$\therefore$ By Cauchy's Integral Formula,

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(w)}{w-z}\, dw \quad \ldots\ldots (1)$$

Take identity

$$\frac{1}{w-z} = \frac{1}{(w-z_0)-(z-z_0)}$$

$$= \frac{1}{(w-z_0)\left[1-\left(\frac{z-z_0}{w-z_0}\right)\right]}$$

$$= \frac{1}{(w-z_0)}\left[1-\left(\frac{z-z_0}{w-z_0}\right)\right]^{-1}$$

We use $(1-x)^{-1} = 1 + x + x^2 + \ldots\ldots + x^n + \ldots\ldots$

$$= \frac{1}{(w-z_0)}\left[1 + \frac{z-z_0}{w-z_0} + \frac{(z-z_0)^2}{(w-z_0)^2} + \ldots\ldots + \frac{(z-z_0)^{n-1}}{(w-z_0)^{n-1}} + \frac{(z-z_0)^n}{(w-z_0)^n}\,\frac{1}{\left[1-\left(\frac{z-z_0}{w-z_0}\right)\right]}\right]$$

$$= \frac{1}{w-z_0} + \frac{z-z_0}{(w-z_0)^2} + \frac{(z-z_0)^2}{(w-z_0)^3} + \ldots\ldots + \frac{(z-z_0)^{n-1}}{(w-z_0)^n} + \frac{(z-z_0)^n}{(w-z_0)^n}\cdot\frac{1}{(w-z)}$$

Multiply by $\frac{f(w)}{2\pi i}$ in both sides & integrating around C, we get,

$$\frac{1}{2\pi i}\int_C \frac{f(w)}{w-z}\,dw = \frac{1}{2\pi i}\int_C \frac{f(w)}{(w-z_0)}\,dw + \frac{z-z_0}{2\pi i}\int_C \frac{f(w)\,dw}{(w-z_0)^2}$$

$$+\frac{(z-z_0)^2}{2\pi i}\int_C \frac{f(w)\,dw}{(w-z_0)^3} + \cdots\cdots + \frac{(z-z_0)^{n-1}}{2\pi i}\int_C \frac{f(w)\,dw}{(w-z_0)^n}$$

$$+\frac{(z-z_0)^n}{2\pi i}\int_C \frac{f(w)}{(w-z)(w-z_0)^n}\,dw$$

$$f(z) = f(z_0) + (z-z_0)\frac{f'(z_0)}{1!} + \frac{(z-z_0)^2}{2!}f''(z_0) +$$

$$\frac{(z-z_0)^3}{3!}f'''(z_0) + \cdots\cdots + \frac{(z-z_0)^{n-1}}{n!}f^{n-1}(z_0) + R_n$$

$$\cdots\cdots (2)$$

where, $R_n = \frac{(z-z_0)^n}{2\pi i}\int_C \frac{f(w)}{(w-z)(w-z_0)}\,dw$

We show that $R_n \to 0$ as $n \to \infty$

$$|R_n| = \left|\frac{(z-z_0)^n}{2\pi i}\int_C \frac{f(w)}{(w-z)(w-z_0)^n}\,dw\right|$$

$$\leq \frac{|z-z_0|^n}{|2\pi i|}\int_C \frac{|f(w)|}{|w-z|\,|w-z_0|^n}\,|dw|$$

{ let $|f(w)| = M$

$\therefore\ |w-z| = |(w-z_0) - (z-z_0)|$

$|w-z| \geq ||w-z_0| - |z-z_0||$

$|w-z| \geq |(R-r)|$

$\therefore\ |w-z| = (R-r)$

$$\therefore |R_n| \le \frac{r^n M}{2\pi (R-r) R^n} \int_C |dw|$$

$$= \frac{r^n \cdot M}{2\pi (R-r) R^n} \int_0^{2\pi} R\, dw$$

$$= \frac{r^n \cdot M}{\cancel{2\pi} (R-r) R^n} \cdot \cancel{2\pi} R$$

$$= \frac{MR}{(R-r)} \left(\frac{r}{R}\right)^n$$

Here $r < R$

$\frac{r}{R} < 1$

$\left(\frac{r}{R}\right)^n \to 0$

$R_n \to 0$ as $n \to \infty$

from eq^n 1

$$f(z) = f(z_0) + (z-z_0) f'(z_0) + \frac{(z-z_0)^2}{2!} f''(z_0) + \dots + \dots\dots$$

$$f(z) = \sum_{n=0}^{\infty} (z-z_0)^n \frac{f^n(z_0)}{n!}$$

It is known as Taylor's Series.

* <u>Maclaurins Series</u> :- When $z_0 = 0$ then Taylor's series reduces to Maclaurins series.

$$f(z) = \sum_{n=0}^{\infty} \frac{f^n(0)}{n!} z^n \quad (|z| < r_0)$$

Ex:- ① Expand the following in Taylor's Series.
$f(z)=e^z$

sol^n:- Since the fu^n $f(z)=e^z$ is entire. It has Maclaurins series representation which is valid for all z.

Here $f^n(z)=e^z$, for $n=0,1,2,\ldots$

&

$$f^n(0)=e^0=1, \text{ for } n=0,1,2,\ldots$$

$$\therefore f(z)=f(0)+zf'(0)+\frac{z^2}{2!}f''(0)+\ldots+$$

$$=1+z+\frac{z^2}{2!}+\frac{z^3}{3!}+\ldots$$

$$\therefore e^z=\sum_{n=0}^{\infty}\frac{z^n}{n!} \quad \ldots(|z|<\infty)$$

② $f(z)=\sin z$; find Maclaurines series.

sol^n:- We know that $\sin z=\dfrac{e^{iz}-e^{-iz}}{2i}$

& also $e^z=\sum_{n=0}^{\infty}\dfrac{z^n}{n!}$

$$\sin z=\frac{1}{2i}\left[\sum_{n=0}^{\infty}\frac{(iz)^n}{n!}-\sum_{n=0}^{\infty}\frac{(-iz)^n}{n!}\right]$$

$$=\frac{1}{2i}\sum_{n=0}^{\infty}\left[1-(-1)^n\right]\frac{i^nz^n}{n!}, |z|<\infty$$

But $1-(-1)^n=0$, when n is even & so we replace n by $2n+1$

$$\therefore \sin z = \frac{1}{2i}\sum_{n=0}^{\infty}\left[1-(-1)^{2n+1}\right]\frac{i^{2n+1}\cdot z^{2n+1}}{(2n+1)!} \quad (|z|<\infty)$$

Here, $1-(-1)^{2n+1} = 2$

& $i^{2n+1} = (i^2)^{(n)}\cdot i = (-1)^n\cdot i$

$$\therefore \sin z = \frac{1}{\cancel{2i}}\sum_{n=0}^{\infty}\frac{\cancel{2}(-1)^n \cancel{i}\; z^{2n+1}}{(2n+1)!} \quad (|z|<\infty)$$

$$\therefore \sin z = \sum_{n=0}^{\infty}\frac{(-1)^n z^{2n+1}}{(2n+1)!}, \quad (|z|<\infty) \;\dots\dots *$$

③ Expand $f(z) = \cos z$ (H.W.)

Soln:-I] We know that $\cos z = \dfrac{e^{iz}+e^{-iz}}{2}$

& also $e^z = \sum_{n=0}^{\infty}\dfrac{z^n}{n!}$

$$\cos z = \frac{1}{2}\left[\sum_{n=0}^{\infty}\frac{(iz)^n}{n!} + \sum_{n=0}^{\infty}\frac{(-iz)^n}{n!}\right]$$

$$= \frac{1}{2}\sum_{n=0}^{\infty}\left[(1+(-1)^n)\frac{(iz)^n}{n!}\right], \quad |z|<\infty$$

But

$1+(-1)^n = 0$ when n is odd & so we replace n by '$2n$'.

$$\therefore \cos z = \frac{1}{2}\sum_{n=0}^{\infty}\left[1+(-1)^{2n}\right]\frac{(i)^{2n}(z)^{2n}}{(2n)!}, \quad |z|<\infty$$

$$= \frac{1}{\cancel{2}}\sum_{n=0}^{\infty}\frac{\cancel{2}(-1)^n\cdot z^{2n}}{(2n)!}, \quad |z|<\infty$$

$\left[\because \quad 1+(-1)^{2n} = 1+1 = 2 \quad \& \right.$

$\left. i^{2n} = (i^2)^n = (-1)^n \right]$

$$\therefore \cos z = \sum_{n=0}^{\infty} \frac{(-1)^n \cdot z^{2n}}{(2n)!} \quad , \; |z| < \infty$$

OR

II] Differentiate both sides of eqⁿ (x)

$$\cos z = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \frac{d}{dz}\left(z^{2n+1}\right)$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} (2n+1) z^{2n}$$

$$\therefore \cos z = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!} \quad , \; |z| < \infty$$

(4) $$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n \quad , \; |z| < 1$$

i.e. $$\frac{1}{1-z} = 1 + z + z^2 + z^3 + \cdots\cdots$$

&

$$\frac{1}{1+z} = \sum_{n=0}^{\infty} (-1)^n \cdot z^n \quad , \; |z| < 1$$

i.e. $$\frac{1}{1+z} = 1 - z + z^2 - z^3 + \cdots\cdots$$

Ex:- Using $e^z = e^{z-1} \cdot e$ obtain Taylor's Series

$$e^z = e \sum_{n=0}^{\infty} \frac{(z-1)^n}{n!} \quad , (|z-1| < \infty)$$

Sol^n:- Replacing z by $z-1$ in expansion of e^z

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} \quad , |z| < \infty$$

We have,

$$e^{z-1} = \sum_{n=0}^{\infty} \frac{(z-1)^n}{n!}$$

$$e^z = e^{z-1} \cdot e = e \sum_{n=0}^{\infty} \frac{(z-1)^n}{n!}$$

Ex:- show that when $0 < |z| < 4$

* $$\frac{1}{4z - z^2} = \frac{1}{4z} + \sum_{n=0}^{\infty} \frac{z^n}{4^{n+2}}$$

Sol^n:- Suppose $0 < |z| < 4$ then $0 < |z/4| < 1$

& we know that $\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n$

when $0 < |z| < 4$

$$\frac{1}{4z - z^2} = \frac{1}{4z\left(1 - \frac{z}{4}\right)} = \frac{1}{4z} \sum_{n=0}^{\infty} \left(\frac{z}{4}\right)^n$$

$$= \sum_{n=0}^{\infty} \frac{z^{n-1}}{4^{n+1}}$$

$$= \frac{1}{4z} + \sum_{n=1}^{\infty} \frac{z^{n-1}}{4^{n+1}}$$

$$= \frac{1}{4z} + \sum_{n=0}^{\infty} \frac{z^n}{4^{n+2}}$$

Ex:- Expand the fu^n $\log(1+z)$ at the pt. $z=1$.

Sol^n:- Here $f(z) = \log(1+z) \Rightarrow f(1) = \log(1+1) = \log 2$

$$\left[\because \frac{d}{dx}(x^{-n}) = \frac{-n}{x^{n+1}}\right]$$

$$f'(z) = \frac{1}{1+z} \Rightarrow f'(1) = \frac{1}{1+1} = \frac{1}{2}$$

$$f''(z) = -\frac{1}{(1+z)^2} \Rightarrow f''(1) = \frac{-1}{(1+1)^2} = \frac{-1}{4}$$

$$f'''(z) = \frac{+2}{(1+z)^3} \Rightarrow f'''(1) = \frac{+2}{(1+1)^3} = \frac{+2}{8} = \frac{+1}{4}$$

$\vdots \qquad \vdots$

Using Taylors series at $z = a$

$$f(z) = f(a) + \frac{(z-a)}{1!} f'(a) + \frac{(z-a)^2}{2!} f''(a) + \ldots$$

$$\therefore f(z) = f(1) + \frac{(z-1)}{1!} f'(1) + \frac{(z-1)^2}{2!} f''(1) + \ldots$$

$$\therefore f(z) = \log 2 + \frac{1}{2}\frac{(z-1)}{1!} - \frac{1}{4}\frac{(z-1)^2}{2!} + \frac{1}{4}\frac{(z-1)^3}{4!} + \ldots$$

Ex:- Expand the fu^n $f(z) = \frac{1}{z+2}$ in Taylor's series at $z=1$.

Sol^n:- $f(z) = \frac{1}{z+2} \Rightarrow f(1) = \frac{1}{3}$

$$f'(z) = \frac{-1}{(z+2)^2} \Rightarrow f'(1) = \frac{-1}{9}$$

$$f''(z) = \frac{2}{(z+2)^3} \Rightarrow f''(1) = \frac{2}{27}$$

$$f'''(z) = \frac{-6}{(z+2)^4} \Rightarrow f'''(1) = \frac{-6}{81} = \frac{-2}{27}$$

Using Taylor's Series, we get,

$$f(z) = \frac{1}{3} - \frac{1}{9}\frac{(z-1)}{1!} + \frac{2}{27}\frac{(z-1)^2}{2!} - \frac{2}{27}\frac{(z-1)^3}{3!} + \ldots\ldots$$

or - Alternate method

OR

We know that $(1+x)^{-1} = 1 - x + x^2 - x^3 + \ldots\ldots |x|<1$

$(1-x)^{-1} = 1 + x + x^2 + x^3 + \ldots\ldots, |x|<1.$

Here $f(z) = \frac{1}{z+2}$, $z=1$ let $z-1=t$

$\Rightarrow z-1=0$ $z=t+1.$

$$f(z) = \frac{1}{t+1+2} = \frac{1}{t+3} = (t+3)^{-1}$$

$$\frac{1}{(t+3)}$$

$$\frac{1}{t(1+\frac{3}{t})} \qquad \frac{1}{3(\frac{t}{3}+1)}$$

$$= \frac{1}{3(t/3+1)}$$

$\therefore |t|<1$

$$= \frac{1}{3}\left[\left(1+\frac{t}{3}\right)^{-1}\right]$$

$$= \frac{1}{3}\left[1 - \frac{t}{3} + \left(\frac{t}{3}\right)^2 - \left(\frac{t}{3}\right)^3 + \ldots\ldots\right]$$

$$= \frac{1}{3}\left[1 - \frac{(z-1)}{3} + \frac{(z-1)^2}{9} - \frac{(z-1)^3}{27} + \ldots\ldots\right]$$

$$= \frac{1}{3} - \frac{(z-1)}{9} + \frac{(z-1)^2}{27} - \frac{(z-1)^3}{81} + \ldots\ldots$$

Ex:- Expand the following funⁿ in Taylor's Series

$f(z) = \dfrac{z-1}{z+1}$ at $z = 0$

at $z = 1$

Solⁿ:-

Here $f(z) = 1 - \dfrac{2}{z+1}$

OR

$$f(z) = \frac{z+1-1+1}{z+1} = \frac{z+1-2}{z+1}$$

$$f(z) = 1 - \frac{2}{z+1}$$

$$\therefore f(z) = 1 - 2(z+1)^{-1}$$

$$f(z) = 1 - 2(1 - z + z^2 - z^3 + \ldots\ldots)$$

$$f(z) = 1 - 2 + 2z - 2z^2 + 2z^3 - \ldots\ldots$$

$$f(z) = -1 + 2z - 2z^2 + 2z^3 - \ldots\ldots$$

OR $f(z) = 1 - \dfrac{2}{z+1} \Rightarrow f(0) = 1 - \dfrac{2}{0+1} = -1$

$$f'(z) = \frac{2}{(z+1)^2} \Rightarrow f'(0) = 2$$

$$f''(z) = -\frac{4}{(z+1)^3} \Rightarrow f''(0) = -4$$

$$f'''(z) = \frac{12}{(z+1)^4} \Rightarrow f'''(0) = 12$$

Using Taylor's series,

$$f(z) = f(0) + z f'(0) + \frac{z^2}{2!} f''(0) + \ldots\ldots$$

$$f(z) = -1 + z(2) + \frac{z^2}{2!}(-4) + \frac{z^3}{3!}(12) + \ldots\ldots$$

$$f(z) = -1 + 2z - 2z^2 + 2z^3 + \ldots\ldots$$

(ii) About $z = 1$

$$f(z) = 1 - \frac{2}{z+1}$$

$z = 1$
$\Rightarrow z - 1 = 0$
let $z - 1 = t$
$\Rightarrow z = t + 1$

$$= 1 - \frac{2}{t+1+1}$$

$$= 1 - \frac{2}{t+2}$$

$$= 1 - \frac{\cancel{2}}{\cancel{2}\left(\frac{t}{2}+1\right)}$$

$$= 1 - (1 + t/2)^{-1}$$

$$= 1 - [1 - t/2 + (t/2)^2 - (t/2)^3 + \ldots\ldots]$$

$$= \frac{t}{2} - \left(\frac{t}{2}\right)^2 + \left(\frac{t}{2}\right)^3 - \ldots\ldots$$

$$= \frac{t}{2} - \frac{t^2}{4} + \frac{t^3}{8} - \ldots\ldots$$

$$\therefore f(z) = \frac{(z-1)}{2} - \frac{(z-1)^2}{4} + \frac{(z-1)^3}{8} - \ldots\ldots$$

Ex:- Find the 1^{st} four terms of the Taylor's series expansion of the complex variable fu^n.

$$f(z) = \frac{(z+1)}{(z-3)(z-4)}$$ about $z=2$. Find the region of convergence.

Sol^n:- Here $f(z) = \frac{z+1}{(z-3)(z-4)}$

$$\therefore f(z) = \frac{3+1}{(z-3)(3-4)} + \frac{4+1}{(4-3)(z-4)}$$

$$f(z) = \frac{-4}{(z-3)} + \frac{5}{(z-4)}$$

$z=2$
$z-2=0$ } let $z-2=t$ & use $(1+x)^{-1} = 1-x+x^2-x^3+\ldots$
$z=t+2$ $(1-x)^{-1} = 1+x+x^2+x^3+\ldots\ldots$, $|x|<1$

$$f(z) = \frac{-4}{t+2-3} + \frac{5}{t+2-4}$$

$$f(z) = \frac{-4}{t-1} + \frac{5}{t-2}$$

$$= \frac{-4}{(-1)(1-t)} + \frac{5}{(-2)(1-t/2)}$$

$$= 4(1-t)^{-1} - \frac{5}{2}(1-t/2)^{-1}$$

$$= 4[1+t+t^2+t^3+\ldots\ldots] - \frac{5}{2}[1+(t/2)+(t/2)^2+(t/2)^3+\ldots]$$

$$= [4+4t+4t^2+4t^3+\ldots\ldots] + [\frac{-5}{2} - \frac{5t}{4} - \frac{5t^2}{8} - \frac{5t^3}{16} - \ldots\ldots]$$

$$= \left(4-\frac{5}{2}\right)+\left(4-\frac{5}{4}\right)t+\left(4-\frac{5}{8}\right)t^2+\left(4-\frac{5}{16}\right)t^3+\dots\dots$$

$$= \frac{3}{2}+\frac{11}{4}t+\frac{27}{8}t^2+\frac{59}{16}t^3+\dots\dots$$

$$\therefore f(z) = \frac{3}{2}+\frac{11}{4}(z-2)+\frac{27}{8}(z-2)^2+\frac{59}{16}(z-2)^3+\dots\dots$$

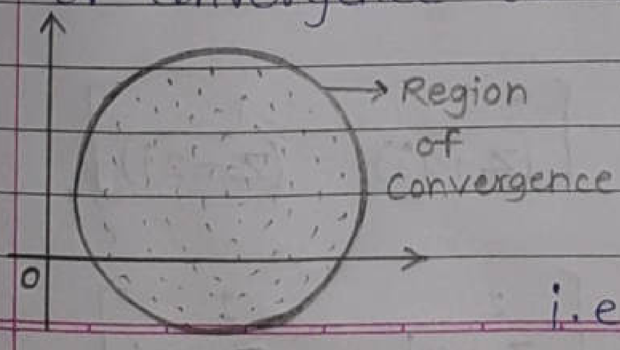
* Region of Convergence :-

① When f is analytic at all points within the circle C_0, the convergence of Taylor's series to $f(z)$ is assured. No test for the convergence of the series is required. The maximum radius of C_0 is the distance from the pt. z_0 to the singular pt. of f that is nearest z_0, since the function is to be analytic at all pts inside C_0.

OR

① For the truth of Taylors theorem it is $\boxed{\text{not}}$ necessary that $f(z)$ be analytic on the boundary of the circle c. It is reqd. is that $f(z)$ should be analytic $\boxed{\text{inside}}$ the boundary.

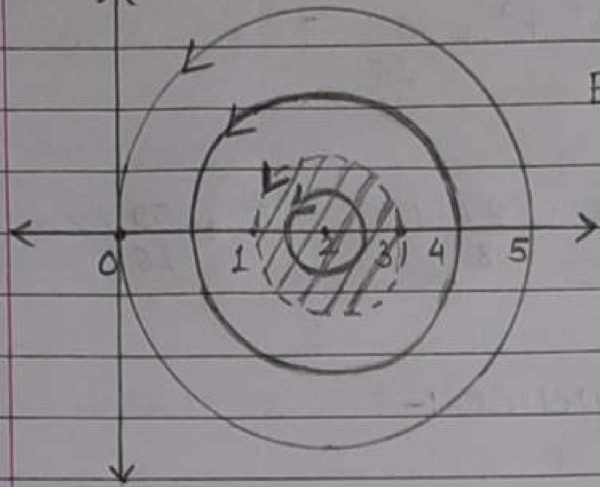
② Since $f(z)$ is analytic at all the points inside the circle c, then it is called region of convergence of Taylor's series of $f(z)$.



$$f(z) = \frac{z+1}{(z-3)(z-4)}$$

$z = 3, 4$ are singular pts.
i.e. fu^n is not analytic.
then about $z = 2$.

Now $z=3 \Rightarrow z = 3+i0 = (3,0)$
$z=4 \Rightarrow z = 4+i0 = (4,0)$



$|z-2|=1$
Eq^n of circle $|z-z_0|=R$
centre $(2,0)$ & radius 1.

Region of convergence
$|z-2|=1$
$|z-2|=1$.

Ex:- Find the 1st three terms of the Taylor's series expansion of the complex variable fu^n $f(z) = \dfrac{1}{z^2+4}$ about $z=-i$. Find the region of convergence.

Sol^n:- Here $f(z) = \dfrac{1}{z^2+4}$

$$f(z) = \frac{1}{(z+2i)(z-2i)}$$

$$f(z) = \frac{1}{(z+2i)(-2i-2i)} + \frac{1}{(2i+2i)(z-2i)}$$

$$f(z) = \frac{1}{-4i(z+2i)} + \frac{1}{4i}\cdot\frac{1}{(z-2i)}$$

$$f(z) = -\frac{1}{4i}\left[\frac{1}{(z+2i)} - \frac{1}{(z-2i)}\right]$$

$$f(z) = \frac{i}{4}\left[\frac{1}{z+2i} - \frac{1}{z-2i}\right]$$

Here $z = -i$

$\Rightarrow z + i = 0$

Let $z + i = t$

$\therefore \quad z = t - i$

$$f(z) = \frac{i}{4}\left[\frac{1}{t-i+2i} - \frac{1}{t-i-2i}\right]$$

$$= \frac{i}{4}\left[\frac{1}{t+i} - \frac{1}{t-3i}\right]$$

$$= \frac{\cancel{i}}{4}\left[\frac{1}{\cancel{i}(\frac{t}{i}+1)} - \frac{1}{(-3\cancel{i})(1 - t/3i)}\right]$$

$$= \frac{1}{4}\left[\frac{1}{(1+\frac{t}{i})} + \frac{1}{3}\left(\frac{1}{(1-t/3i)}\right)\right]$$

$$= \frac{1}{4}\left[(1 - t(i))^{-1} + 1/3\,(1 + ti/3)^{-1}\right]$$

$$= \frac{1}{4}\left[[1 + (it) + (it)^2 + \ldots] + \frac{1}{3}\left(1 - \frac{it}{3} + \frac{i^2t^2}{9} + \ldots\right)\right]$$

$$= \frac{1}{4}\left[[1 + it - t^2 + \ldots] + \frac{1}{3}\left(1 - \frac{it}{3} - \frac{t^2}{9} + \ldots\right)\right]$$

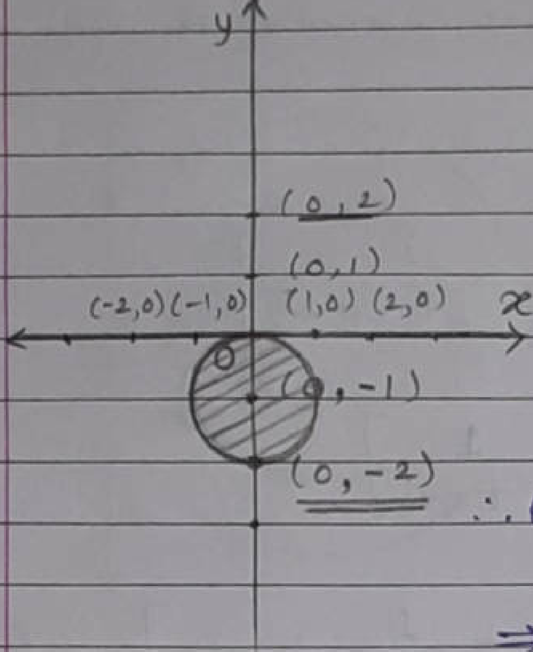
$$= \frac{1}{4}\left[(1 + it - t^2 + \ldots) + \left(\frac{1}{3} - \frac{it}{9} - \frac{t^2}{27} + \ldots\right)\right]$$

$$= \frac{1}{4}\left[\frac{4}{3} + \frac{8}{9}(it) - \frac{28}{27}t^2 + \ldots\right]$$

$$= \left[\frac{1}{3} + \frac{2}{9}(it) - \frac{7}{27}t^2 + \ldots\right]$$

$$= \frac{1}{3} + \frac{2}{9}i(z+i) - \frac{7}{27}(z+i)^2 + \ldots$$

Region of convergence



$z^2+4=0$

$\Rightarrow z=\pm 2i \Rightarrow z=2i \Rightarrow (0+2i) = (0,2)$

& $z=-2i \Rightarrow z=0-2i$

$=(0,-2)$

are singular pts.

About $z=-i$ i.e. centre

$z=-i \Rightarrow z=0-i \Rightarrow (0,-1)$

$\therefore$ Region of convergence:-

$|z-z_0|=R$

$\Rightarrow |z-(-i)|=1 \Rightarrow |z+i|=1$

★ Laurent Series :-

* Laurent Theorem :- If f is analytic of C_1 & C_2 and throughout the region between those two circles then at each point z betn them $f(z)$ is represented by a convergent series of positive & negative powers of $(z-z_0)$,

$$f(z)=\sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n}$$

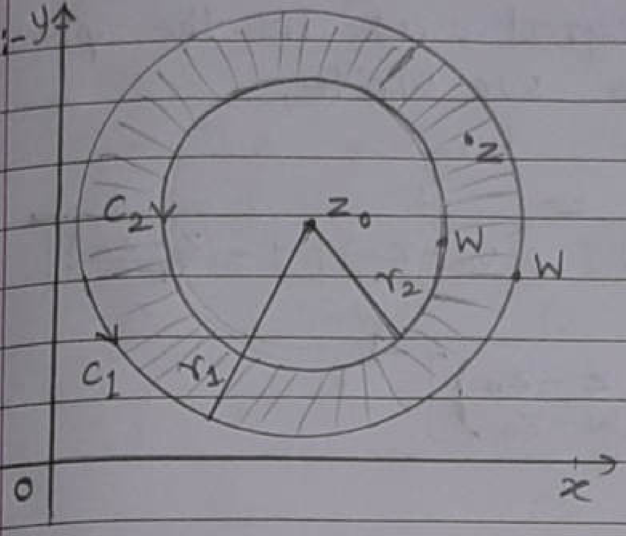
where

$$a_n = \frac{1}{2\pi i}\int_{C_1} \frac{f(w)}{(w-z_0)^{n+1}}\,dw \quad (n=0,1,2,\ldots)$$

$$\& \; b_n = \frac{1}{2\pi i}\int_{C_2} \frac{f(w)}{(w-z_0)^{-n+1}}\,dw, \quad (n=1,2,\ldots)$$

each integral being taken Counterclockwise.

Proof:-



Here $|w - z_0| = r_1$
$|w - z_0| = r_2$
about a pt. z_0 where $r_2 < r_1$.

To prove the theorem, let C_1 & C_2 two concentric circles with radius r_1 & r_2 ($r_2 < r_1$).

→ annular domain
→ two Concentric Circles

Observe that f is analytic on C_1 & C_2, as well as in the annular domain betn them.

It follows from the adaptation of the Cauchy-Goursat theorem to integrals of analytic funs. around oriented boundries of multi-connected domains. that

$$\int_{C_1} \frac{f(w)\,dw}{w-z} - \int_{C_2} \frac{f(w)}{w-z}\,dw - \int_{C} \frac{f(w)}{w-z}\,dw = 0.$$

But according to the (using) Cauchy's Integral formula the value of third integral here is $2\pi i f(z)$. Hence

$$f(z) = \frac{1}{2\pi i} \int_{C_1} \frac{f(w)\,dw}{w-z} - \frac{1}{2\pi i} \int_{C_2} \frac{f(w)\,dw}{w-z} \quad \ldots\ldots (1)$$

$$f(z) = I_1 + I_2 \quad \ldots\ldots (2)$$

Since C_1 & C_2 form the boundary of a closed region throughout which f is analytic.

In the 1^{st} integral, as in the proof of Taylor's theorem we write,

$$\frac{1}{W-z} = \frac{1}{(W-z_0)-(z-z_0)} = \frac{1}{(W-z_0)\left[1-\left(\frac{z-z_0}{W-z_0}\right)\right]}$$

$$= \frac{1}{(W-z_0)}\left[1-\left(\frac{z-z_0}{W-z_0}\right)\right]^{-1}$$

We use $(1-x)^{-1} = 1+x+x^2+\cdots\cdots, |x|<1.$

$$= \frac{1}{(W-z_0)}\left[1+\frac{z-z_0}{W-z_0}+\frac{(z-z_0)^2}{(W-z_0)^2}+\cdots\cdots+\frac{(z-z_0)^{n-1}}{(W-z_0)^{n-1}}\right.$$

$$\left.+\frac{(z-z_0)^n}{(W-z_0)^n\left[1-\left(\frac{z-z_0}{W-z_0}\right)\right]}\right]$$

$$= \frac{1}{W-z_0}+\frac{z-z_0}{(W-z_0)^2}+\frac{(z-z_0)^2}{(W-z_0)^3}+\frac{(z-z_0)^3}{(W-z_0)^4}+\cdots\cdots$$

Multiply by $\frac{f(W)}{2\pi i}$ in both sides & integrating around C_1, we get,

$$\frac{1}{2\pi i}\int_{C_1}\frac{f(w)\,dW}{W-z} = \frac{1}{2\pi i}\int_{C_1}\frac{f(\omega)}{(W-z_0)}\,dw+\frac{z-z_0}{2\pi i}\int_{C_1}\frac{f(\omega)\,dw}{(\omega-z_0)^2}$$

$$+\frac{(z-z_0)^2}{2\pi i}\int_{C_1}\frac{f(w)\,dw}{(W-z_0)^3}+\frac{(z-z_0)^3}{2\pi i}\int_{C_1}\frac{f(w)\,dw}{(W-z_0)^4}+\cdots$$

$$+\cdots\cdots$$

$$I_1 = \sum_{n=0}^{\infty}(z-z_0)^n\frac{1}{2\pi i}\int_{C_1}\frac{f(\omega)}{(\omega-z_0)^{n+1}}\,dw$$

i.e. $I_1 = \sum_{n=0}^{\infty} a_n (z-z_0)^n \quad \ldots\ldots (3)$

In the second integral we write

$$-\frac{1}{w-z} = \frac{1}{z-w} = \frac{1}{(z-z_0)-(w-z_0)}$$

$$= \frac{1}{(z-z_0)\left[1-\frac{(w-z_0)}{(z-z_0)}\right]^{-1}}$$

$$= \frac{1}{(z-z_0)}\left[1-\left(\frac{w-z_0}{z-z_0}\right)\right]^{-1}$$

$$= \frac{1}{(z-z_0)}\left[1+\left(\frac{w-z_0}{z-z_0}\right)+\left(\frac{w-z_0}{z-z_0}\right)^2+\left(\frac{w-z_0}{z-z_0}\right)^3+\cdots\right]$$

$$= \frac{1}{z-z_0}+\frac{w-z_0}{(z-z_0)^2}+\frac{(w-z_0)^2}{(z-z_0)^3}+\frac{(w-z_0)^3}{(z-z_0)^4}$$

$$+ \ldots\ldots$$

Multiply $\frac{f(w)}{2\pi i}$ in both sides & integrating around C_2 we get,

$$I_2 = \frac{1}{2\pi i}\int_{C_2}\frac{f(w)\,dw}{(z-z_0)} + \frac{1}{2\pi i}\int_{C_2}\frac{(w-z_0)}{(z-z_0)^2}f(w)\,dw +$$

$$\frac{1}{2\pi i}\int_{C_2}\frac{(w-z_0)^2}{(z-z_0)^3}f(w)\,dw + \ldots\ldots$$

$$= \frac{1}{2\pi i}\int_{C_2}\frac{f(w)\,dw}{(z-z_0)} + \frac{1}{2\pi i}\cdot\frac{1}{(z-z_0)^2}\int_{C_2}(w-z_0)f(w)\,dw$$

$$+\frac{1}{2\pi i}\frac{1}{(z-z_0)^3}\int_{C_2}(w-z_0)^2 f(w)\,dw$$

$$+\frac{1}{2\pi i}\frac{1}{(z-z_0)^4}\int_{C_2}(w-z_0)^3 f(w)\,dw.$$

$$=\frac{1}{(z-z_0)}\cdot\frac{1}{2\pi i}\int_{C_2} f(w)\,dw+\frac{1}{(z-z_0)^2}\cdot\frac{1}{2\pi i}\int(w-z_0)f(w)\,dw$$

$$+\frac{1}{(z-z_0)^3}\cdot\frac{1}{2\pi i}\int_{C_2}(w-z_0)^2\cdot f(w)\,dw+\cdots\cdots$$

$$=\sum_{n=1}^{\infty}\frac{1}{(z-z_0)^n}\cdot\frac{1}{2\pi i}\int_{C_2}(w-z_0)^{n-1} f(w)\,dw$$

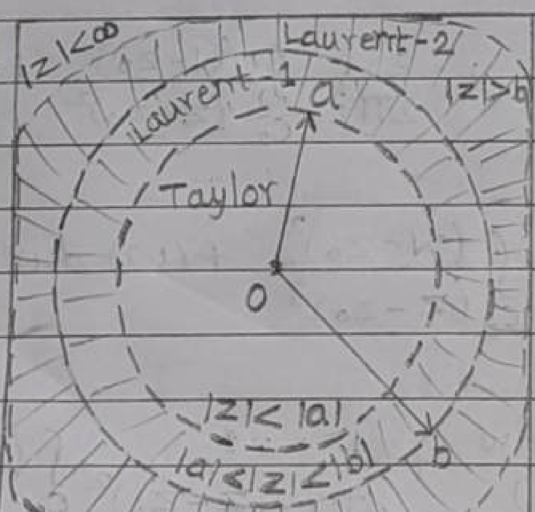
$$=\sum_{n=1}^{\infty}(z-z_0)^{-n}\cdot\frac{1}{2\pi i}\int_{C_2}\frac{f(w)}{(w-z_0)^{-n+1}}\,dw\quad, n=1,2,\ldots$$

$$=\sum_{n=1}^{\infty} b_n (z-z_0)^{-n}\quad\ldots\ldots (4)$$

$\therefore$ From (3) & (4); (2) becomes;

$$\therefore f(z)=\sum_{n=0}^{\infty} a_n (z-z_0)^n+\sum_{n=1}^{\infty} b_n (z-z_0)^{-n}$$

Examples :-



(1) Expand $\frac{1}{z^2-3z+2}$

When

(i) $0<|z|<1$

(ii) $1<|z|<2$

(iii) $|z|>2$

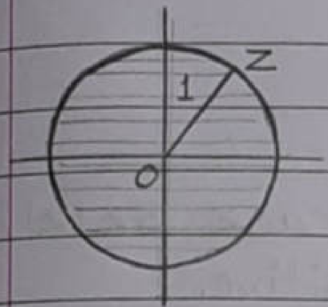
Sol^n:- let $f(z) = \dfrac{1}{z^2 - 3z + 2} = \dfrac{1}{(z-1)(z-2)}$

then

$$f(z) = \frac{1}{(2-1)(z-2)} + \frac{1}{(1-2)(z-1)} \quad \text{.....[By Partial fractions].}$$

$$= \frac{1}{z-2} - \frac{1}{z-1} \quad \text{.....} \; (1)$$

(1) When $0 < |z| < 1$



$|z| < 1$

$|z - 0| < 1$ Centre $(0,0)$ & radius 1.

Hence, $|z| < 1$

Taylor's series

We know that

$(1+x)^{-1} = 1 - x + x^2 - x^3 + \ldots$

& $(1-x)^{-1} = 1 + x + x^2 + x^3 + \ldots$

$|x| < 1$

$$= \frac{1}{-2(1 - z/2)} - \frac{1}{-1(1-z)}$$

$$\frac{1}{z-2} \;<\; \frac{1}{z(1-\frac{2}{z})} \;\text{X} \qquad \frac{1}{-2(1-z/2)} \;✓$$

$|z| < 1$
Put $z = 0.5$
& check

&

$$\frac{1}{z-1} \;<\; \frac{1}{z(1-1/z)} \;\text{X} \qquad \frac{1}{-1(1-z)} \;✓$$

from (1),

$$f(z) = \frac{1}{-2(1-z/2)} - \frac{1}{-1(1-2)}$$

i.e. $f(z) = \dfrac{1}{1-z} - \dfrac{1}{2(1-z/2)}$

$$= (1-z)^{-1} - \frac{1}{2}(1 - z/2)^{-1}$$

$$= (1+z+z^2+z^3+\ldots\ldots) - \frac{1}{2}\left(1+\frac{z}{2}+\left(\frac{z}{2}\right)^2+\left(\frac{z}{2}\right)^3+\ldots\ldots\right)$$

$$= \left(1-\frac{1}{2}\right)+\left(z-\frac{z}{4}\right)+\left(z^2-\frac{z^2}{8}\right)+\left(z^3-\frac{z^3}{16}\right)+\ldots\ldots$$

$$= \frac{1}{2}+\frac{3z}{4}+\frac{7z^2}{8}+\frac{15z^3}{16}+\ldots\ldots$$

$$\therefore f(z) = \frac{1}{2}+\frac{3z}{4}+\frac{7z^2}{8}+\frac{15z^3}{16}+\ldots\ldots$$

OR $\rightarrow$ $$= \sum_{n=0}^{\infty}(z^n) - \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n$$

$$= \sum_{n=0}^{\infty}\left[1-\frac{1}{2^{n+1}}\right]z^n.$$

This is Taylor's series in case $0<|z|<1$. i.e. All powers of z are positive.

(ii) when $1<|z|<2$

Then $\frac{1}{|z|}<1$, $\frac{|z|}{2}<1$.

Now expression (1) becomes;

$$f(z) = \frac{1}{z-2} - \frac{1}{z-1}$$

$$\frac{1}{z-2} \quad \begin{cases} \frac{1}{z(1-2/z)} \\ \frac{1}{-2(1-z/2)} \checkmark \end{cases}$$

$\underline{z = 1.5}$ check

&

$$\frac{1}{z-1} \quad \begin{cases} \frac{1}{z(1-1/z)} \checkmark \\ \frac{1}{-1(1-z)} \end{cases}$$

$\therefore$ from (1),

$$f(z) = \frac{1}{-2(1-z/2)} - \frac{1}{z(1-1/z)}$$

$$= -\frac{1}{2}(1-z/2)^{-1} - \frac{1}{z}(1-1/z)^{-1}$$

$$= -\frac{1}{2}\left[1+\frac{z}{2}+\left(\frac{z}{2}\right)^2+\cdots\right] - \frac{1}{z}\left[1+\frac{1}{z}+\frac{1}{z^2}+\cdots\right]$$

$$= -\frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n - \frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n$$

$$= -\frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n - \frac{1}{z}\sum_{n=0}^{\infty}(z^{-1})^n$$

This is Laurent series expansion in the annulus $1 < |z| < 2$.

③ When $|z| > 2$ then $\frac{2}{|z|} < 1$

So that $\frac{1}{|z|} < \frac{1}{z} < 1$

Now expression ① becomes;

$$f(z) = \frac{1}{z-2} - \frac{1}{z-1}$$

$$\frac{1}{z-2} < \begin{cases} \frac{1}{z(1-\frac{2}{z})} \checkmark \\ \frac{1}{-2(1-\frac{z}{2})} \end{cases}$$

$$f(z) = \frac{1}{z(1-2/z)} - \frac{1}{z(1-1/z)}$$

$$\frac{1}{z-1} < \begin{cases} \frac{1}{z(1-1/z)} \checkmark \; z=4 \\ \frac{1}{-1(1-z)} \end{cases}$$

$$= \frac{1}{z}\left(1-\frac{2}{z}\right)^{-1} - \frac{1}{z}\left(1-\frac{1}{z}\right)^{-1}$$

$$= \frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{2}{z}\right)^n - \frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n$$

This is Laurent's expansion in the annulus $2 < |z| < R$

Ex:- Obtain the Taylor & Laurent's series which represents the fun $f(z) = \dfrac{z^2-1}{(z+2)(z+3)}$ in the regions.

i) $|z| < 2$ ii) $2 < |z| < 3$ iii) $|z| > 3$.

Soln:- $$f(z) = \frac{z^2-1}{(z+2)(z+3)}$$

$$\begin{array}{r|l} z^2+5z+6 & z^2-1 \quad (1 \\ & \underline{z^2+5z+6} \\ & -5z-7 \end{array}$$

$$= 1 - \frac{5z+7}{(z+2)(z+3)}$$

$$= 1 - \frac{5(-2)+7}{(z+2)(-2+3)} - \frac{5(-3)+7}{(-3+2)(z+3)}$$

$$= 1 + \frac{3}{(z+2)} - \frac{8}{z} \quad \ldots\ldots (1)$$

Case (1) When $|z| < 2$ then $\frac{|z|}{2} < 1$

$$f(z) = 1 + \frac{3}{2\left(1+\frac{z}{2}\right)} - \frac{8}{3\left(1+z/3\right)}$$

$$= 1 + \frac{3}{2}\left(1+\frac{z}{2}\right)^{-1} - \frac{8}{3}\left(1+z/3\right)^{-1}$$

$$= 1 + \frac{3}{2}\left[1 - (z/2) + (z/2)^2 - \left(\frac{z}{2}\right)^3 + \ldots\right]$$

$$- \frac{8}{3}\left[1 - (z/3) + (z/3)^2 - (z/3)^3 + \ldots\right]$$

$$= 1 + \frac{3}{2}\sum_{n=0}^{\infty} (-1)^n \frac{z^n}{2^n} - \frac{8}{3}\sum_{n=0}^{\infty} (-1)^n \frac{z^n}{3^n}$$

$$= 1 + \sum_{n=0}^{\infty} (-1)^n \left[\frac{3}{2^{n+1}} - \frac{8}{3^{n+1}}\right] z^n.$$

This is Taylor's Series valid for $|z| < 2$.

② When $2 < |z| < 3$ then $\frac{2}{|z|} < 1$, $\frac{|z|}{3} < 1$

$\therefore$ from ①,

$$f(z) = 1 + \frac{3}{z+2} - \frac{8}{z+3}$$

$$\frac{3}{z+2} \to \frac{3}{z(1+2/z)} \checkmark, \quad \frac{3}{2(1+z/2)}$$

&

$$\frac{8}{z+3} \to \frac{8}{z(1+3/z)}, \quad \frac{8}{3(1+z/3)} \checkmark$$

$$= 1 + \frac{3}{z(1+\frac{2}{z})} - \frac{8}{3(1+z/3)}$$

$$= 1 + \frac{3}{z}\left(1+\frac{2}{z}\right)^{-1} - \frac{8}{3}\left(1+\frac{z}{3}\right)^{-1}$$

$$= 1 + \frac{3}{z}\left[1 - \frac{2}{z} + \left(\frac{2}{z}\right)^2 - \left(\frac{2}{z}\right)^3 + \cdots\right]$$

$$- \frac{8}{3}\left[1 - \frac{z}{3} + \left(\frac{z}{3}\right)^2 - \left(\frac{z}{3}\right)^3 + \cdots\right]$$

$$= 1 + \frac{3}{z}\sum_{n=0}^{\infty} (-1)^n \cdot \frac{2^n}{z^n} - \frac{8}{3}\sum_{n=0}^{\infty} (-1)^n \cdot \frac{z^n}{3^n}$$

$$= 1 + \sum_{n=0}^{\infty} (-1)^n \left[\frac{3 \cdot 2^n}{z^{n+1}} - \frac{8 \cdot z^n}{3^{n+1}}\right]$$

This is Laurent Series.

③ When $|z| > 3$ then $\frac{3}{|z|} < 1$

$$\Rightarrow \frac{2}{|z|} < \frac{2}{3} < 1.$$

from ①

$$f(z) = 1 + \frac{3}{z+2} - \frac{8}{z+3}$$

$$\frac{1}{z+2} < \begin{matrix} \frac{1}{z(1+2/z)} \checkmark \\ \frac{1}{2(1+\frac{z}{2})} \end{matrix}$$

$$= 1 + \frac{3}{z(1+\frac{2}{z})} - \frac{8}{z(1+\frac{3}{z})}$$

8

$$\frac{1}{z+3} < \begin{matrix} \frac{1}{z(1+3/z)} \checkmark \\ \frac{1}{3(1+\frac{z}{3})} \end{matrix}$$

$$= 1 + \frac{3}{z}\left(1+\frac{2}{z}\right)^{-1} - \frac{8}{z}(1+3/z)^{-1}$$

$$= 1 + \frac{3}{z}\sum_{n=0}^{\infty}(-1)^n\left(\frac{2}{z}\right)^n - \frac{8}{z}\sum_{n=0}^{\infty}(-1)^n\left(\frac{3}{z}\right)^n$$

$$= 1 + \sum_{n=0}^{\infty}\frac{(-1)^n}{z^{n+1}}\left[3 \cdot 2^n - 3^n \cdot 8\right]$$

$\therefore$ This is Laurent series.

③ Expand $e^{\frac{1}{z}}$

$\Rightarrow$ We know that $e^z = \sum_{n=0}^{\infty}\frac{z^n}{n!} = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \dots\dots$

Put $z = \frac{1}{z}$ in above series.

$$\therefore e^{1/z} = \sum_{n=0}^{\infty}\frac{1}{n!\,z^n} = \frac{1}{1!\,z} + \frac{1}{2!\,z^2} + \frac{1}{3!\,z^3} + \dots\dots$$

No positive powers of z appear here the coeff. of positive powers being zero. & coeff. of $\frac{1}{z}$ is unity.

$\therefore$ According to Laurent's theorem.

$$b_n = \frac{1}{2\pi i}\int_C \frac{f(w)}{(z-z_0)^{n+1}} \quad (n = 1, 2, \dots\dots)$$

$b_1 = \frac{1}{2\pi i}\int_C e^{1/z}\,dz$, where C is positively oriented simple closed contour around the origin. since $b_1 = 1$.

$$\therefore \int_C e^{1/z}\,dz = 2\pi i$$

④ Find the Laurent Series that represent the fu^n $f(z) = z^2 \sin\left(\frac{1}{z^2}\right)$ in the domain $0<|z|<\infty$

Sol^n:- We know that $\sin z = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!}$ $(|z|<\infty)$.

$$\therefore \sin\left(\frac{1}{z^2}\right) = \sum_{n=0}^{\infty} \frac{(-1)^n (1/z^2)^{2n+1}}{(2n+1)!}$$

$$= \sum_{n=0}^{\infty} (-1)^n \cdot \frac{1}{(2n+1)!} \cdot \frac{1}{(z^2)^{2n+1}}$$

$$\sin(1/z^2) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n+1)!} \cdot \frac{1}{z^{4n+2}}$$

$$z^2 \cdot \sin(1/z^2) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n+1)!} \cdot \frac{1}{z^{4n}}$$

$$\therefore z^2 \cdot \sin\left(\frac{1}{z^2}\right) = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n+1)!\, z^{4n}}$$

⑤ Find the representation for the fu^n $f(z) = \frac{1}{1+z}$ when $1 < |z| < \infty$ in negative powers of z.

Sol^n:- $f(z) = \frac{1}{1+z} = \frac{1}{z(1+1/z)} = \frac{1}{z}\left(1+\frac{1}{z}\right)^{-1}$

$$= \frac{1}{z}\left[1 - \frac{1}{z} + \frac{1}{z^2} - \frac{1}{z^3} + \cdots\cdots\right]$$

$$= \frac{1}{z} - \frac{1}{z^2} + \frac{1}{z^3} - \frac{1}{z^4} + \cdots\cdots$$

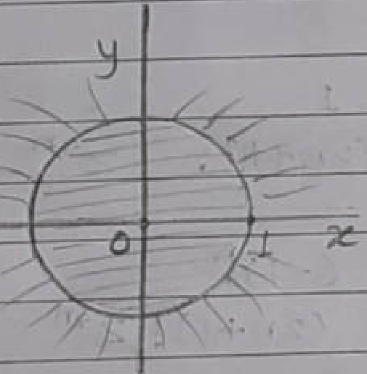
$$\therefore \frac{1}{1+z} = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{z^n}, \text{ when } 1<|z|<\infty.$$

Ex:- ⑥ Give two Laurent's series expansion in powers of z for the function. $f(z) = \frac{1}{z^2(1-z)}$ and specify the regions in which those expansions are valid.

Sol^n:- Here $f(z) = \frac{1}{z^2(1-z)}$

$z=0$ & $z=1$ are the singular points.

Hence there are Laurent series in powers of z for the domains $0<|z|<1$ & $1<|z|<\infty$



i) For $0<|z|<1$

$$f(z) = \frac{1}{z^2(1-z)}$$

$$= \frac{1}{z^2}(1-z)^{-1}$$

$$= \frac{1}{z^2}\sum_{n=0}^{\infty} z^n = \sum_{n=0}^{\infty} z^{n-2}$$

$$= \frac{1}{z^2} + \frac{1}{z} + \sum_{n=2}^{\infty} z^{n-2}$$

$$= \frac{1}{z^2} + \frac{1}{z} + (1+z+z^2+\cdots)$$

$$\therefore f(z) = \frac{1}{z^2} + \frac{1}{z} + \sum_{n=0}^{\infty} z^n$$

ii) For $1 < |z| < \infty$

$$f(z) = \frac{1}{z^2(1-z)} = \frac{1}{z^2(-z)(1-1/z)}$$

$$= \frac{1}{-z^3}(1-1/z)^{-1}$$

$$= -\frac{1}{z^3}(1 + 1/z + 1/z^2 + \ldots\ldots)$$

$$= -\frac{1}{z^3}\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n$$

$$= -\sum_{n=0}^{\infty}\frac{1}{z^{n+3}} = -\sum_{n=3}^{\infty}\frac{1}{z^n}.$$

Ex:- 7) Represent the fu^n $f(z) = \frac{z+1}{z-1}$

a) by its Maclaurins series, & state where the representation is valid.

b) by its Laurent's series in the domain $1 < |z| < \infty$.

Sol^n:-

a) The Maclaurin's series for the fu^n $\frac{z+1}{z-1}$ is valid when $|z| < 1$.

$$f(z) = \frac{z+1}{z-1} = \frac{z-1+2}{z-1} = 1 + \frac{2}{z-1}$$

$$= 1 + \frac{2}{(-1)(1-z)}$$

$$= 1 - 2(1-z)^{-1}$$

$$= 1 - 2[1 + z + z^2 + \ldots\ldots]$$

$$= 1 - 2\sum_{n=0}^{\infty} z^n$$

$$= 1 - 2 - 2\sum_{n=1}^{\infty} z^n$$

$$= -1 - 2\sum_{n=1}^{\infty} z^n, \quad (|z| < 1).$$

(b) When $1 < |z| < \infty$

$$\therefore \frac{1}{|z|} < 1.$$

$$f(z) = \frac{z+1}{z-1} = 1 + \frac{2}{z-1}$$

$$= 1 + \frac{2}{z(1 - 1/z)}$$

$$= 1 + \frac{2}{z}(1 - 1/z)^{-1}$$

$$= 1 + \frac{2}{z}\left[1 + 1/z + 1/z^2 + 1/z^3 + \ldots\right]$$

$$= 1 + 2\left[1/z + 1/z^2 + 1/z^3 + 1/z^4 + \ldots\ldots\right]$$

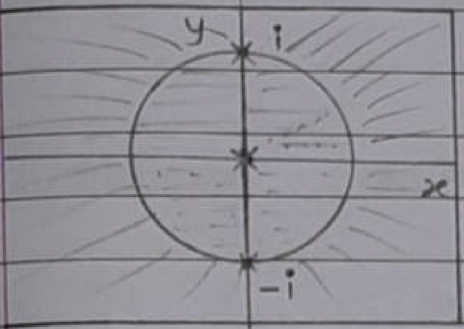
$$= 1 + 2\sum_{n=1}^{\infty} \frac{1}{z^n} \quad \text{for } 1 < |z| < \infty.$$

(8) Write the two Laurent Series in powers of z that represent the fu^n $f(z) = \frac{1}{z(1+z^2)}$ in certain domains & specify those domains.

Sol^n:-

The fu^n $f(z) = \frac{1}{z(1+z^2)}$ has singularity at $z = 0$ & $z = \pm i$

$\boxed{\begin{aligned} \& \; 1+z^2 &= 0 \\ z^2 &= -1 \end{aligned}}$



Hence there is Laurent series representation for the domain $0 < |z| < 1$ & also for the domain $1 < |z| < \infty$.
which is exterior of circle $|z| = 1$.

1) When $0 < |z| < 1$

$$f(z) = \frac{1}{z} \cdot \frac{1}{(1+z^2)} = \frac{1}{z}(1+z^2)^{-1}$$

$$= \frac{1}{z}\left[1 - (z^2) + (z^2)^2 - (z^3)^3 + \ldots\ldots\right]$$

$$= \frac{1}{z} - z + z^3 - z^5 + z^7 - \ldots\ldots$$

$$= \frac{1}{z} - \left[z - z^3 + z^5 - \ldots\ldots\right]$$

$$= \frac{1}{z} + \sum_{n=1}^{\infty} (-1)^n z^{2n-1}.$$

2) When $1 < |z| < \infty \Rightarrow \frac{1}{|z|} < 1$

$$f(z) = \frac{1}{z \cdot z^2(1+1/z^2)}$$

$$f(z) = \frac{1}{z^3}(1+1/z^2)^{-1}$$

$$= \frac{1}{z^3}\left[1 - \frac{1}{z^2} + \frac{1}{z^4} - \frac{1}{z^6} + \dots\dots\right]$$

$$= \frac{1}{z^3}\sum_{n=0}^{\infty}\left(-\frac{1}{z^2}\right)^n$$

$$= \sum_{n=0}^{\infty}\frac{(-1)^n}{z^{2n+3}}$$

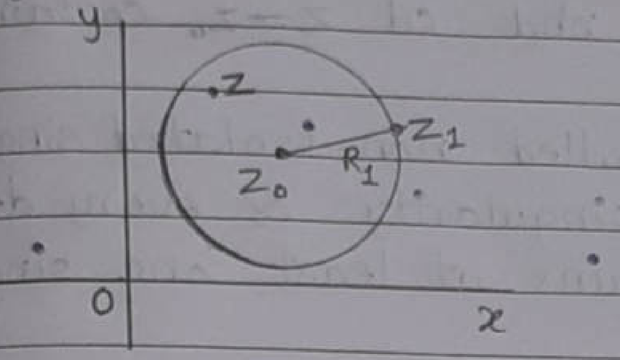
$$= \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{z^{2n+1}}$$

★ Absolute and uniform convergence of Power series:-

• Defn:- If the series $\sum_{n=1}^{\infty} z_n$ is absolutely convergent that is, if the series $\sum_{n=1}^{\infty}|z| = \sum_{n=1}^{\infty}\sqrt{x_n^2+y_n^2}$ converges then it follows from the comparison test for the series of positive real no.s that the two series $\sum_{n=1}^{\infty}|x_n|$ & $\sum_{n=1}^{\infty}|y_n|$ both converges.

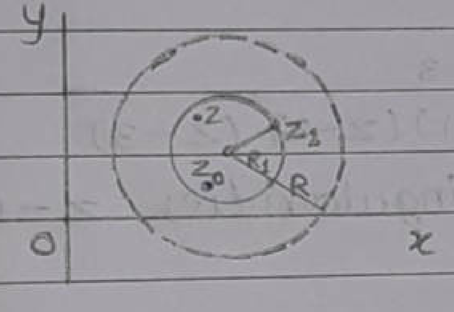
Thus, $X=\sum_{n=1}^{\infty} x_n$ & $Y=\sum_{n=1}^{\infty} y_n$ are absolutely convergent.

• Theorem :- If a power series $\sum_{n=0}^{\infty} a_n (z-z_0)^n$ converges when $z = z_1$, $(z_1 \neq z_0)$ then it is absolutely convergent at each point z in open disk $|z-z_0| < R_1$ where $R_1 = |z_1 - z_0|$.



* Uniform Convergence :-

*Th^m :- If z_1 is a point inside the circle of convergence $|z-z_0| = R$ of a power series $\sum_{n=0}^{\infty} a_n (z-z_0)^n$ then that series must be uniformly convergent in the closed disk $|z-z_0| \leq R_1$, where $R_1 = |z_1 - z_0|$



* Singular Points :- A singularity (or singular pt.) of a fu^n is the pt. at which the fu^n ceases to be analytic.

e.g.- $f(z) = \frac{1}{z-2}$ then $z=2$ is a singularity of $f(z)$.

* **Isolated Singular Points :-**

A pt. z_0 is said to be isolated singularity of $f(z)$ if

1. $f(z)$ is not analytic at z_0.
2. $f(z)$ is analytic in the deleted nhd. of z_0.

i.e. there exists a nhd of $z=z_0$ containing no other singularity.

If $z=z_0$ is called non-isolated singularity of $f(z)$ if $z=z_0$ singularity & every deleted nhd of $z=z_0$ contains at least one singularity of $f(z)$.

e.g.- ① The fun $f(z) = \dfrac{z+1}{z^3(z+1)}$ has three isolated singular pts. $z=0,\ z=\pm i$.

② The fun $f(z) = \dfrac{1}{z}$ is analytic everywhere except at $z=0$.

$\therefore z=0$ is an **isolated** singularity.

③ The fun $f(z) = \dfrac{z+3}{(z-1)(z-2)(z-3)}$

has three isolated singularities $z=1,2,3$.