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Hilbert Space

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Inner product:-

Let X be a complex vector space. Let $:X \times X \to \mathbb{C}$ be an inner product defined by (x, y), $x, y \in X$ satisfying.

i] $(x, y) \geq 0$ and $(x, x) = 0$ iff $x = 0$.

ii] $(\alpha x + \beta y, z) = \alpha(x, z) + \beta(y, z)$, $\alpha, \beta \in \mathbb{C}$.

iii] $\overline{(x, y)} = (y, x)$ $\forall\ x, y \in X$.

Space forming vector space with inner product is called inner product space (IPS). [(X, I) is an IPS].

Remark:-

1] Let X is an inner product space then show that $(x, \alpha y + \beta z) = \bar{\alpha}(x, y) + \bar{\beta}(x, z)$, $\forall$, $x, y, z \in X$, $\alpha, \beta \in \mathbb{C}$.

$\rightarrow$ Let,

Given that,

X is an inner product space.

consider,

$$
\begin{aligned}
(x, \alpha y + \beta z) &= \overline{(\alpha y + \beta z, x)} \\
&= \overline{\alpha (y\ x) + \beta(z\ x)} \\
&= \bar{\alpha}\ \overline{(y\ x)} + \bar{\beta}\ \overline{(z\ x)} \\
&= \bar{\alpha}\ (x\ y) + \bar{\beta}\ (x\ z).
\end{aligned}
$$

2] Let (X, I) be an inner product space. Let $x \in X$, define $\|\cdot\|$ on X by, $\|x\| = \sqrt{(x, x)}$ then clearly, $\|\cdot\|$ is an norm on X.

$\therefore$ X is an normed linear space.

$\Rightarrow$ Every IPS in nls.

$V \times V \to F.$
$x \cdot y = \|x\| \|y\| \cos\theta$
$x \times y = \begin{vmatrix} i & j & k \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix}$

$z \cdot \bar{z} = |z|^2$

3] If $(x, y) = 0$ then x is orthogonal to y, $x \perp y$
$x, y \in X$, and
0 is orthogonal to X, as,
$(x, 0) = (x, y-y) = (x, y) - (x, y) = 0$
$(0, x) = (y-y, x) = (y, x) - (y, x) = 0$
$\forall x \in X$.

Hilbert space :- ~~and~~
An complete inner product space is Hilbert space
Every Hilbert space is Banach space but converse need not be true.

*] Examples of inner product space.
l_2 1] Space (l_2^n) of n tuples of scalars,
$(x_1, x_2, \ldots x_n) \in l_2^n$ where $\sum_{i=1}^{n} |x_i|^2 < \infty$
Define ~~inner~~ (x, y) on $l_2^n \times l_2^n$ by,
$(x, y) = \sum_{i=1}^{n} x_i \bar{y_i}$ $\forall x, y \in l_2^n$. Show that with defined fun^n l_2^n is an inner product space.

→ Proof:-
Let l_2^n be the space of n-tuples of scalars $(x_1, x_2, \ldots x_n) \in l_2^n$ where $\sum_{i=1}^{n} |x_i|^2 < \infty$.
i] Consider,
$(x, x) \Leftrightarrow \sum_{i=1}^{n} x_i \bar{x}_i$

$\Leftrightarrow \sum_{i=1}^{n} |x_i|^2$

If $(x, x) = 0$

$\Leftrightarrow \sum_{i=1}^{n} |x_i|^2 = 0$

$\Leftrightarrow \quad |x_i|^2 = 0 \quad \forall \; x_i$

$\Leftrightarrow \quad |x_i| = 0 \quad \forall \; x_i$

$\Leftrightarrow \quad x_i = 0 \quad \forall \; x_i$

$\Leftrightarrow \quad x = 0$

ii) consider,

$\alpha, \beta \in \mathbb{C}, \; x, y, z \in X.$

$$(\alpha x + \beta y, z) = \sum_{i=1}^{n} (\alpha x_i + \beta y_i)\bar{z}_i$$

$$= \sum_{i=1}^{n} \alpha x_i \bar{z}_i + \beta y_i \bar{z}_i$$

$$= \sum_{i=1}^{n} \alpha x_i \bar{z}_i + \sum_{i=1}^{n} \beta y_i \bar{z}_i$$

$$= (\alpha x, y) + (\beta y, z)$$

$$= ~~(\alpha x + \beta y,~~$$

$$= \alpha(x, z) + \beta(y, z).$$

iii) consider,

$$\overline{(x, y)} = \sum_{i=1}^{n} \overline{x_i \bar{y}_i}$$

$$= \sum_{i=1}^{n} \bar{x}_i y_i$$

$$= \sum_{i=1}^{n} y_i \bar{x}_i$$

$$\therefore \overline{(x, y)} = (y, x)$$

$\therefore$ From (i), (ii) & (iii),

l_2^n forms inner product space with inner product (function).

Schwart'z inequality:-

If x, y are any two vectors in Hilbert space (H) then $|(x, y)| \leq ||x|| \cdot ||y||$.

→ **Proof:-**

Let H be the Hilbert space.

$x, y \in H$.

Consider, $y = 0$ or $x = 0$.

$(x, y) = (x, 0) = 0$

$\Rightarrow |(x, y)| = 0$ — (I)

and $||x|| \cdot ||y|| = ||x|| \cdot 0 = 0$ — (II)

From (I) & (II),

If one of x, y is zero,

$|(x, y)| = ||x|| \cdot ||y||$.

Consider $x \neq 0$ and $y \neq 0$ in H.

Then cosider $\alpha \in F$,

$0 \leq ||x - \alpha y||^2$ then,

$0 \leq (x - \alpha y, x - \alpha y)$

$0 \leq (x, x) - \bar{\alpha}(x, y) - \alpha(y, x) + \alpha\bar{\alpha}(y, y)$.

$$0 \le (x,x) - \bar{\alpha}(x,y) - \alpha(y,x) + |\alpha|^2 (y,y)$$
$$0 \le (x,x) - \bar{\alpha}(x\ y) - \alpha(y,x) + |\alpha|^2 (y,y)$$

Put $\alpha = \frac{(x,y)}{(y,y)} \Rightarrow, \quad \bar{\alpha} = \frac{\overline{(x,y)}}{(y,y)} = \frac{(y,x)}{(y,y)}$

$$0 \le (x,x) - \frac{(y,x)}{(y,y)}(x,y) - \frac{(x,y)}{(y,y)} \cdot (y,x) + \left(\frac{|(x,y)|}{|(y,y)}\right)^2 (y,y)$$

$$\le \|x\|^2 - \frac{(y,x)}{\|y\|^2}(x,y) - \frac{(x,y)}{\|y\|^2}(y,x) + \left(\frac{|(x,y)|}{|(y,y)|}\right)^2 \|y\|^2$$

$$\le \|x\|^2 - \frac{\overline{(x\ y)}}{\|y\|^2}(x,y) - \frac{(x,y)}{\|y\|^2}\overline{(x\ y)} + \left(\frac{|(x,y)|}{\|y\|^2}\right)^2 \cdot \|y\|^2$$

$$\le \|x\|^2 - 2\frac{|(x,y)|^2}{\|y\|^2} + \frac{|(x,y)|^2}{\|y\|^2}$$

$$0 \le \|x\|^2 - \frac{|(x,y)|^2}{\|y\|^2}$$

$$|(x,y)|^2 \le \|x\|^2 \cdot \|y\|^2 .$$

Thm 1:-

The inner product is jointly continuous on any Hilbert space.

⇒ Proof:-

Let H be an Hilbert space.
Let $x, y \in H$.
consider,
$x_n \to x$ as $n \to \infty$ in H
and $y_n \to y$ as $n \to \infty$ in H.

Now,

$|(x_n\ y_n) - (x, y)|$

$= |(x_n\ y_n) - x_n y + x_n y - (x, y)|$

$= |(x_n\ \ y_n - y) + (x_n - x, y)|$

By schwart'z inequality,

$|(x_n\ y_n) - (x, y)| \leq \|x_n\| \cdot \|y_n - y\| + \|x_n - x\| \cdot \|y\|$

consider as $n \to \infty$.

$|(x_n\ y_n) - (x\ y)| \longrightarrow 0$

$|(x_n\ y_n) \longrightarrow |(x, y)|$ as $n \to \infty$

$\therefore$ This shows that inner product in Hilbert space is jointly continuous.

Parallelogram Law:-

In any Hilbert Space,

$2\|x\|^2 + 2\|y\|^2 = \|x+y\|^2 + \|x-y\|^2$

$\forall\ x, y \in H$.

→ **Proof:-**

Let $x, y \in H$.

consider,

$\|x+y\|^2 + \|x-y\|^2 = (x+y, x+y) + (x-y, x-y)$

$= \cancel{(x+x+x+y,\ x+y + x-x-y)}$

$= \cancel{(2x,}$

$= (x, x) + (x\ y) + (y, x) + (y, y)$

$(x\ x) - (x\ y) - (y, x) + (y, y)$

$= 2(x, x) + 2(y, y)$

$= 2 \cdot \|x\|^2 + 2 \cdot \|y\|^2$

Convex set:-

If a subset C of Hilbert space H satisfies $tx + (1-t)y \in C \quad \forall x, y \in C, t \in [0,1]$ then C is convex.

Thm 2:-

A closed convex subset C of Hilbert space H contains a unique vector of smallest norm.

Proof:-

Let C be a closed convex subset of Hilbert space H.

To prove that $\exists$ unique $x \in H$ such that $\|x\| = \inf\{\|y\|, y \in C\}$.

Since C is a convex subset,

$t = \frac{1}{2}$
$tx + (1-t)y \in C$

for $x, y \in C$, $\frac{x+y}{2} \in C$

be any number

Let $d = \inf\{\|y\|, y \in C\}$.

By definition,

be any no.

$\exists \{x_n\}$ is C such that $\|x_n\| \to d$ as $n \to \infty$.

Vector

As C is convex,

$$\frac{x_n + x_m}{2} \in C$$

be any elements in $\inf\{\|y\| \; y \in C\}$.

Then $\left\| \frac{x_n + x_m}{2} \right\| \geq d$

$$\|x_n + x_m\| \geq 2d \Rightarrow -\|x_n + x_m\|^2 \leq -4d^2$$

This will hold for any x_n and x_m in seq^n $\{x_n\}$ in C.

Now by Parallelogram law,

$$\|x_n - x_m\|^2 = 2\|x_n\|^2 + 2\|x_m\|^2 - \|x_n + x_m\|^2.$$

$$\|x_n - x_m\|^2 \le 2\|x_n\|^2 + 2\|x_m\|^2 - 4d^2$$

as $m, n \to \infty$

$\|x_n - x_m\|^2 \to 0$ as $\|x_n\| \to d$, $\|x_m\| \to d$

This shows that, $\{x_n\}$ is cauchy seq^n.
As C is closed subspace of complete space it is complete.

$\therefore$ $\{x_n\}$ converges in C.

Let $x_n \to x$ as $n \to \infty$ in C

$\Rightarrow$ $\|x_n\| \to \|x\|$

$\Rightarrow$ $\cancel{\|x\|} \Leftrightarrow \|x\| = d$ $\qquad (\because \|x_n\| \to d)$.

~~Uniquen~~ $\therefore \exists\ x \in C$ such that,

$$\|x\| = \inf\{\|y\| \quad y \in C\}$$

<u>Uniqueness</u> :-

Let $\exists\ x, x'$ in C such that,

$\|x\| = \|x'\| = d$. $\qquad$ where $x \neq x'$

Now,

$$x, x' \in C \Rightarrow \frac{x + x'}{2} \in C$$

be any +ve no.

By Parallelogram law,

$$\left\|\frac{x}{2} + \frac{x'}{2}\right\|^2 = 2\left\|\frac{x}{2}\right\|^2 + 2\left\|\frac{x'}{2}\right\|^2 - \left\|\frac{x}{2} - \frac{x'}{2}\right\|^2$$

$$\le 2\left\|\frac{x}{2}\right\|^2 + 2\left\|\frac{x'}{2}\right\|^2$$

$$= 2 \cdot \frac{d^2}{4} + 2 \cdot \frac{d^2}{4}$$

$$= d^2$$

$$\left\|\frac{x}{2} + \frac{x'}{2}\right\| \le d$$

infimum

$\therefore \left\|\frac{x}{2} + \frac{x'}{2}\right\|$ is less than infimum.

which is a contradiction.

$\therefore$ x must be equal to x'.

In any Hilbert Space H,

$\|x+y\|^2 \rightarrow (x+y, x+y)$

$$4(x, y) = \|x+y\|^2 - \|x-y\|^2 + i\|x+iy\|^2 - i\|x-iy\|^2.$$

Proof:-

Let Given,

$$4(x, y) = \|x+y\|^2 - \|x-y\|^2 + i\|x+iy\|^2 - i\|x-iy\|^2$$

Consider,

$$\begin{aligned}
\text{R.H.S.} &= \|x+y\|^2 - \|x-y\|^2 + i\|x+iy\|^2 - i\|x-iy\|^2 \\
&= (x+y, x+y) - (x-y, x-y) + i(x+iy, x+iy) \\
&\quad - i(x-iy, x-iy) \\
&= (x,x) + (x,y) + (y,x) + (y,y) - (x,x) + (x,y) \\
&\quad + (y,x) - (y,y) + i[(x,x) + i(y,x) + i(x,y) \\
&\quad - i^2(y,y)] - i[(x,x) + i(x,y) - i(y,x) - i^2(y,y)] \\
&= 2(x,y) + 2(y,x) - 2(y,x) + 2(x,y) \\
&= 4(x,y)
\end{aligned}$$

R.H.S. = L.H.S.

$$\therefore 4(x, y) = \|x+y\|^2 - \|x-y\|^2 + i\|x+iy\|^2 - i\|x-iy\|^2$$

Note:- satisfies parallelo law.

1] If b is complex banach space whose norm where parallelogram law holds and if an inner product on B is defined as,

$$(x, y) = \frac{\|x+y\|^2}{4} - \frac{1}{4}\|x-y\|^2 + \frac{i}{4}\|x+iy\|^2 - \frac{i}{4}\|x-iy\|^2$$

Then B is Hilbert Space.

Orthogonal Complement:-

1] Orthogonal elements:-

Two vectors x and y in Hilbert space H are orthogonal if,

$(x, y) = 0$ i.e. $x \perp y$.

2] $x \perp y \Rightarrow y \perp x$

3] $0 \perp x \quad \forall x \in H$

4] zero is only element orthogonal to itself as $(x, x) = 0 \Leftrightarrow x = 0$.

5] Let $x, y \in H$ and $x \perp y$ then,

$$\begin{aligned}\|x-y\|^2 &= 2\|x\|^2 + 2\|y\|^2 - \|x+y\|^2 \\ &= 2\|x\|^2 + 2\|y\|^2 - [\|x\|^2 + \|y\|^2] \\ &= \|x\|^2 + \|y\|^2\end{aligned}$$

$$\begin{aligned}\|x+y\|^2 &= 2\|x\|^2 + 2\|y\|^2 - \|x-y\|^2 \\ &= 2\|x\|^2 + 2\|y\|^2 - [\|x\|^2 + \|y\|^2] \\ &= \|x\|^2 + \|y\|^2.\end{aligned}$$

Orthogonal complement:- (to a set)

Let S be a non-empty subset of Hilbert space H.

A vector $x \in H$ is orthogonal to S if

$x \perp s \quad \forall s \in S$ and the orthogonal complement of S ($S^\perp$) is set of such $x \in H$,

$$S^\perp = \{x \in H \mid x \perp S\}$$
$$= \{x \in H \mid x \perp y \quad \forall y \in S\}$$
$$= \{x \in H \mid (x, y) = 0 \quad \forall y \in S\}.$$

Remark :-

1] $\{0\}^\perp = \{x \in H \mid (x, 0) = 0\} = H$.

2] $H^\perp = \{x \in H \mid (x, y) = 0 \quad \forall y \in H\} = \{0\}$.

3] Show that $S \cap S^\perp \subseteq \{0\}$.

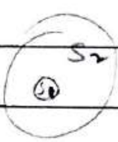
$\rightarrow$ Let

$$x \in S \cap S^\perp$$
$$\Rightarrow x \in S, \text{ and } x \in S^\perp$$
$$\Rightarrow (x, x) = 0$$
$$\Rightarrow x = 0$$
$$\Rightarrow S \cap S^\perp \subseteq \{0\}$$

4] Show that if $S_1 \subseteq S_2 \Rightarrow S_2^\perp \subseteq S_1^\perp$.

$\rightarrow$ Let,

$$S_1 \subseteq S_2$$



Let, $x \in S_2^\perp$

$$\Rightarrow (x, \cancel{y}\,y) = 0 \quad \forall y \in S_2$$
$$\Rightarrow (x, y) = 0 \quad \forall y \in S_1 \quad (\because S_1 \subseteq S_2).$$
$$\Rightarrow x \in S_1^\perp$$
$$\Rightarrow S_2^\perp \subseteq S_1^\perp$$

5] $S^\perp$ is show that $S^\perp$ is closed linear subspace of H.

→ Let,

S be any non-empty set.

Let $S^{\perp} = \{x \in H \mid (x, y) = 0 \quad \forall y \in S\} \subseteq H$.

i) $S^{\perp}$ is linear.

Let, $x, y \in S^{\perp}$, $z \in S$

$$(\alpha x + \beta y), z) = \alpha(x, z) + \beta(y, z) \qquad \forall z \in S$$

$$= \alpha \cdot 0 + \beta \cdot 0$$

$$= 0 \qquad \forall z \in S, \alpha, \beta \in \mathbb{C}.$$

$\therefore (\alpha x + \beta y, z) \in S^{\perp}$

ii) $S^{\perp}$ is closed.

Let, $x_n \in S^{\perp}$ then $\forall n$

$\Rightarrow (x_n, y) = 0 \qquad ; \forall y \in S, \forall n$

We know that, inner product is jointly continuous in Hilbert space.

$\therefore (\cdot, \cdot)$ is continuous.

$\Rightarrow (x_n, y) \to (x, y) \qquad \forall y \in S$

$\Rightarrow \lim_{n \to \infty} (x_n, y) = 0 = (x, y) \qquad \forall y \in S$

$\Rightarrow (x, y) = 0 \qquad \forall y \in S$

$\Rightarrow x \in S^{\perp}$

$\therefore S^{\perp}$ is closed.

6) Prove that $S \subseteq S^{\perp\perp}$, $S^{\perp\perp} = (S^{\perp})^{\perp}$

→ Let,

$x \in S$

$(x, y) = 0 \qquad \forall y \in S^{\perp}$

$x \in (S^{\perp})^{\perp} \qquad \Rightarrow S \subseteq S^{\perp\perp}$

7) If M is closed linear subspace of Hilbert space H then we know $M^\perp$ is closed linear subspace and $M \cap M^\perp = \{0\}$

Th^m 3:-

Let M be a closed linear subspace of Hilbert space H, let a vector $x \notin M$ then define $d = d(x, M) = \inf\{\|x-m\|, m \in M\}$ there exist unique vector y_0 in M such that $\|x - y_0\| = d$.

→ Proof:-

Let M be a closed linear subspace of Hilbert space H.

$x+m_1$, $4\ x+m_2$

$t(x+m_1)+(1-t)(x+m)$

consider the set,

$C = x + M = \{x+m, m \in M\}$

x be any ... it doesn't change.

Then C is convex ~~subspace~~ set of H.

To prove that C is closed.

Let $z_n \to z$ as $n \to \infty$ be seq^n in C.

since $z_n \in C \Rightarrow z_n = x + m_n \quad \forall n, m_n \in M$

As M is closed,

$m_n \to m \quad$ as $n \to \infty$, in M

$x+m_n \to x+m \quad$ as $n \to \infty$

$\therefore z_n \to x+m$

$\therefore z = x+m \in C$

$\Rightarrow C$ is closed.

As C is closed convex subset of H

$\exists$ a unique vector of smallest norm.

$\Rightarrow \exists\ z_0$ in C such that $\|z_0\| = d'$.

$z_0 \in C \Rightarrow z_0 = x+m$.

$z_0 \in$ ... $\|z_0\|$ ... M

$x - z_0$
$= x - (x+m)$
$= -m \in M$

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where $d' = \inf\{\|x+m\|,\ m \in M\}$
consider the vector $y_0 = x - z_0 \in M$ and
$\|x - y_0\| = \|z_0\| = d'$

$\therefore$ There exist y_0 in M such that $\|x-y_0\| = d'$.

Uniqueness:
Let $\exists\ y_0', y_0 \in M$ such that, $y_0 \neq y'$
$y_0 = y_0' = x - z_0$
$\Rightarrow \|y_0'\| = \|x - y_0'\| = d'$
$\Rightarrow \|x - y'\| = d'$

$\therefore \exists\ \|z'\| = d'$ in M
$\Rightarrow$ which contradicts to uniquess of z_0.
$\Rightarrow$ y_0 is unique.

Th^m 4:-
If M is proper closed linear subspace of Hilbert space H then there exist a non-zero vector z_0 in H such that $z_0 \perp M$.

$\rightarrow$ Proof:-
Since is proper closed linear subspace of H such that $x \notin M$.
By previous th^m, $\exists$ unique y_0 in M s.t.
$\|x - y_0\| = d = \inf\{\|x - m\|,\ m \in M\} > 0$
Take $z_0 = x - y_0 \in H$,
$\|z_0\| = d$
and as $x \notin M$, $z_0 \neq 0$.
To show that z_0 is orthogonal to M.
Consider $y \in M$ be any element.
consider $\|z_0 - \alpha y\|$ $\forall\ \alpha \in \mathbb{C}$.

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$y \in M$
$y_0 \in M$
$\alpha y + y_0 \in M$

$$\|z_0 - \alpha y\| = \| x - y_0 - \alpha y\|$$
$$= \| x - \underbrace{(\alpha y + y_0)}_{\in M}\| \geq d = \|z_0\|$$
$$\Rightarrow \|z_0 - \alpha y\| \geq \|z_0\|.$$

$$\Rightarrow \|z_0 - \alpha y\|^2 \geq \|z_0\|^2$$
$$\Rightarrow \|z_0 - \alpha y\|^2 - \|z_0\|^2 \geq 0$$
$$\Rightarrow (z_0 - \alpha y \,, z_0 - \alpha y) - (z_0 \,, z_0) \geq 0$$
$$\Rightarrow (z_0 \, z_0) - \bar{\alpha}(z_0 \, y) - \alpha(y \, z_0) + \bar{\alpha}\alpha(y, y)$$
$$- (z_0 \, z_0) \geq 0$$

$\alpha\bar{\alpha} = |\alpha|^2$

$$\Rightarrow -\bar{\alpha}(z_0 \, y) - \alpha(y \, z_0) + \alpha\bar{\alpha}(y \, y) \geq 0$$
$$\Rightarrow -\bar{\alpha}(z_0 \, y) - \alpha(y \, z_0) + |\alpha|^2(y \, y) \geq 0$$

α be any no.

This holds for any $y \in M$, $\alpha \in K$.
Inparticular for $\alpha = \beta(z_0, y)$, $\beta \in R$ $\forall \alpha$.
be any no. (y, z_0) be any no.

β is real
$\bar{\beta} = \beta$

$$\therefore -\beta\overline{(z_0 \, y)}\,(z_0 \, y) - \beta(z_0, y) + |\beta(z_0 \, , y)|^2 (y, y) \geq 0.$$
$$\therefore -\beta(y, z_0)(z_0 \, y) - \beta(z_0 \, y)(y, z_0) + |\beta(z_0 \, y)|^2 (y, y) \geq 0$$

$$\Rightarrow -2\beta(y, z_0)(z_0 \, y) + |\beta(z_0 \, y)|^2 (y, y) \geq 0$$

$$\Rightarrow +\beta\,|(z_0 \, y)|^2 \left[-2 + \beta\,\|y\|^2\right] \geq 0$$
non-zero, positive

If β is so small such that,
$$-2 + \beta\,\|y\|^2 < 0$$
Then also above inequality must hold, as β is any real number.
Which gives,
$$|(z_0 \, , y)|^2 = 0$$
$$\Rightarrow (z_0 \, y) = 0$$
$$\Rightarrow z_0 \perp y \qquad y \in M.$$
$$\Rightarrow z_0 \perp M.$$

Th^m 5:-

If M and N are subspaces of H then $M+N = \{m+n \mid m \in M,\ n \in N\}$. If $M \perp N$ then $M+N$ is orthogonal so on.

Show that $M+N$ is linear subspace of H.

→ **Proof:-**

Let M and N

Let $x, y \in M+N$

$\Rightarrow \quad x = m_1 + n_1, \qquad m_1 \in M, \quad n_1 \in N$

$\quad\quad y = m_2 + n_2 \qquad m_2 \in M, \quad n_2 \in N$

Consider, $\quad \alpha, \beta \in F$

$$\begin{aligned}
\alpha x + \beta y &= \alpha(m_1 + n_1) + \beta(m_2 + n_2) \\
&= \alpha m_1 + \alpha n_1 + \beta m_2 + \beta n_2 \\
&= \alpha m_1 + \beta m_2 + \alpha n_1 + \beta n_2 \\
&= m_3 + n_3 \qquad , m_3 \in M,\ n_3 \in N
\end{aligned}$$

$\therefore \alpha x + \beta y \in M+N$

$\therefore$ $M+N$ is linear subspace.

Th^m 6:-

If M and N are closed linear subspaces of Hilbert space H such that $M \perp N$ then show that $M+N$ is closed.

→ **Proof!-**

Let N and M are closed linear subspace of H.

state & prove projection Th^m

Projection Th^m 6:-

If M is closed linear subspace of Hilbert space H then $H = M \oplus M^{\perp}$.

→ **Proof:-**

M is closed linear subspace of H.

$M^{\perp}$ is also closed linear subspace of H.

" If M and N are closed linear subspace of H such that $M \perp N$ then

We have,

$M + M^{\perp}$ is closed subspace of H

also $M \cap M^{\perp} = \{0\}$ always.

Claim:- $H = M + M^{\perp}$

Let us assume that,

$H \neq M + M^{\perp}$

$M + M^{\perp}$ is proper subspace of H.

$\Rightarrow \exists\, z_0 \in H$ such that $z_0 \notin M + M^{\perp}$.

We have, $M + M^{\perp}$ is closed linear subspace of H then $\exists\ z_0 \neq 0$ in H such that $z_0 \perp M$.

$$z_0 \perp (M + M^{\perp})$$

As ~~M~~ ~~con~~ $M \subseteq M + M^{\perp}$ and $M^{\perp} \subseteq M + M^{\perp}$

$z_0 \perp M$ also $z_0 \perp M^{\perp}$.

$\Rightarrow (z_0\ m) = 0 \quad \forall\ m \in M$ & $(z_0\ m') = 0 \quad \forall m \in M^{\perp}$.

$\Rightarrow \quad z_0 \in M^{\perp}$ & $z_0 \in (M^{\perp})^{\perp}$

$\Rightarrow \quad z_0 \in M^{\perp} \cap (M^{\perp})^{\perp}$

$\Rightarrow \quad z_0 \in M^{\perp} \cap M = 0 \qquad \because ((M^{\perp})^{\perp} = M$ For M is closed).

$\Rightarrow \quad z_0 = 0$

which is a contradiction.

$\therefore\ H = M + M^{\perp}$

As $H = M + M^{\perp}$ and $M \cap M^{\perp} = 0$

$\Rightarrow \quad H = M \oplus M^{\perp}$.

(my)? If S is non-empty subset of Hilbert space H then show that $S^{\perp} = S^{\perp\perp\perp}$.

$\rightarrow$ Let,

S is non-empty subset of Hilbert space H.

We know that,

If M is closed,

$$M = (M^{\perp})^{\perp}$$

$S \neq \{0\} \Rightarrow S^{\perp}$ is closed.

As $M = S^{\perp}$ $\quad$ ($\because M$ is closed

$$M = (M^{\perp})^{\perp}$$

$$S^{\perp} = ((S^{\perp})^{\perp})^{\perp}$$

$$\therefore\ S^{\perp} = S^{\perp\perp\perp}$$

2] If M is a linear subspace of H then show that M is closed iff $M = (M^{\perp})^{\perp}$.

→ Let,

i] Suppose $M = (M^{\perp})^{\perp}$

then as $M^{\perp}$ is closed subspace of H

$\Rightarrow$ $(M^{\perp})^{\perp}$ is also closed subspace of H

$\Rightarrow$ M is closed.

ii] suppose M is closed.

We know that, "If M is closed linear subspace of Hilbert space H then $H = M \oplus M^{\perp}$ — (1)

Now for any non-empty set S,

$$S \subseteq S^{\perp\perp}$$

$$\Rightarrow \quad M \subseteq M^{\perp\perp}$$

Now let,

$$x \in M^{\perp\perp} \subset H = M \oplus M^{\perp}$$

$$\Rightarrow \quad x = z + y \qquad \text{where } z \in M^{\perp}, y \in M.$$

As M is contained in $M^{\perp\perp}$,

$$\Rightarrow \quad y \in M^{\perp\perp}$$

and then as $M^{\perp\perp}$ is closed linear subspace of H.

$$\Rightarrow \quad x, y \in M^{\perp\perp}$$

$$\Rightarrow \quad x - y = z \in M^{\perp\perp}$$

As $\quad z \in M^{\perp} \cap M^{\perp\perp} = \{0\}$

$$z = 0$$

$$\Rightarrow \quad x = y$$

$$\Rightarrow \quad \text{as } y \in M, \ x \in M$$

$$\Rightarrow \quad M^{\perp\perp} \subseteq M$$

$$\Rightarrow \quad M = M^{\perp\perp}$$

Note:-

1) If S is a non-empty subset of Hilbert space H. Show that the set of all linear combinations of vectors in S is dense in H iff $S^{\perp} = \{0\}$.

orthonormal set:-

An orthonormal set S in Hilbert space H is non-empty subset which consists of mutually orthogonal unit vectors.
ie. non-empty subset e_i of Hilbert space H is ~~x~~orthonormal if,

i) $(e_i \ \ e_j) = 0 \qquad \forall i \neq j$.

ii) $(e_i \ \ e_i) = 1 \qquad \forall i$

Remark:-

1) If $H = \{0\}$, H has no orthonormal set.

2) If $0 \neq x \in H$ then $\frac{x}{\|x\|}$ is orthonormal set.

iii) If $\{x_i\}$ is set of non-zero orthogonal elements of H then $\left\{\frac{x_i}{\|x_i\|}\right\}$ is orthonormal set, where x_i is orthogonal set.

Bessel's inequality for finite orthonormal set:-

Let $e_1, e_2, \cdots e_n$ be finite orthonormal set in Hilbert space H. If x is any vector in H then,

$$\sum_{i=1}^{n} |(x \ \ e_i)|^2 \leq \|x\|^2$$

Further $\left(x - \sum_{i=1}^{n} (x, e_i)\, e_i\right) \perp e_j \qquad \forall j$.

→ **Proof:-**

Let H be any vector in Hilbert space H

$$0 \leq \left\| x - \sum_{i=1}^{n} \underbrace{(x, e_i)}_{\text{be any no. in field}} e_i \right\|^2,$$

$$0 \leq \left(x - \sum_{i=1}^{n} (x, e_i) e_i ,\ x - \sum_{i=1}^{n} (x, e_i) e_i \right)$$

$$\leq (x\ x) - \overline{\left(x, \sum_{i=1}^{n} (x, e_i) e_i \right)} - \left(\sum_{i=1}^{n} (x, e_i) e_i, x \right)$$

$$+ \left(\sum_{i=1}^{n} (x, e_i) e_i ,\ \overline{\sum_{i=1}^{n} (x, e_i) e_i} \right)$$

$$\leq \|x\|^2 - \sum_{i=1}^{n} \overline{(x, e_i)} (x, e_i) - \sum_{i=1}^{n} (x, e_i)(e_i, x)$$

$$+ \sum_{i=1}^{n} \sum_{i=1}^{n} (x, e_i) \overline{(\cancel{x}\ e_i)} (e_i\ e_i)$$

$$\leq \|x\|^2 - \sum_{i=1}^{n} \overline{(x, e_i)} (x, e_i) - \sum_{i=1}^{n} (x, e_i) \underset{\overline{(x, e_i)}}{(e_i, x)}$$

$$+ \sum_{i=1}^{n} \sum_{i=1}^{n} (x, e_i) \overline{(x, e_i)} (e_i\ e_i)$$

$$\leq \|x\|^2 - \sum_{i=1}^{n} |(x, e_i)|^2 - \sum_{i=1}^{n} |(x, e_i)|^2$$

$$+ \sum_{i=1}^{n} |(x, e_i)|^2$$

$$0 \leq \|x\|^2 - \sum_{i=1}^{n} |(x, e_i)|^2$$

$$\therefore \sum_{i=1}^{n} |(x, e_i)|^2 \leq \|x\|^2 .$$

Now,

$$\left(x - \sum_{i=1}^{n} (x, e_i) e_i ,\ e_j \right)$$

$$= (x, e_j) - \sum_{i=1}^{n} (x, e_i)(e_i\ e_j)$$

$\sum (x, e_i)(e_i, e_j) = (x, e_1)\underbrace{(e_1, e_j)}_{0} + (x, e_2)\underbrace{(e_2, e_j)}_{0} + \cdots + (x, e_j)\underbrace{(e_j, e_j)}_{1}$

$= (x, e_j) \cdot 1$

$$= (x, e_j) - \sum_{i=1}^{n} (x, e_i)\cdot(e_i, e_j)$$

$$= (x, e_j) - (x, e_j)\cdot 1$$

$$= 0$$

Th^m 8 :-

If e_i is an orthonormal set in a H and if x is any vector in H then the set $s = \{e_i \mid (x, e_i) \neq 0\}$, is either empty or countable.

Proof:-

Consider $s = \{e_i \mid (x, e_i) \neq 0\}$

i) If s is empty then nothing to prove.

ii) If s is non-empty.

We have to prove s is countable.

For each positive integer n define

$s_n = \left\{ e_i \mid |(x, e_i)|^2 \geq \frac{\|x\|^2}{n} \right\}$. n be any fixed no.

If s_n contains $m \geq n$ number of elements then,

$$|(x, e_1)|^2 \geq \frac{\|x\|^2}{n}$$

$$|(x, e_2)|^2 \geq \frac{\|x\|^2}{n}$$

$$\vdots$$

$$|(x, e_m)|^2 \geq \frac{\|x\|^2}{n}$$

and $\sum_{i=1}^{m} |(x, e_i)|^2 \geq \frac{m}{n}\|x\|^2 \geq \|x\|^2$.

But by Bessel's inequality,

$$\sum_{i=1}^{m} |(x, e_i)|^2 \leq \|x\|^2.$$

which is contradiction.

$\Rightarrow$ S_n is finite for each positive integer n.

since, $S = \bigcup_{i=1}^{\infty} S_n$

$\therefore$ S is countable.

Generalized Bessel's inequality is:-

If $\{e_i\}$ is an orthonormal set in a Hilbert space H then $\sum_{i=1}^{\infty} |(x, e_i)|^2 \leq \|x\|^2 \quad \forall x \in H$.

→ **Proof:-**

Let $\{e_i\}$ is an orthonormal set in a Hilbert space H.

Then by " If e_i is an orthonormal set in H and if x be any vector in H then the set $S = \{e_i \mid (x, e_i) \neq 0\}$ is empty or countable.

i) If S is empty,

Then $(x, e_i) = 0 \quad \forall e_i$

$\Rightarrow \quad \sum |(x, e_i)|^2 = 0 \leq \|x\|^2$

Inequality holds.

ii) If S is countably finite set,

$\Rightarrow S = \{e_1, e_2, \ldots, e_n\}$

Then by Bessel's inequality for finite orthonormal set,

$$\sum_{i=1}^{n} |(x, e_i)|^2 \leq \|x\|^2$$

iii) If S is countably infinite set.

i.e. $S = \{e_1, e_2, \ldots\}$

If $\sum_{i=1}^{\infty} |(x, e_i)|^2$ converges then every rearrangement series converges and has same sum.

Define

$$s_n = \sum_{i=1}^{n} |(x, e_i)|^2$$

ie. $s_n \le s_{n+1}$

ie $\{s_n\}$ is increasing sequence of non-negative numbers.

and by case (ii)

$$s_n \le \|x\|^2 \qquad \forall n.$$

$\therefore$ s_n is bounded increasing sequence.

$\therefore$ s_n converges.

$\therefore$ $\lim_{n\to\infty} s_n \le \|x\|^2$

$$\sum_{i=1}^{\infty} |(x, e_i)|^2 \le \|x\|^2.$$

Th^m 10 :-

If set $\{e_i\}$ is an orthonormal set in a Hilbert space H then show that for any $x \in H$,

$x - \sum (x, e_i) e_i \perp e_j \quad \forall j.$

→ Proof :-

Let $\{e_i\}$ is an orthonormal set in H.

Define $s = \{ e_i / (x, e_i) \neq 0 \}$

1] If s is empty then

$$(x, e_i) = 0 \qquad \forall e_i$$

$$\Rightarrow \sum (x, e_i) = 0$$

$$\Rightarrow x - \sum (x, e_i) = x - 0 = x \therefore x \perp e_i \quad \forall i$$

ii) If S is non-empty and finite.
$\therefore$ $S = \{e_1, e_2, \dots e_n\}$

We define,
$$\sum (x, e_i) e_i = \sum_{i=1}^{n} (x, e_i) e_i$$

$$\therefore \quad x - \sum (x, e_i) e_i \perp e_j \quad \forall j$$ ($\because$ By Bessels inequal)

iii) If S is countably infinite.
$S = \{e_1 \ e_2 \dots e_n, \dots\}$

Define, $s_n = \sum_{i=1}^{n} (x, e_i) e_i$

Now for $m > n$
$$\|s_m - s_n\|^2 = (s_m - s_n)(s_m - s_n)$$

As $s_m - s_n = \sum_{i=n+1}^{m} (x, e_i) e_i)$

$$\therefore \|s_m - s_n\|^2 = \left(\sum_{i=n+1}^{m} (x, e_i) e_i , \sum_{i=n+1}^{m} (x \ e_j) e_j \right)$$

$e_j)$
$$= \left(\sum_{i=n+1}^{m} \sum_{i=n+1}^{m} (x, e_i) (\overline{x \ e_j}) (e_i \ e_j) \right)$$

$$= \sum_{i=n+1}^{m} |(x, e_i)|^2$$

$\because$ By Bessel's inequality,

$\sum_{i=1}^{\infty} |(x \ e_i)|^2$ is convergent.

$\therefore \{s_n\}$ is cauchy in H, $\&$ as H is complete
$\Rightarrow \{s_n\}$ is cgt.

Let $s_n \to s$ as $n \to \infty$ in H.

$$\lim_{n\to\infty} \sum_{i=1}^{n} (x, e_i) e_i \to s$$

Now we will show that,

$$\& \; x - \sum_{i=1}^{\infty} (x \; e_i) e_i \perp e_j \qquad \forall j$$

$$\left(x - \sum_{i=n+1}^{\infty} (x \; e_i) e_i, e_j\right) = (x \; e_j) - \sum_{i=1}^{\infty} (x \; e_i)(e_i \; e_j)$$

$$= (x \; e_j) - \left(\lim_{n\to\infty} s_n, e_j\right)$$

$$= (x \; e_j) - \lim_{n\to\infty} (s_n, e_j)$$

$$= (x \; e_j) - \lim_{n\to\infty} \left(\sum_{i=1}^{n} (x, e_i) e_i, e_j\right)$$

$$= (x \; e_j) - \lim_{n\to\infty} (x, e_j)$$

$\left(\because \text{as } (e_j \; e_j) = 1 \text{ o.w.} = 0\right)$

$$= (x, e_j) - (x, e_j)$$

$$= 0$$

$$\therefore \left(x - \sum_{i=1}^{\infty} (x \; e_i) e_i, e_j\right) \perp e_j \qquad \forall j.$$

Note:-

1) Let H be a non-zero Hilbert space H and C be the collection of all orthonormal sets in H. Then 'C' is non-empty and 'C' $(C \leq)$ is a poset where '$\leq$' is rel^n of inclusion of sets.

Complete orthonormal set:-

An orthonormal set $\{e_i\}$ in H is set to be complete if it is maximal in poset of orthonormal sets.

Th^m 11:-

Every non-zero Hilbert space contains a complete orthonormal set.

→ **Proof:-**

Let H be a non-zero Hilbert space.

Let 'c' be the collection of orthonormal sets in H.

We know, $(c \le)$ is an poset.

Let $\mathcal{C}$ be any arbitrary chain in c.

Now, we know union $\cup_{S \in \mathcal{C}} \mathcal{C}$ is an upper bound for any $\mathcal{C}$.

The 'Zorn's' Lemma c has a maximal element.

$\Rightarrow \exists$ complete orthonormal set in c.

Th^m 12:-

Let H be a Hilbert space $\langle e_i \rangle$ be an orthonormal set in H then following conditions are equivalent to one another.

i) set $\{e_i\}$ is complete.

ii) $x \perp \{e_i\} \Rightarrow x = 0$

iii) If x is an any element in H then $x = \sum (x, e_i) e_i$.

iv) If x is any element in H then $\|x\|^2 = \sum |(x, e_i)|^2$.

Proof:-

Let H be Hilbert space & $\{e_i\}$ be orthonormal set

i) $\Rightarrow$ ii) Assume set $\{e_i\}$ is complete.

on contrary suppose if possible,

$\exists\ x \in H$ s.t. $x \neq 0$ and $x \perp e_i \ \forall i$.

Then consider, $e = \dfrac{x}{\|x\|}$ and $e \perp e_i \ \forall e_i$

and $e \perp e_i \quad \forall e_i$ with $\|e\| = 1$

$\Rightarrow \{e_i \ e\}$ is an orthonormal set

which is contradiction to $\{e_i\}$ is an maximal orthonormal set

$\Rightarrow \quad x = 0$

$\Rightarrow \quad x \perp \{e_i\} \Rightarrow x = 0$.

ii) $\Rightarrow$ iii) Assume $x \perp \{e_i\} \Rightarrow x = 0$

since e_i is orthonormal set in H.

Then we have,

$$\left(x - \sum (x, e_i)\, e_i \,,\, e_j\right) = 0 \qquad \forall e_j$$

By assumption,

$$x - \sum (x \ e_i)\, e_i = 0$$

$$\Rightarrow \quad x = \sum (x \ e_i)\, e_i$$

iii) $\Rightarrow$ iv) Suppose $x \in H$ and $x = \sum (x \ e_i) e_i$

$$\|x\|^2 = (x, x)$$

$$= \left(\sum (x \ e_i) e_i, \quad \sum (x \ e_j)\, e_j\right)$$

$$= \sum\sum (x, e_i)\, \overline{(x, e_i)} \quad (e_i \ e_j)$$

$$= \sum (x, e_i) \overline{(x, e_i)}$$

$$\|x\|^2 = \sum |(x \ \ e_i)|^2$$

iv] $\Rightarrow$ i] Assume $x \in H \ni \|x\|^2 = \sum |(x, e_i)|^2$

If possible assume $\{e_i\}$ is not complete

$\Rightarrow \exists \, e \in H$ s.t. $(e_i \ \ e)$ is orthonormal in H.

Then,

$\Rightarrow \|e\|^2 = \sum |(e \ \ e_i)|^2$ $\quad \because e \in H$

$\Rightarrow 1 = 0$.

which is contradiction.

$\therefore \{e_i\}$ is complete.

Note:-

1] In above th^m expression $x = \sum (x \ \ e_i) e_i$ is called the fourier expansion of x, and the inner product (x, e_i) are fourier coefficients of x.

2] In above th^m eq^n $\|x\|^2 = \sum |(x, e_i)|^2$ is called the parseval's eq^n.

Th^m 13:-

Hilbert space H is separable iff every orthonormal set in H is countable.

Conjugate of an Hilbert space H^* :- OR conjugate Space :-

Let H be a HS & $H^* = B(H, \mathbb{C})$ is conjugate space H^* i.e. set containing all bounded, linear functionals from H to $\mathbb{C}$.

Let $y \in H$ be fixed element.
Define $f_y : H \to \mathbb{C}$ such that

$$f_y(x) = (x, y) \qquad \forall x \in H$$

i] f_y is linear.

Let $a, b \in H$, $\alpha, \beta \in \mathbb{C}$

$$f_y(\alpha a + \beta b) = (\cancel{\alpha a} + \cancel{\beta b},\ \cancel{\alpha a} + \cancel{\beta b})$$
$$= (\cancel{\alpha a}\ \cancel{\alpha a}) + (\cancel{\alpha a}\ \cancel{\beta b}) + (\cancel{\beta b}\ \cancel{\alpha a}) + (\beta b\ \ \beta b)$$
$$= \cancel{\alpha\bar{\alpha}}(\cancel{a}\ .\ \cancel{a}) + \cancel{\alpha\bar{\beta}}(\cancel{a}\ \ \cancel{b}) + \cancel{\beta\bar{\alpha}}(\cancel{b}\ \ \cancel{a}) + \cancel{\beta\bar{\beta}}(\cancel{b}\ \ \cancel{a})$$

$$f_y(\alpha a + \beta b) = (\alpha a + \beta b\ \ y)$$
$$= \alpha(a\ \ y) + \beta(b\ \ y)$$
$$= \alpha f_y(a) + \beta f_y(b)$$

ii] f_y is bounded.

Consider,

$|f_y(x)|$

Now, $|f_y(x)| = |(x, y)|$
$$\le \|x\| \cdot \|y\|$$
$$= \|y\| \cdot \|x\| \qquad \forall x \in H$$
$$|f_y(x)| \le k \cdot \|x\| \qquad \forall x \in H$$

f_y is bdd.

iii] $\|f_y\| = \|y\|$.

consider,

$\|f_y\| = \sup\{\, |f_y(x)| \,/\, x \in H,\ \ \|x\| \le 1 \}$

as, $|f_y(x)| \le \|x\| \cdot \|y\|$

We have,

$$\|f_y\| \le \|y\| \qquad \text{—— (I)}$$

Now consider,

$$z = \frac{y}{\|y\|}, \quad y \neq 0$$

$$f_y(z) = (z, y)$$

$$|f_y(z)| = \left| f_y\left(\frac{y}{\|y\|}\right)\right|$$

We know,

$$\|f_y\| \geq |f_y(z)| = \left| f_y\left(\frac{y}{\|y\|}\right)\right|$$

$$\|f_y\| \geq \left|\left(\cancel{y, \frac{y}{\|y\|}}\right)\right| \; \left|\left(\frac{y}{\|y\|}, y\right)\right|$$

$$= \|y\|$$

$$\|f_y\| \geq \|y\| \qquad \text{— ②}$$

Now for $y = 0$

$$f_y(x) = (\cancel{0, x}) = (x, 0) = 0 = \|y\| \qquad \text{— ③}$$

From ①, ②, ③

$$\|f_y\| = \|y\|.$$

Riesz Lemma:-

Let H be a Hilbert Space, f be a arbitrary functional in H^* then $\exists$ unique $y \in H$ such that $f(x) = (x, y)$ $\forall x \in H$. Further $\|f\| = \|y\|$.

→ Proof:-

Let H be Hilbert space.

If $f = 0$ choose $y = 0$ then thm trivially holds.

If $f \neq 0$ then define,

$$M = \ker f = \{x \in H, \mid f(x) = 0\}$$

Then we know M is proper linear closed subspace of H.
Then by thm,

$\exists$ non-zero vector, $0 \neq z_0 \in H$ such that $z_0 \perp M$.

Now for any $x \in H$ consider,
$f(z_0)x - z_0 f(x) \in H$
and $f[f(z_0)x - z_0 f(x)] = 0$ $\quad$ $(\because f(\_\_) \in \ker f$ $\therefore f(\quad) = 0)$
$\Rightarrow f(z_0)x - z_0 f(x) \in M$.
since $z_0 \perp M$
$\Rightarrow z_0 \perp (f(z_0)x - z_0 f(x))$
$\Rightarrow (f(z_0)x - z_0 f(x)\}, z_0) = 0$

$\Rightarrow f(z_0)(x, z_0) - f(x)(z_0\ z_0) = 0$
$\Rightarrow f(z_0)(x, z_0) = f(x)(z_0\ z_0)$
$$\Rightarrow f(x) = \frac{f(z_0)}{\|z_0\|^2}(x, z_0)$$
$$= \left(x, \frac{\overline{f(z_0)}}{\|z_0\|^2} z_0\right)$$

Put $y = \dfrac{\overline{f(z_0)}}{\|z_0\|^2} z_0$

$\therefore f(x) = (x, y)$.

Uniqueness:-
Let consider $\exists\ y' \neq y$ such that,
$f(x) = (x, y) = (x, y')$
$\Rightarrow (x, y) = (x, y')$
$\Rightarrow (x, y) - (x, y') = 0$ $\quad$ $(x, \alpha y + \beta y')$
$\Rightarrow (x, y - y') = 0$ $\quad$ $\Rightarrow (x, \alpha y) + (x, \beta y)$
$\Rightarrow x \perp y - y'$ $\quad \forall x \in H$.

all Hilbert space is $\perp$ to $y - y'$

$\Rightarrow \quad y - y' = 0$
$\Rightarrow \quad y = y'$

To show that $\|y\| = \|f\|$
Inparticular, consider,
$f(y) = (y, y) = \|y\|^2$
i.e. $\|y\|^2 = |f(y)| = \|f\| \cdot \|y\|$
$\Rightarrow \quad \|y\| \leq \|f\|$ —— (I)

Now consider,
$|f(x)| = |(x, y)| \leq \|x\| \cdot \|y\|$
and
$\|f\| = \sup\{|f(x)|, x \in H, \|x\| \leq 1\}$
$\leq |f(x)|, x \in H$
$\leq \|x\| \cdot \|y\|$

For,
$\|x\| \leq 1$
$\Rightarrow \|f\| \leq \|y\|$ —— (II)

$\therefore$ From (I) & (II),
$\|f\| = \|y\|$

Note:-

1] In above th^m we can say that norm-preserving mapping of H into H^* defined by,
$f_y(x) = (x, y)$ i.e. $y \to f_y \quad \forall x \in H$.
is actually mapping of H onto H^*

2] The norm preserving mapping $y \to f_y$ is not linear. $y \to f_y \quad \Rightarrow \alpha x + \beta y \to f_{\alpha x + \beta y}$
$f_{\alpha x + \beta y}(z) = (z, \alpha x + \beta y)$
$= (z, \alpha x) + (z, \beta y)$
$= \bar{\alpha}(z, x) + \bar{\beta}(z, y)$
$= \bar{\alpha} f_x(z) + \bar{\beta} f_y(z)$
$= (\bar{\alpha} f_x + \bar{\beta} f_y) z$ $\Rightarrow$ not linear $\because \bar{\alpha} \neq \alpha, \bar{\beta} \neq \beta$

3] For $x, y \in H$, $\|f_x - f_y\| = \|f_{x-y}\| = \|x-y\|$

Thm 14:-

Let H be a Hilbert space. Show that H^* is an Hilbert space, w. r. to inner product defined by,

$(f_x, f_y) = (y, x)$.

In just the same way the fact that H^* is Hilbert space implies H^{**} is an also Hilbert space, with respect to inner product given by,

$(F_{f_x}, F_{f_y}) = (f_y, f_x)$.

→ Proof:-

Let H be Hilbert Space.

As every Hilbert space is Banach space.

H is Banach space.

$\Rightarrow$ H^* is a Banach space.

$\therefore$ H^* is a complete space.

Now to show that H^* is Hilbert space it is remains to show that, H^* is inner product space w. r. to

$(f_x \; f_y) = (y, x)$.

Consider,

$f_x, f_y, f_z \in H^*$, $x, y \in H$

i) consider,

$(f_x, f_x) = 0$

$\Leftrightarrow$ $(x, x) \cancel{f_x} = 0$

$\Leftrightarrow$ $x = 0$

$\Leftrightarrow$ $f_x = 0$

ii] consider,

$(\alpha f_x + \beta f_y, f_z)$

$= \cancel{\alpha} (\cancel{\alpha f_x}\ f_z) + \beta (\beta f_y\ f_z)$

$f_{\alpha x} = \bar{\alpha} f_x$

$= (f_{\bar{\alpha} x} + f_{\bar{\beta} y}, f_z)$

$= (z, \bar{\alpha} x + \bar{\beta} y)$

$= \alpha (z, \bar{\alpha} x) + (z, \bar{\beta} y)$

$= \alpha (z\ x) + \beta (z\ y)$

$= \alpha (f_x, f_z) + \beta (f_y\ f_z)$

$= \alpha (f_x\ f_z) + \beta (f_y\ f_z)$

iii) Now consider,

$(f_x\ f_y) = (\overline{y\ x}) = (x\ y) = (f_y\ f_x)$

From i), ii), & iii).

H^* is inner product space.

$\Rightarrow$ H^* is Hilbert Space.

HW

Now we have to show that,

H^{**} is Hilbert space w.r. to norm

$(F_{f_x}, F_{f_y}) = (f_y\ f_x)$.

We know,

H^* is Hilbert space.

$\Rightarrow$ H^{**} is Hilbert space.

$\Rightarrow$ H^{**} is complete.

It's remain to show H^{**} is inner product w.r. to $(F_{f_x}, F_{f_y}) = (f_y\ f_x)$.

consider,

$F_{f_x}, F_{f_y}, F_{f_z} \in H^{**}$, $x, y, \in H$,

i) consider,

$(Ff_x, Ff_x) = 0$

$\Leftrightarrow (f_x \; f_x) = 0$

$\Leftrightarrow (x \; x) = 0$

$\Leftrightarrow x = 0$

$\Leftrightarrow Ff_x = 0$

$(Ff_x + f_y)(f_z)$

$= f_z Ff_x$

$: (f_z \; f_x + f_y)$

$= (f_z \; f_x) + (f_z \; f_y)$

$= Ff_x f_z + Ff_y f_z$

$= \cancel{f_z} (Ff_x + Ff_y)(f_z)$

ii) consider,

$(\alpha Ff_x + \beta Ff_y \; , \; Ff_z)$

$= (F\bar{\alpha} f_x + F\bar{\beta} f_y \; , \; Ff_z)$

$Ff_x)\alpha(f_z)$

$= (\alpha f_z \; f_x)$

$= \alpha (f_z \; f_x)$

$= \alpha \; Ff_x(f_z)$

$= (f_z \; , \; \bar{\alpha} f_x + \bar{\beta} f_y)$

$= (f_z \; , \; \bar{\alpha} f_x) + (f_z \; , \; \bar{\beta} f_y)$

$= \alpha (f_z \; f_x) + \beta (f_z \; f_y)$

$= \alpha (Ff_x \; Ff_z) + \beta (Ff_y \; Ff_z)$

iii) Now consider,

$(Ff_x, \; Ff_y) = (\overline{f_y \; f_x}) = (f_x \; f_y) = (Ff_y \; Ff_x)$

From (i) + (ii) + (iii),

H^{**} is inner product

$\Rightarrow H^{**}$ is Hilbert Space.

$Ff_x(\alpha f_z + \beta f_y)$

Th^m 15 :-

Every Hilbert space is Reflexive.

→ Proof:-

Let H be an Hilbert space

To prove that H is reflexive.

ie. to show that $H = H^{**}$.

Define,

$\phi : H \longrightarrow H^{**}$ by $\phi(y) = Ff_y \quad \forall y \in H$

$f_y \in H^*$

where $F_{f_y}(g) = (g, f_y)$

i) ϕ is one one

consider,

$\phi(x) = \phi(y) \qquad x, y \in H$

$\Rightarrow F_{f_x} = F_{f_y}$

apply on element. they are i.e. F are fun^n

$\Rightarrow F_{f_x}(g) = F_{f_y}(g) \qquad \forall g \in H^*$

$\Rightarrow (g, f_x) = (g, f_y)$

$\Rightarrow (g, f_x) - (g, f_y) = (0, 0)$

$\Rightarrow (g, f_x - f_y) = (0, 0)$

$\Rightarrow (g, f_x - f_y) = 0$

$\Rightarrow f_x - f_y = 0$

$\Rightarrow f_x = f_y \qquad \ldots$ it is $\perp$ to all H

$\Rightarrow f_x(z) = f_y(z) \qquad \forall z \in H$

$\Rightarrow (z, x) = (z, y)$

$\Rightarrow (z, x) - (z, y) = 0$

$\Rightarrow (z, x - y) = 0$

$\Rightarrow x - y = 0 \qquad -$ it is $\perp$ to all H $\qquad F \in H^{**}$

$\Rightarrow x = y$

ii) ϕ is onto.

Let $F \in H^{**}$.

Then by Riesz Lemma $\exists$ unique $g \in H^*$

s.t. $F = F_g \qquad f(x) = (x, y)$ i.e. $f_x(z) = (z, x)$

Simillarly, for $g \in H^*$ $\exists$ unique $y \in H$ s.t.

s.t. $g_y = g \qquad F = F_g \Rightarrow F = F_{g_y} \quad \because g = g_y$.

$\Rightarrow \therefore \exists\, y \in H$ s.t. $\phi(y) = F_{g_y} = F$.

$\therefore$ y is preimage of F.

$\therefore$ ϕ is onto.

FacaLook

iii] ϕ is linear.

consider, $x, y \in H$, $\alpha, \beta \in ¢$

Let, $x, y \in H$, $\alpha, \beta \in C$.

consider

$\phi(x+y) = F_{f_{x+y}}$

Now,

$F_{f_{x+y}}(g) = (g, f_{x+y})$ $\forall g \in H^*$

$= (g, f_x + f_y)$

$= (g\ f_x) + (g\ f_y)$

$= F_{f_x}(g) + F_{f_y}(g)$.

$F_{f_{x+y}}(g) = (F_{f_x} + F_{f_y})(g)$

$\therefore \phi(x+y) = (F_{f_x} + F_{f_y})$

$= \phi(x) + \phi(y)$

consider,

$\phi(\alpha x) = F_{f_{\alpha x}}$

$F \in H^{**}$

$F_{f_{\alpha x}}(g) = (g, f_{\alpha x})$

$= (g, \bar{\alpha} f_x)$

$= \bar{\bar{\alpha}}(g\ f_x)$

$= \alpha (g\ f_x)$

$\therefore F_{f_{\alpha x}}(g) = \alpha F_{f_x}(g)$

$\therefore \phi(\alpha x) = F_{f_{\alpha x}} = \alpha F_{f_x}$

iv] ϕ is norm preserving.

consider $x \in H$

Norm of $\phi(x)$ is,

$\|\phi(x)\| = \|F_{f_x}\|$

$= \|f_x\|$

$\therefore \|\phi(x)\| = \|x\|$ $\because$ By Riesz's lemma

From (i) (ii) (iii), (iv).

$\therefore \phi$ is isometric isomorphism

$\therefore H \cong H^{**}$

$\therefore H$ is Reflexive

Conjugate of an operator :-

Let H be a Hilbert space, H^* be a dual space. Let $y \in H$ and f_y be its corresponding funcitional in H^*. operate T^* on f_y to obtain functional f_z i.e.

$$T^*(f_y) = f_z$$

Then the mapping f_z gives $z \in H$ back. We write forming product of three mapping as,

$$T^*(y) = z$$

and the mapping $T^* : H \longrightarrow H$ defined by,

$$T^*(y) = z \qquad z \in H$$

is called adjoint of T. $(T : H \to H)$.

Thus for any vector $x \in H$,

$$\begin{aligned}(T^* f_y)(x) &= (f_y \circ T)(x) \\ &= f_y(T(x)) \\ &= (T(x), y) \qquad \forall x \in H\end{aligned}$$

and

$$\begin{aligned}(T^* f_y)(x) &= f_z(x) \\ &= (x, z) \\ &= (x, T^*(y))\end{aligned}$$

$$\therefore (T(x), y) = (x, T^*(y)) \qquad \forall x, y \in H.$$

* For $T \in B(H)$ the mapping $T^* : H \to H$ defined by $(T(x), y) = (x, T^*(y))$ $\forall x, y \in H$ is called adjoint of T.

Note:-

1] Adjoint of T is unique for given $T \in B(H)$. Suppose T^* and T' are too adjoints of $T \in B(H)$

$\Rightarrow$ For $x, y \in H$

$$(T(x), y) = (x, T^*(y)) = (x, T'(y))$$

$$\Rightarrow (x, T^*(y) - T'(y)) = 0 \qquad \forall x, y \in H$$

For $y \in H$ be any vector.

$$\therefore T^*(y) - T'(y) = 0 \qquad \forall x \in H \text{ but fixed as in } H.$$

$$\Rightarrow T^*(y) = T'(y) \qquad \text{only } (x, 0) = 0 \ \forall x.$$

$$\Rightarrow T^* = T'$$

Properties of Adjoint:-

1] T^* is linear.

Let, $z, x, y \in H$, $\alpha, \beta \in \mathbb{C}$

consider,

$$(z, T^*(\alpha x + \beta y)) = (T(z), \alpha x + \beta y)$$
$$= \bar{\alpha}(T(z), x) + \bar{\beta}(T(z), y)$$
$$= \bar{\alpha}(z, T^*(x)) + \bar{\beta}(z, T^*(y))$$
$$= (z, \alpha T^*(x)) + (z, \beta T^*(y))$$
$$= (z, \alpha T^*(x) + \beta T^*(y))$$

$$(z, T^*(\alpha x + \beta y) - \alpha T^*(x) - \beta T^*(y)) = 0 \qquad \forall z \in H$$

$$\Rightarrow T^*(\alpha x + \beta y) = \alpha T^*(x) + \beta T^*(y) \qquad \forall x, y \in H, \ \alpha, \beta \in \mathbb{C}.$$

2] T^* is Bounded.

$$\|T^*(y)\|^2 = (T^*(y), T^*(y))$$
$$= (T(T^*(y)), y)$$

Taking modulus,

$$\|T^*(y)\|^2 = |(T(T^*(y)), y)|$$
$$\leq \|T(T^*(y))\| \cdot \|y\|$$
$$\leq \|T\| \cdot \|T^*(y)\| \cdot \|y\|$$

$$\Rightarrow \|T^*(y)\|^2 \leq \|T\| \cdot \|y\| \qquad \forall y \in H$$

$\therefore$ T^* is bdd, and hence continuous.

$T^* : H \to H$ - linear bdd.

$\Rightarrow$ $T^* \in B(H)$ —— ①

3] $\|T^*\| = \sup\{\|T^*(x)\| \,/\, x \in H, \|x\| = 1\} \leq \|T\|$

$\therefore$ $\|T^*\| \leq \|T\|$ —— ②

$\therefore$ The mapping $T \to T^*$ of $B(H)$ into $B(H)$ is called an adjoint operation on $B(H)$.

Th^m 16 :-

The adjoint operation $T \to T^*$ on $B(H)$ then the following properties.

1] $(T_1 + T_2)^* = T_1^* + T_2^*$

2] $(\alpha T)^* = \bar{\alpha} T^*$

3] $(T_1 T_2)^* = T_2^* T_1^*$

4] $T^{**} = T$

5] $\|T^*\| = \|T\|$

6] $\|T^* T\| = \|T\|^2$.

→ Proof :-

$T_2, T_1 \in B(H)$, $T_2^*, T_1^* \in B(H)$ for $x, y, z \in H$, $\alpha, \beta \in \mathbb{C}$

1] Consider,

$(x, (T_1 + T_2)^*(y)) = ((T_1 + T_2)(x), y)$

$= (T_1(x) + T_2(x), y)$ $\quad \because$ T is linear

$= (T_1(x), y) + (T_2(x), y)$ $\quad \because$ Def^n of IP

$= (x, T_1^*(y)) + (x, T_2^*(y))$ $\quad \because$ Def^n of T^*

$= (x, T_1^*(y) + T_2^*(y))$ $\quad \because$ Def^n of IP

$= (x, (T_1^* + T_2^*)(y))$ $\quad \because$ T^* is linear

$\Rightarrow$ $(T_1 + T_2)^* = T_1^* + T_2^*$

2] Consider,

$(x, (\alpha T)^*(y)) = (\alpha T(x), y)$ $\quad \because$ def^n of T^*

$$= \alpha\,(T(x), y) \qquad \because T \text{ is linear}$$
$$= \alpha\,(x, T^*(y)) \qquad \because \text{def}^n \text{ of } T^*$$
$$= (x, \bar{\alpha}\, T^*(y)) \qquad \because \text{def}^n \text{ of IP}$$
$$\Rightarrow (\alpha T)^* = \bar{\alpha}\, T^*$$

3] consider,

$$(x, (T_1T_2)^*(y)) = (T_1T_2(x), y)$$
$$= (T_1(T_2(x)), y) \qquad (\text{def}^n \text{ of } T^*)$$
$$= (T_2(x), T_1^*(y)) \qquad \because T_1, T_2 \in B(H)$$
$$= (x, T_2^* T_1^*(y)) \qquad \because (\text{def}^n \text{ of } T^*)$$
$$= T_2^* T_1^* \qquad \because (\text{def}^n \text{ of } T^*)$$

4] consider,

$$(x, T^{**}(y)) = (T^*(x), y)$$
$$= (x, T(y)) \qquad \because \text{Def}^n \text{ of } T^*$$
$$\therefore T^{**} = T \qquad \because \text{Def}^n \text{ of } T^*$$

<u>OR</u>

$$(T^*(x), y) = (x, T^{**}(y))$$
$$(T^*(x), y) = \overline{(y, T^*(x))}$$
$$= \overline{(T(y), x)}$$
$$= (x, T(y))$$
$$\Rightarrow (x, T(y)) = (x, T^{**}(y))$$
$$\Rightarrow T = T^{**}$$

5] We know that,

For $T \in B(H)$

$$\|T^*\| \leq \|T\| \qquad \text{—(I)}$$

$\|^{ly}$, $\|T^{**}\| \leq \|T^*\|$

But $T^{**} = T$

$\Rightarrow \|T\| \leq \|T^*\|$

$\Rightarrow \|T^*\| = \|T\|$

6] Since for any $T, S \in B(H)$ we know that

$\|TS\| \leq \|T\| \cdot \|S\|$

$\therefore$ As $T, T^* \in B(H)$

$\|T^*T\| \leq \|T^*\| \cdot \|T\| = \|T\| \cdot \|T\|$ $\quad \because$ By $\|T^*\| = \|T\|$

$= \|T\|^2$

$\therefore \|T^*T\| \leq \|T\|^2$ — ①

Now for any $x \in H$

$\|T(x)\|^2 = (T(x), T(x))$

$= |(T^*(T(x)), x)|$

$\leq \|T^*T(x)\| \cdot \|x\|$ -- Schwartz inequal

$\leq \|T^*T\| \cdot \|x\|^2$

Then, $\|T(x)\|^2 \leq \|T^*T\| \cdot \|x\|^2$ $\quad \forall x \in H$

$\|T(x)\|^2 \leq \|T^*T\| \cdot \|x\|^2$ $\quad \forall 0 \neq x \in H$

$\Rightarrow \left(\frac{\|Tx\|}{\|x\|}\right)^2 \leq \|T^*T\|$ $\quad \forall 0 \neq x \in H$

$\left(\sup\left\{\frac{\|T(x)\|}{\|x\|} / x \in H, \|x\| \neq 0\right\}\right)^2 \leq \|T^*T\|$

$\Rightarrow \|T\|^2 \leq \|T^2T\|$ — ②

$\Rightarrow \|T\|^2 = \|T^*T\|$ $\quad \because$ From ① & ②.

7] Show that $0^* = 0$, $I^* = I$ $\quad$ 0: Zero operator.

→ Let

$x, y \in H$

i] Consider

$(x, 0y) = (x, 0) = 0$ $\quad \forall x, y \in H$

$(x, 0^*y) = (0x, 0y) = (0, y) = 0$ $\quad \forall x, y \in H$

$\Rightarrow (x, 0y) = (x, 0^*y)$ $\quad \forall x, y \in H$.

ii] Consider,

$(x, Iy) = (x, y)$ $\quad \forall x, y \in H$

$(x, I^*y) = (Ix, y) = (x, y)$ $\quad \forall x, y \in H$

$\Rightarrow (x, Iy) = (x, I^*y)$ $\quad \forall x, y \in H$.

Date:- 12-12-2022

Self Adjoint Operator

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Self Adjoint operator:-

An operator $T \in B(H)$ is said to be self adjoint if $T^* = T$.

Thm 1:-

The self adjoint operator in $B(H)$ forms closed linear subspace of $B(H)$. (on field of Reals).

→ **Proof:-**

Consider $S(H) = \{T \in B(H) \mid T^* = T\}$

Clearly, $S(H) \subseteq B(H)$.

i) As $0^* = 0$ and $I^* = I$ for $0 \in B(H)$, $I \in B(H)$

$\Rightarrow 0, I \in S(H)$

$\Rightarrow S(H)$ is non empty.

ii) Let $\alpha, \beta \in F = \mathbb{R}$

Consider, $T_1, T_2 \in S(H)$

$$(\alpha T_1 + \beta T_2)^* = \bar{\alpha} T_1^* + \bar{\beta} T_2^*$$
$$= \alpha T_1 + \beta T_2$$

as $\bar{\alpha} = \alpha$, $\bar{\beta} = \beta$, $T_1^* = T_1$, $T_2^* = T_2$

$\therefore \alpha T_1 + \beta T_2 \in S(H)$.

iii) Let $\{T_n\}$ be any seq in $S(H)$ converging to T as $n \to \infty$.

Consider,

$$\|T - T^*\| = \|T - T_n + T_n - T_n^* + T_n^* - T^*\|$$
$$\leq \|T - T_n\| + \|T_n - T_n^*\| + \|T_n^* - T^*\|$$
$$\leq \|T - T_n\| + \|(T_n - T)^*\|$$

As $n \to \infty$ $T_n \to T$ $\Rightarrow \|T_n - T\| \to 0$

$\Rightarrow$ As $n \to \infty$ RHS $\to 0$ $\Rightarrow \|T - T^*\| = 0$.

$\Rightarrow \quad T = T^*$

$\Rightarrow \quad T \in S(H)$

$\Rightarrow \quad S(H)$ is closed.

Th^m 2:-

Let A_1 and A_2 be two self adjoint operators on H then their product $a_1 a_2$ is self adjoint iff $A_1 A_2 = A_2 A_1$.

→ Proof:-

Let $A_1, A_2 \in S(H)$

$$A_1 = A_1^*, \quad A_2 = A_2^*$$

Consider,

i) $A_1 A_2 \in S(H)$

$$(A_1 A_2)^* = A_1 A_2$$

$$\Rightarrow A_2^* A_1^* = A_1 A_2$$

$$\Rightarrow A_2 A_1 = A_1 A_2$$

ii) Consider, $A_1 A_2 = A_2 A_1$

$$(A_1 A_2) = A_2^* A_1^*$$

$$= (A_1 A_2)^*$$

$$A_1 A_2 = (A_1 A_2)^*$$

$$\Rightarrow A_1 A_2 \in S(H).$$

Th^m 3:-

Let $T \in B(H)$ if $(Tx, x) = 0 \quad \forall x \in H$ $\Rightarrow T = 0$.

→ Proof:-

Let $T \in B(H)$ such that $(Tx, x) = 0 \quad \forall x \in H$

Let $\alpha, \beta \in C$

Consider,

$(T(\alpha x+\beta y), \alpha x+\beta y) = 0$ $\quad \because \alpha x+\beta y \in H.$

$\Rightarrow (\alpha T(x) + \beta T(y), \alpha x+\beta y) = 0$

$\Rightarrow \alpha\bar{\alpha}(T(x), x) + \alpha\bar{\beta}(T(x), y) + \beta\bar{\alpha}(T(y), x) + \beta\bar{\beta}(T(y), y)$

$\Rightarrow \alpha\bar{\beta}.(T(x), y) + \beta\bar{\alpha}(T(y), x) = 0.$

For $\alpha, \beta = 1$ we get,

$(Tx, y) + (Ty, x) = 0 \quad \forall x, y \in H$ — ①

$\bar{i} = \bar{i} = -i$

For $\alpha = i$, $\beta = +1$ we get,

$\therefore i(Tx, y) - i(T(y), x) = 0 \quad \forall x, y \in H$ — ②

$(Tx, y) - (Ty, x) = 0 \quad \forall x, y \in H.$ — ③

① + ③

$\Rightarrow 2(Tx, y) = 0 \quad \forall x, y \in H$

$\Rightarrow (Tx, y) = 0 \quad \forall x, y \in H$

$\Rightarrow (Tx, y) = 0 \quad \forall y \in H$

$\Rightarrow Tx = 0 \quad \forall x \in H$

$\Rightarrow T = 0$

Remark:- ie. $T \equiv 0$

1] If T is zero then $(Tx, x) = 0 \quad \forall x \in H.$

2] $T \in B(H)$, $T \equiv 0$ iff $(Tx, x) = 0.$

Th^m 3:-

An operator $T \in B(H)$ is self adjoint ie. $T = T^*$ iff (Tx, x) is an real number, for $x \in H$.

→ Proof:-

Let $T \in B(H)$

Consider $T = T^*$

Let $x \in H$

i) Consider, $\overline{(Tx, x)} = (x \; Tx)$

$= (x \; T^* x)$ $\forall x \in H$

$= (Tx \; x)$ $\forall x \in H$

$\Rightarrow \overline{(Tx, x)} = (Tx \; x)$

$\Rightarrow (Tx, x)$ is real.

$\bar{z} = z$ then z is real.

ii) Consider (Tx, x) is real no. $\forall x \in H$.

$\overline{(Tx, x)} = (Tx \; x)$

$\Rightarrow (x \; Tx) = (Tx \; x)$ $\forall x \in H$

$\Rightarrow (x \; T^{**}x) = (Tx \; x)$ $\forall x \in H \because T = T^{**}$

$\Rightarrow (T^* x, x) = (Tx \; x)$ $\forall x \in H$

$\Rightarrow (T^* x - Tx, x) = 0$ $\forall x \in H$

$\Rightarrow ((T^* - T) x, x) = 0$ $\forall x \in H$

$\Rightarrow T^* - T = 0$ $\forall x \in H$

$\Rightarrow T^* = T$

Th^m 4:-

Real Banach space of set of all self adjoint operators on set of H i.e. S(H) is Poset whose linear structure and order structure are related by following properties.

i) $A_1 \le A_2$ then $A_1 + A \le A_2 + A$ $\forall A \in S(H)$.

ii) If $A_1 \le A_2$ and $\alpha \geq 0$, $\alpha A_1 \le \alpha A_2$.

Proof:-

If A_1 and $A_2 \in S(H)$.

We know that,

$T \in S(H) \Leftrightarrow (Tx, x) \in \mathbb{R}$ $\forall x \in H$.

Then $(A_1 x, x)$ and $(A_2 x, x)$ are real no.

Define $A_1 \leq A_2$ if $(A_1x, x) \leq (A_2x, x)$.
To show that,
Rel^n '$\leq$' is a partial ordered.

i) Reflexivity:-
Let $A_1 \in S(H)$.
As $(A_1 x, x) = (A_1x, x)$
$\Rightarrow A_1 = A_1$
ie A_1 is comparable with A_1.

ii) Anti-symmetric :-
Let, $A_1, A_2 \in S(H)$ such that,
$A_1 \leq A_2$ and $A_2 \leq A_1$
$\Rightarrow (A_1x, x) \leq (A_2x, x)$ and
$(A_2x, x) \leq (A_1x, x)$. $\forall x \in H$

$\Rightarrow (A_1x, x) = (A_2x, x)$ as they are real
$\Rightarrow A_1 = A_2$

iii) Transitive:-
Let, $A_1, A_2, A_3 \in S(H)$ such that,
$A_1 \leq A_2 \quad A_2 \leq A_3$
$\Rightarrow (A_1x, x) \leq (A_2x \; x)$
$(A_2x, x) \leq (A_3x \; x)$

$\Rightarrow (A_1x \; x) \leq (A_2x \; x)$
$(A_2x \; x) \leq (A_3x \; x)$

$\Rightarrow (A_1x, x) \leq (A_3x \; x)$
$\Rightarrow A_1 \leq A_3$
$\therefore$ From (i) & (ii) (iii) '$\leq$' is partial ordering.

$\therefore (S(H), \leq)$ is poset.

iv] Let $A_1 \leq A_2 \in S(H)$

$(A_1x \quad x) \leq (A_2x \quad x)$

Let, $A \in S(H)$

linear structure

$\Rightarrow (A_1x \quad x) + (Ax \quad x) \leq (A_2x \quad x) + (Ax \quad x)$

$\Rightarrow (A_1x + Ax, \quad x) \leq (A_2x + Ax, x)$

$\Rightarrow ((A_1+A)x, \quad x) \leq ((A_2+A)x, x)$

$\Rightarrow A_1 + A \leq A_2 + A$

$\Rightarrow A_1 + A \leq A_2 + A$

v] Let $A_1 \leq A_2$

ordered structure

$(A_1x \quad x) \leq (A_2x \quad x)$

Let $\alpha \geq 0$

$\alpha(A_1x \quad x) \leq \alpha(A_2x \quad x)$

$\Rightarrow (\alpha A_1x, x) \leq (\alpha A_2x \quad x)$

$\Rightarrow \alpha A_1 \leq \alpha A_2$

Positive operator:-

A self adjoint operator A is said to be positive if $A \geq 0$ i.e. $(Ax, x) \geq (0x \quad x)$ $\forall x \in H$.

Remark:-

1] Zero is an positive operator as $(0x \quad x) \geq (0x \quad x)$ $\forall x \in H$.

2] I is an positive operator $(Ix \quad x) = (x \quad x) \geq \|x\|^2 \geq (0x, x) = 0$.

3] For any $T \in B(H)$, T^*T is an positive operator.

as $(T^*Tx, x) = (Tx, T^{**}x) \quad \forall x \in H.$

$\Rightarrow$

$\Rightarrow (T^*Tx, x) = (Tx, Tx) \quad \because T^{**} = T$

$(T^*Tx, x) = \|T(x)\|^2 \geq (0x, x) = 0$

Simillarly, TT^* is positive operator as

$(TT^*x, x) = (T^*x, T^*x)$

$\Rightarrow (TT^*x, x) = \|T^*(x)\|^2 \geq (0x, x) = 0.$

4] If A is positive operator on H then $I+A$ is non-singular.

Normal Operators:-

An operator ~~B &~~ $N \in B(H)$ is normal if $NN^* = N^*N$ where $*$ represents adjoint of N.

Show that set of all normal operators of H i.e. $N(H)$ is closed subset of $B(H)$, contains $s(H)$ and is closed under scalar multiplication.

$\rightarrow$ Let,

$N(H)$ be set of all normal operators on H.

$0^* = 0^* 0 = 0$

$I^* = I^* I = 0$

i] $0, I \in N(H)$ clearly. $\Rightarrow N(H)$ is non-empty

ii] Let $T \in s(H)$

We have $T = T^*$

$\Rightarrow TT^* = T^* T^*$

$\Rightarrow TT^* = T^* T \quad$ as $\because (T^* = T)$

$\Rightarrow T \in N(H)$

$\therefore s(H) \subseteq N(H).$

iii] Let $T \in N(H)$ and $\alpha \in \mathbb{C}$

$(\alpha T)(\alpha T)^* = \alpha T \cdot \bar{\alpha} T^*$

$\alpha T \quad \alpha \alpha$

$$(\alpha T)(\alpha T)^* = \alpha\bar{\alpha} \cdot T \cdot T^*$$
$$= \alpha\bar{\alpha} \cdot T^* T \quad \because T^*T = TT^*$$
$$= \bar{\alpha}\,\alpha T^* T$$
$$= \bar{\alpha} T^* \alpha T$$
$$\therefore (\alpha T)(\alpha T)^* = (\alpha T)^* \alpha T$$
$$\therefore \alpha T \in N(H).$$

iv) Let, $\{T_n\}$ be seq^n in $N(H)$ such that $T_n \to T$ as $n \to \infty$

Consider,

$$\| TT^* - T^*T \|$$
$$= \| TT^* - T_nT_n^* + T_nT_n^* - T_n^*T_n + T_n^*T_n - T^*T \|$$
$$\leq \| TT^* - T_nT_n^* \| + \| T_nT_n^* - T_n^*T_n \| + \| T_n^*T_n - T^*T \| \quad \text{--- (1)}$$

As $T_n \in N(H)$

$T_nT_n^* = T_n^* T_n$ and $T_n \to T$, $T_n^* \to T^*$

Then $T_nT_n^* \to TT^*$ and $T_n^* T_n \to T^*T$ as $n \to \infty$

As $n \to \infty$ RHS of (1) is tending to zero.

$\therefore$ As $n \to \infty$,

$$\| TT^* - T^*T \| \to 0$$
$$\Rightarrow \| TT^* - T^*T \| = 0$$
$$\Rightarrow TT^* = T^*T$$
$$\Rightarrow T \in N(H)$$
$\Rightarrow$ $N(H)$ is closed.

Th^m 5:-

If N_1 and N_2 are normal operators on H with property that either commutes with the adjoint of other then $N_1 + N_2$ and N_1N_2 are normal.

→ Proof:-

Let N_1 and N_2 are normal operators on H, and suppose

$$\left.\begin{aligned} & N_1 N_2^* = N_2^* N_1 \\ \Leftrightarrow\ & (N_1 N_2^*)^* = (N_2^* N_1)^* \\ \Leftrightarrow\ & N_2^{**} N_1^* = N_1^* N_2^{**} \\ \Leftrightarrow\ & N_2 N_1^* = N_1^* N_2 \end{aligned}\right\} \quad \text{--- (1)}$$

(: either commutes with adjoint of other)

$(\because T^{**} = T)$

i) Now consider,

$$(N_1 + N_2)(N_1 + N_2)^*$$
$$= (N_1 + N_2)(N_1^* + N_2^*)$$
$$= (N_1 N_1^*) + (N_1 N_2^*) + (N_2 N_1^*) + (N_2 N_2^*). \quad \text{--- (2)}$$

Now

$$(N_1 + N_2)^* (N_1 + N_2)$$
$$= (N_1^* + N_2^*)(N_1 + N_2)$$
$$= N_1^* N_1 + N_1^* N_2 + N_2^* N_1 + N_2^* N_2 \quad \text{--- (3)}$$

$N_1^* N_2 = N_2 N_1^*$
$N_2^* N_1 = N_1 N_2^*$

From (1), (2), (3),

$$(N_1 + N_2)(N_1 + N_2)^* = (N_1 + N_2)^* (N_1 + N_2)$$

ii)

$$(N_1 N_2)(N_1 N_2)^*$$
$$= (N_1 N_2)(N_2^* N_1^*)$$
$$= (N_1 N_2^*) \cdot (N_1 N_1^*) \cdot (N_2 N_2^*) \cdot (N_2 N_1^*) \quad \text{--- (4)}$$

$$(N_1 N_2)^* (N_1 N_2)$$
$$= (N_2^* N_1^*)(N_1 N_2)$$
$$= (N_2^* N_1) \cdot (N_2^* N_2)(N_1^* N_1)(N_1^* N_2) \quad \text{--- (5)}$$

From (4) & (5)

$$(N_1 N_2)(N_1 N_2)^* = (N_1 N_2)^* (N_1 N_2).$$

$N_2^* N_1 = N_1 N_2^*$
$N_1^* N_2 = N_2 N_1^*$

Th^m 6:-

An operator T on H is normal iff $\|T^*(x)\| = \|T(x)\| \quad \forall\, x \in H$

→ Proof:-

i) Assume T is normal.

$$T^*T = TT^*$$

$$\Rightarrow T^*T - TT^* = 0$$

(a-b) x, x)

$$\Rightarrow ((T^*T - TT^*)x, x) = 0$$

$$\Rightarrow ((T^*T)x, x) - ((TT^*)x, x) = 0$$

$$\Rightarrow ((T^*T)x, x) = ((TT^*)x, x)$$

$$\Rightarrow (Tx \;\; Tx) = (T^*x, T^*x)$$

$$\Rightarrow \|Tx\|^2 = \|T^*x\|^2$$

$$\Rightarrow \|Tx\| = \|T^*x\|$$

ii) Assume

$$\|Tx\| = \|T^*x\|$$

$$\Rightarrow \|Tx\|^2 = \|T^*x\|^2$$

$$\Rightarrow (Tx \;\; Tx) = (T^*x, T^*x)$$

$$\Rightarrow ((T^*T)(x), x) = ((TT^*)(x), x)$$

$$\Rightarrow ((T^*T)x, x) - ((TT^*)x, x) = 0$$

$$\Rightarrow ((T^*T - TT^*)x, x) = 0$$

$$\Rightarrow T^*T - TT^* = 0$$

$$\Rightarrow T^*T = TT^*$$

Th^m 7:-

If N is normal operator on H then $\|N^2\| = \|N\|^2$.

→ Proof:-

Let N be normal operator,

$$\Rightarrow \|Nx\| = \|N^*x\|.$$

Consider,

$$\|N^2 x\| = \|(N \cdot N)x\|$$

Remark:-

1) Let T be an operator on H which is continuous and linear and let T^* be adjoint of T on H. Define,

$$A_1 = \frac{T+T^*}{2} \quad \text{and} \quad A_2 = \frac{T-T^*}{2i}$$

Then A_1 and A_2 are self adjoint operator as,

$$A_1^* = \left[\frac{T+T^*}{2}\right]^* = \frac{T^{**}+T^*}{2} = \frac{T+T^*}{2} = A_1$$

$$A_2^* = \left[\frac{T-T^*}{2i}\right]^* = \frac{T^*-T^{**}}{-2i} = \frac{-T^*+T}{2i}$$

$$A_2^* = \frac{T-T^*}{2i}$$

$$A_2^* = A_2$$

Also $A_1 + iA_2 = T$ and $A_1 - iA_2 = T^*$.

Note:-

If T is an operator on H then T is normal iff its real and imaginary parts commute.

→ Let,

$$T \in B(H)$$

We know that, T can be written as,

$A_1 + iA_2 = T$ and $A_1 - iA_2 = T^*$

where $A_1 = \dfrac{T + T^*}{2}$ and $A_2 = \dfrac{T - T^*}{2i}$.

Suppose T is normal.

$$T^* T = (A_1 - iA_2)\cdot(A_1 + iA_2)$$
$$= (A_1A_1) + iA_1A_2 - iA_2A_1 + A_2A_2$$
$$= A_1A_1 + A_2A_2 + iA_1A_2 - iA_2A_1$$
$$= A_1A_1 + A_2A_2 + i(A_1A_2 - A_2A_1) \quad \text{— (I)}$$

$$TT^* = (A_1 + iA_2)(A_1 - iA_2)$$
$$= A_1A_1 - iA_1A_2 + iA_2A_1 + A_2A_2$$
$$= A_1A_1 + A_2A_2 + i(A_2A_1 - A_1A_2) \quad \text{— (II)}$$

$A_1A_1 = A_1A_1$ $A_2A_2 = A_2A_2$

T is normal, real and imaginary parts are equal

From (I) & (II)

$$A_1A_2 - A_2A_1 = A_2A_1 - A_1A_2 \quad \because T \text{ is normal.}$$
$$A_1A_2 + A_1A_2 = A_2A_1 + A_2A_1$$
$$\Rightarrow 2A_1A_2 = 2A_2A_1$$
$$\Rightarrow A_1A_2 = A_2A_1$$

Conversely,

Suppose $A_1A_2 = A_2A_1$.

Then from (I) & (II),

$$TT^* = T^*T$$

$\Rightarrow$ T is normal.