UNIT - I - INTERPOLATION : EQUAL INTERVALS

* Interpolation -

Defn - Interpolation is defined as the technique of calculating the values of a function for any intermediate values of independent variable.

* Forward Difference operator (Δ) -

Let $y = f(x)$ be a function then x is the independent variable and is called 'arguement'. Also y is the dependent variable and is called 'entry'.

Suppose x assumes values such that the differences betn. any two values is constant as

$x : a, a+h, a+2h, \ldots$

then 'h' is called the 'interval difference'.

Thus the corresponding values of y or $f(x)$ will be,

$y = f(x) : f(a), f(a+h), f(a+2h), f(a+3h), \ldots$

Now consider the differences of y as

$f(a+h) - f(a)$

$f(a+2h) - f(a+h)$

$f(a+3h) - f(a+2h)$

$\vdots$

These differences are called 'first forward differences' of the function $y = f(x)$ and is denoted by,

$\Delta f(a) = f(a+h) - f(a)$

$\Delta f(a+h) = f(a+2h) - f(a+h)$

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$\Delta f(a+2h) = f(a+3h) - f(a+2h)$

$\vdots$

Where, Δ is forward difference operator.

similarly the dibberences,

$\Delta f(a+h) - \Delta f(a)$

$\Delta f(a+2h) - \Delta f(a+h)$

$\Delta f(a+3h) - \Delta f(a+2h)$

$\vdots$

is called 'second forward difference' and is denoted by,

$\Delta^2 f(a) = \Delta f(a+h) - \Delta f(a)$

$\Delta^2 f(a+h) = \Delta f(a+2h) - \Delta f(a+h)$

$\Delta^2 f(a+2h) = \Delta f(a+3h) - \Delta f(a+2h)$

$\vdots$

Imp

Q. Define Forward Differences.

Ans i) The first forward difference ob $f(x)$ at x is defined as, $f(x+h) - f(x)$ & is denoted by $\Delta f(x)$.

$\therefore \Delta f(x) = f(x+h) - f(x)$

Where, Δ is called 'Forward difference operator'.

ii) We define,

$$\Delta^2 f(x) = \Delta[\Delta f(x)]$$
$$= \Delta[f(x+h) - f(x)]$$
$$= \Delta f(x+h) - \Delta f(x)$$

is called 'second forward difference' & so on.

Note -

1) The third order difference ob $f(x)$ is the difference ob second differences of $f(x)$.

ii) on similar lines we define $4^{th}, 5^{th}, \ldots$ n^{th} order dibberences.

iii) We also represent the values ob x and y in the function $y = f(x)$ as follows.

ⓐ x : $1, 2, 3, 4, \ldots$

y : $f(1), f(2), f(3), f(4), \ldots$ [OR $f_1, f_2, \ldots$]

ⓑ x : $x_0, x_1, x_2, x_3, \ldots$

y : $y_0, y_1, y_2, y_3, \ldots$

Using these notations we have,

i) $\Delta f(1) = f(2) - f(1)$

$\Delta f(3) = f(4) - f(3)$

ii) $\Delta^2 f(2) = \Delta [\Delta f(2)]$

$= \Delta [f(3) - f(2)]$

$= \Delta f(3) - \Delta f(2)$

$= f(4) - f(3) - [f(3) - f(2)]$

$= f(4) - f(3) - f(3) + f(2)$

$= f(4) - 2f(3) + f(2)$

iii) $\Delta y_2 = y_3 - y_2$

Now,

$\Delta^2 y_1 = \Delta [\Delta y_1]$

$= \Delta [y_2 - y_1]$

$= \Delta y_2 - \Delta y_1$

$= y_3 - y_2 - [y_2 - y_1]$

$= y_3 - y_2 - y_2 + y_1$

$= y_3 - 2y_2 + y_1$

* Forward Difference Table -

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$ ...
a	$f(a)$				
		$\Delta f(a)$ $= f(a+h) - f(a)$			
$a+h$	$f(a+h)$		$\Delta^2 f(a)$		
		$\Delta f(a+h)$		$\Delta^3 f(a)$	
$a+2h$	$f(a+2h)$		$\Delta^2 f(a+h)$		$\Delta^4 f(a)$ ...
		$\Delta f(a+2h)$		$\Delta^3 f(a+h)$	⋮
$a+3h$	$f(a+3h)$		$\Delta^2 f(a+2h)$	⋮	⋮
		$\Delta f(a+3h)$	⋮	⋮	
$a+4h$	$f(a+4h)$	⋮	⋮		
⋮	⋮	⋮			

Note -

i) The term $f(a)$ is called 'leading term'.

ii) The other differences at the top of each column $\Delta f(a)$, $\Delta^2 f(a)$, $\Delta^3 f(a)$, --- are called 'leading differences'.

* Algebra of Forward Differences.

i) If $f(x) = c$ then $\Delta f(x) = 0$

ii) $\Delta [c \cdot f(x)] = c \cdot \Delta f(x)$

iii) $\Delta [f(x) \pm g(x)] = \Delta f(x) \pm \Delta g(x)$

iv) $\Delta^m [\Delta^n f(x)] = \Delta^{m+n} f(x)$
Where m & n are positive integers.

* The operator E

Defn -

i) Let $y = f(x)$ be a function of x. Let a, a+h, a+2h, a+3h, ... be conjugative values ob x, we define operator E as, $E f(x) = f(x+h)$

E is called 'shift operator'. Thus for special values of x we define.

i) $E f(a) = f(a+h)$

ii) $E f(a+h) = f(a+2h)$

iii) $E f(a+2h) = f(a+3h)$

⋮

ii) The We define

$$E^2 f(x) = E[E f(x)]$$
$$= E[f(x+h)]$$
$$= f(x+2h)$$

similarlly,

$$E^3 f(x) = E[E^2 f(x)]$$
$$= E[f(x+2h)]$$
$$= f(x+3h)$$

In general,

$$E^n f(x) = f(x+nh)$$

where, n is +ve integer.

Where h is interval difference.

*

* Properties ob E -

MCQ i) Ib $f(x) = c$ where c is constant then

$$E f(x) = c$$

ii) Ib $f(x)$ and $g(x)$ are two functions ob x then,

$$E[f(x) \pm g(x)] = E f(x) \pm E g(x)$$

iii) $E[c \cdot f(x)] = c \cdot E f(x)$.

iv) If m & n are +ve integers then,

$$E^m[E^n f(x)] = E^{m+n} f(x)$$

* Relation betⁿ. operators E and Δ -

We have,

$$\Delta f(x) = f(x+h) - f(x)$$

and

$$E f(x) = f(x+h)$$

$$\therefore \Delta f(x) = E f(x) - f(x) \quad \cdots (1)$$

We define, an operator I as

$$I f(x) = f(x)$$

hence eqn. (1) become

$$\Delta f(x) = E f(x) - I f(x)$$

$$\Rightarrow \Delta f(x) = [E - I] f(x)$$

$$\boxed{\Delta \equiv E - I}$$

OR ~~$E \equiv \Delta$~~ $\boxed{E - \Delta \equiv I}$ OR $\boxed{E \equiv \Delta + I}$

* Backward Difference operator (∇) -

Defⁿ -

I) The first backward difference of $f(x)$ at x is defined as,

$$f(x) - f(x-h)$$

and is denoted by $\nabla f(x)$

$$\therefore \nabla f(x) = f(x) - f(x-h)$$

Where, ∇ is called 'backward difference operator' and h is called interval difference.

II) We define,

$$\nabla^2 f(x) = \nabla[\nabla f(x)]$$

$$= \nabla[f(x) - f(x-h)]$$

$$= \nabla f(x) - \nabla f(x-h)$$

be the 'second backward difference of $f(x)$ & x.

iii) We define,

$$\nabla^3 f(x) = \nabla[\nabla^2 f(x)]$$
$$= \nabla[\nabla f(x) - \nabla f(x-h)]$$
$$= \nabla^2 f(x) - \nabla^2 f(x-h)$$

be the 'third backward difference ob $f(x)$ and x.

Note -
The forward differences are defined by points to the right ob 'a' where backward differences are defined by points to the lebt ob 'a'.

* Backward Difference Table -.

x	$f(x)$	$\nabla f(x)$	$\nabla^2 f(x)$	$\nabla^3 f(x)$	$\nabla^4 f(x)$
a	$f(a)$				
		$\nabla f(a+h)$			
$a+h$	$f(a+h)$		$\nabla^2 f(a+2h)$		
		$\nabla f(a+2h)$		$\nabla^3 f(a+3h)$	
$a+2h$	$f(a+2h)$		$\nabla^2 f(a+3h)$		$\nabla^4 f(a+4h)$
		$\nabla f(a+3h)$		$\nabla^3 f(a+4h)$	
$a+3h$	$f(a+3h)$		$\nabla^2 f(a+4h)$		
		$\nabla f(a+4h)$			
$a+4h$	$f(a+4h)$				

Note -
1) Ib backward difference table the last entry $f(a+4h)$ is called 'leading term' & the differences, $\nabla f(a+4h)$, $\nabla^2 f(a+4h)$, $\nabla^3 f(a+4h)$ ... are called leading back-ward differences.

II) From the definations of forward & back-ward differences it follows that,

ⓐ $\Delta f(a) = f(a+h) - f(a)$
$= \nabla f(a+h)$

In other notations,
$\Delta y_1 = y_2 - y_1$
$= \nabla y_2$

ⓑ $\Delta^2 f(a) = \Delta[\Delta f(a)]$
$= \Delta[f(a+h) - f(a)]$
$= \Delta f(a+h) - \Delta f(a)$
$= f(a+2h) - f(a+h) - [f(a+h) - f(a)]$
$\Delta^2 f(a) = f(a+2h) - 2f(a+h) + f(a)$ ...(I)

consider,
$\nabla^2 f(a+2h) = \nabla[\nabla f(a+2h)]$
$= \nabla[f(a+2h) - f(a+h)]$
$= \nabla f(a+2h) - \nabla f(a+h)$
$= f(a+2h) - f(a+h) - [f(a+h) - f(a)]$
$= f(a+2h) - 2f(a+h) + f(a)$ -(II)

By (I) and (II),
$\Delta^2 f(a) = \nabla^2 f(a+2h)$

In general,
$\Delta^n f(a) = \nabla^n f(a+nh)$

* Algebra of Backward Differences -

Let $f(x)$ and $g(x)$ be two function of x and c be a constant then,

I) If $f(x) = c$ then
$\nabla f(x) = 0$

II) $\nabla[c.f(x)] = c.\nabla f(x)$

III) $\nabla[f(x) \pm g(x)] = \nabla f(x) \pm \nabla g(x)$

IV) $\nabla^m[\nabla^n f(x)] = \nabla^{m+n} f(x)$,
where m & n are +ve integers.

* The operator E^{-1} -

We define the operator E^{-1} by,

$E^{-1}f(x) = f(x-h)$

Thus,

I) $E^{-1}f(a+h) = f(a)$

$E^{-1}f(a+2h) = f(a+h)$

$\vdots \qquad \vdots$

II) Also we define,

$EE^{-1}\{f(a) = E[E^{-1}f(a)]$

$= E[f(a-h)]$

$= f(a)$

And $E^{-1}Ef(a) = E^{-1}[Ef(a)]$

$= E^{-1}[f(a+h)]$

$= f(a)$

Thus,

$EE^{-1}f(a) = f(a) = E^{-1}Ef(a)$

III) We further define,

$E^{-2}f(x) = E^{-1}[E^{-1}f(x)]$

$= E^{-1}[f(x-h)]$

$= f(x-2h)$

In general,

$E^{-n}f(x) = f(x-nh)$.

* Relation betn operator ∇ & E^{-1} -

We have,

$\nabla f(x) = f(x) - f(x-h)$

and $E^{-1}f(x) = f(x-h)$

$\therefore \nabla f(x) = f(x) - E^{-1}f(x)$

We define operator I as

$I f(x) = f(x)$

Thus,

$\nabla f(x) = If(x) - E^{-1}f(x)$

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$$\Rightarrow \nabla f(x) = (I - E^{-1})\, f(x)$$

$$\Rightarrow \boxed{\nabla \equiv I - E^{-1}}$$

OR $\boxed{E^{-1} \equiv I - \nabla}$

OR $\boxed{I \equiv \nabla + E^{-1}}$

Note -

We have,

1) $\nabla[E f(a)] = \nabla[f(a+h)]$

$$= f(a+h) - f(a)$$

$$= \Delta f(a)$$

$$\Rightarrow \nabla E \equiv \Delta$$

ii) $\nabla[E f(a)] = \nabla[f(a+h)]$

$$= f(a+h) - f(a) \quad \text{-- (i)}$$

Also,

$$E[\nabla f(a)] = E[f(a) - f(a-h)]$$

$$= E f(a) - E f(a-h)$$

$$= f(a+h) - f(a) \quad \text{-- (ii)}$$

By (i) & (ii)

$$\nabla E \equiv E \cancel{\Delta} \nabla$$

iii) Similarly,

(a) $\Delta \nabla \equiv \nabla \Delta$

(b) $(I + \Delta)(I - \nabla) \equiv I$

(c) $\nabla \equiv \Delta E^{-1}$

Ex- Solve $\Delta^2 3 . e^x$

Sol^n- Let, h be interval dibberence.

$$\therefore \Delta^2 3 e^x = \Delta\, 3 \{ \Delta [\Delta e^x] \}$$

$$= 3 \{ \Delta [e^{x+h} - e^x] \}$$

$$= 3 \{ \Delta [e^h - 1] e^x \}$$

$$= 3 (e^h - 1) \{ \Delta e^x \}$$

$$= 3 (e^h - 1) \{ e^{x+h} - e^x \}$$

$$= 3 (e^h - 1)^2 e^x$$

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2) solve, $\Delta^2 bx^2$. where b is constant.

soln. Let, h be interval difference.

$$\therefore \Delta^2 bx^2 = b\{\Delta[\Delta x^2]\}$$
$$= b\{\Delta[(x+h)^2 - x^2]\}$$
$$= b\{\Delta(x+h)^2 - \Delta x^2\}$$
$$= b\{[(x+2h)^2 - (x+h)^2]$$
$$-[(x+h)^2 - x^2]\}$$

$$= b\{\Delta[x^2 + 2xh + h^2 - x^2]\}$$
$$= b\{\Delta(2hx + h^2)\}$$
$$= b\{2h\Delta x + \cancel{h^2}\Delta h^2\}$$
$$= 2bh[x + h - x + 0]$$
$\because f(x) = c$ then $\Delta f(x) = 0$
$$= 2bh^2.$$

imp 3) If $f(x) = x^2 + x + 1$ construct forward difference table by taking $x = 0$ to 5. Hence find $\Delta f(0)$, $\Delta^2 f(1)$ & $\Delta^3 f(2)$.

soln. Given, function is,
$$f(x) = x^2 + x + 1$$
If $x = 0 \Rightarrow f(0) = 1$
if $x = 1 \Rightarrow f(1) = 3$
if $x = 2 \Rightarrow f(2) = 7$
if $x = 3 \Rightarrow f(3) = 13$
if $x = 4 \Rightarrow f(4) = 21$
if $x = 5 \Rightarrow f(5) = 31$

Thus, forward difference table is,

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
0	1			
		$\boxed{2}$		
1	3		2	
		4		0

			$\boxed{2}$		
2	7			0	
		6			
			2		
3	13			$\boxed{0}$	
		8			
4	21		2		
		10			
5	31				

which is required forward difference table.

From table,

$\Delta f(0) = 2$

$\Delta^2 f(1) = 2$

$\Delta^3 f(2) = 0$

2) construct a backward difference table for the following data.

x : 1 2 3 4 5

$y = f(x)$: 2 -2 2 -2 2

Soln- The backward difference table is constructed as follows

x	$f(x)$	$\nabla f(x)$	$\nabla^2 f(x)$	$\nabla^3 f(x)$	$\nabla^4 f(x)$
1	2				
		-4			
2	-2		8		
		4		-16	
3	2		-8		32
		-4		16	
4	-2		8		
		$\boxed{4}$			
5	2				

which is required backward dibberence table.
From backward difference table-
$\nabla f(5) = 4$, $\nabla^3 f(1) =$ not define
$\nabla^2 f(4) = -8$

3) By constructing a forward difference table find the 7 and 8 term ob the sequence, 8, 14, 22, 32, 44, 58, ...

Soln- We construct forward difference table as

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$
1	8		
		6	
2	14		2
		8	
3	22		2
		10	
4	32		2
		12	
5	44		2
		14	
6	58		2
		$\boxed{16}$	
7	$\boxed{74}$		2
		$\boxed{18}$	
8	$\boxed{92}$		

Thus, 7^{th} term ob seq^n. is 74 and 8^{th} term ob seq^n. is 92.

4) Find 20th term ob the seq^n. 2, 6, 12, 20, 30 ...

Soln- First we construct forward dibberence table as follows.

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$
1	2		
		4	
2	6		2
		6	
3	12		2
		8	
4	20		2
		10	
5	30		

We are required to find $f(20)$.

We have,

$$f(20) = f(19+1)$$
$$= E\, f(19)$$
$$= E^{19} f(1)$$
$$= (I+\Delta)^{19} f(1).$$
$$= \left(I + 19\Delta + \frac{19.18}{2!}\Delta^2 + \frac{19\times18\times17}{3!}\Delta^3 + \cdots\right) f(1)$$
$$= f(1) + 19\Delta f(1) + 171\,\Delta^2 f(1) + \cdots$$
$$= 2 + 19\times 4 + 171 \times 2$$
$$= 420$$

Thus, 20th term of seq^n. is 420.

5) $f(x)$ is a polynomial of degree 2 in x if $f(0) = 8$, $f(1) = 12$, $f(2) = 18$ find $f(x)$.

Sol^n- First we construct forward difference table.

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$
0	8		
		4	
1	12		2
		6	
2	18		

Now,

$$f(x) = E^x f(0)$$
$$= (I+\Delta)^x f(0)$$
$$= \left(I + x\Delta + \frac{x(x-1)}{2}\Delta^2 + \cdots\right) f(0)$$
$$= f(0) + x\Delta f(0) + \frac{x(x-1)}{2}\Delta^2 f(0) + \cdots$$
$$= 8 + x(4) + x^2 - x$$
$$= x^2 + 3x + 8.$$

* Newton's Forward Interpolation Formula –

Let, the function $y = f(x)$ assume the values $f(a)$, $f(a+h)$, $f(a+2h)$, ... $f(a+nh)$ at the $(n+1)$ equidistant point a, $a+h$, $a+2h$, ..., $a+nh$ of the arguement x.

suppose that, we want to estimate $f(x)$ for $x = a+mh$

$$\Rightarrow m = \frac{x-a}{h}$$

Now, $f(x) = f(a+mh)$

$$= E^m f(a)$$
$$= (I+\Delta)^m . f(a)$$
$$= \left[I + m\Delta + \frac{m(m-1)}{2!}\Delta^2 + \frac{m(m-1)(m-2)}{3!}\Delta^3 + \cdots \frac{m(m-1)\cdots(m-n+1)}{n!}\Delta^n\right] f(a)$$

$$= f(a) + m\Delta f(a) + \frac{m(m-1)}{2!}\Delta^2 f(a) + \cdots + \frac{m(m-1)\cdots(m-n+1)}{n!}\Delta^n f(a)$$

this is known as 'Newton's formula' for forward interpolation.

Note -
This formula can also be expressed as,

$$f_x = f_0 + m\Delta f_0 + \frac{m(m-1)}{2!}\Delta^2 f_0 + \cdots + \frac{m(m-1)\cdots(m-n+1)}{n!}\Delta^n f_0$$

Ex-1) Using Newton's forword interpolation formula estimate f(5) for the following data.

x :	2	4	6	8
$f(x)$:	4	7	11	18

Soln. First we construct forward difference table,

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
2	4			
		3		
4	7		1	
		4		2
6	11		3	
		7		
8	18			

From forward difference table,

$a = 2$, $h = 2$, $x = 5$

$$\therefore\ m = \frac{x-a}{h} = \frac{5-2}{2} = \frac{3}{2} = 1.5$$

Also, $f(2) = 4$, $\Delta f(2) = 3$, $\Delta^2 f(2) = 1$

$\Delta^3 f(2) = 2$

Thus, by using Newton's forward interpolation formula

$$f(x) = f(a) + m\,\Delta f(a) + \frac{m(m-1)}{2!}\Delta^2 f(a) + \frac{m(m-1)(m-2)}{3!}\Delta^3 f(a) + \cdots$$

Thus,

$$f(5) = 4 + 1.5\times 3 + \frac{1.5(1.5-1)}{2}\times 1 + \frac{1.5\times 0.5\times -0.5}{\cancel{3!}\ 6}\times 2 + \cdots$$

$f(5) = 8.75$ (approx.)

2) Estimate $f(1.5)$ from the following data using Newton's forward interpolation formula

x :	0	1	2	3	4
$f(x)$:	0	1	8	27	64

Soln- First we construct Newton's forward difference table,

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
0	0				
		1			
1	1		6		
		7		6	
2	8		12		0
		19		6	
3	27		18		
		37			
4	64				

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$a=0$, $h=1$, $x=1.5$

$\therefore m = \frac{x-a}{h} = 1.5$

Also, $f(0)=0$, $\Delta f(0)=1$, $\Delta^2 f(0)=6$
$\Delta^3 f(0) = 6$

Thus by using newton's forward interpolation formula,

$$f(x) = f(a) + m\Delta f(a) + \frac{m(m-1)}{2!}\Delta^2 f(a) + \frac{m(m-1)(m-2)}{3!}\Delta^3 f(a) + \dots$$

$$f(1.5) = 0 + 1.5\times1 + \frac{1.5\times0.5}{2}\times6 + \frac{1.5\times0.5\times-0.5}{3\times2}\times6 + \dots$$

$\therefore f(1.5) = 3.375$

3) A 2nd degree polynomial passes through the points $(0,3)$, $(1,6)$, $(2,11)$, $(3,18)$. Find the polynomial.

Soln. First we construct Newton's forward difference table.

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
0	3			
		3		
1	6		2	
		5		0
2	11		2	
		7		
3	18			

From forward difference table.

$a=0$, & $h=1$, $f(0)=3$, $\Delta f(0)=3$
$\Delta^2 f(0)=2$,

$$m = \frac{x-a}{h} = x$$

Thus by using newton's forward difference ~~tab~~ interpolation formula,

$$f(x) = f(a) + m\Delta f(a) + \frac{m(m-1)}{2!}\Delta^2 f(a) + \frac{m(m-1)(m-2)}{3!}\Delta^3 f(a) + \cdots$$

$$f(x) = 3 + 3x + \frac{x(x-1)}{2} \times 2 + \frac{x(x-1)(x-2)}{6} \times 6$$

$$= \cancel{3 + 3x + x^2 - x^2} \quad 3 + 3x + x^2 - x$$

$$= x^2 + 2x + 3$$

x6 f..

$$f(x) = x^3 - 2x + 3$$

which is required polynomial.

4) For the given data..

x :	0.2	0.4	0.6	0.8	1.0
f :	3.2	3.6	2.8	3	2.4

prepare the table of forward diff. & hence interpolate $f(0.3)$.

Sol^n. First we construct forward difference table.

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
0.2	3.2				
		0.4			
			-1.2		
0.4	3.6				
		-0.8		2.2	
			1		-4
0.6	2.8				
		0.2		-1.8	
			-0.8		
0.8	3				
		-0.6			
1	2.4				

From forward diff. table,

$a = 0.2$, $h = 0.2$, $f(0.2) = 3.2$, $\Delta f(0.2) =$

$\Delta^2 f(0.2) = -1.2$, $\Delta^3 f(0.2) = 2.2$, $\Delta^4 f(0.2) =$

$$\therefore m = \frac{x-a}{h} = \frac{0.3-0.2}{0.2} = 0.5$$

Thus, by using newton's formula,

$$f(x) = f(a) + m\Delta f(a) + \frac{m(m-1)}{2!}\Delta^2 f(a) + \frac{m(m-1)(m-2)}{3!}\Delta^3 f(a) + \frac{m(m-1)(m-2)(m-3)}{4!}\Delta^4 f(a) + \cdots$$

$$f(0.3) = 3.2 + 0.5 \times 0.4 + \frac{0.5(0.5-1)}{2} \times -1.2 + \frac{0.5 \times 0.5 \ldots}{3 \times 2} \times 2.2 + \frac{0.5 \times -0.5 \times -1.5 \times -2.5}{4 \times 3 \times 2} \times -4$$

$f(0.3) = 3.8438$ (approx).

5) Prepare the forward dibberence table for the data

x	:	-1	0	1	2	3
f	:	10	2	0	10	62

Using newton's forward formula find the approximate value of $f(-0.5)$.

Sol^n – First we construct forward dibberence table,

x	f	Δf	$\Delta^2 f$	$\Delta^3 f$	$\Delta^4 f$
-1	10				
		-8			
0	2		6		
		-2		6	
1	0		12		24

		10		30	
2	10		42		
		52			
3	62				

From forward difference table,

$a = -1$, $f(a) = 10$, $x = -0.5$, $h = 1$

$$\therefore\ m = \frac{x-a}{h} = \frac{-0.5+1}{1} = 0.5$$

$\Delta^4 f(a) + \cdots$

Also, $\Delta f(a) = -8$, $\Delta^2 f(a) = 6$,

$\Delta^3 f(a) = 6$, $\Delta^4 f(a) = 24$

Thus, by Newton's forward interpolation formula,

$$f(x) = f(a) + m\Delta f(a) + \frac{m(m-1)}{2!}\Delta^2 f(a)$$

$$+ \frac{m(m-1)(m-2)}{3!}\Delta^3 f(a) + \frac{m(m-1)(m-2)(m-3)}{4!}\Delta^4 f(a)$$

$$\therefore\ f(-0.5) = 10 + 0.5 \times -8 + (-0.125 \times 6) + 0.0625 \times 6 + (-0.08908 \times 24)$$

$$\therefore\ f(-0.5) = 4.6871 \quad \text{(approx.)}$$

6) From the following data find the no. of students who have obtain less than 45 marks using Newton's forward interpolation formula.

Marks :	30-40	40-50	50-60	60-70	70-80
No. of student :	31	42	51	35	31

Soln. First we construct forward difference table,

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x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
below 40	31				
		42			
~~below~~ 5			9		
below 50	73			-25	
		57			
			-16		37
" 60	124			12	
		35			
			-4		
70	159				
		31			
80	190				

Step 2 - calculation

From forward difference table,

$h=10$, $a=40$, $f(a)=31$, $\Delta f(a)=42$,

$\Delta^2 f(a)=9$, $\Delta^3 f(a)=-25$, $\Delta^4 f(a)=37$

By using Newton's forward interpolation formula,

$$f(x)=f(a)+m\Delta f(a)+\frac{m(m-1)}{2!}\Delta^2 f(a)+\frac{m(m-1)(m-2)}{3!}\Delta^3 f(a)$$

$$+\frac{m(m-1)(m-2)(m-3)}{4!}\Delta^4 f(a)$$

Now,

$$m=\frac{x-a}{h}=\frac{45-40}{10}=\frac{5}{10}=0.5$$

$$\therefore f(x)=31+0.5\times 42+\frac{0.5\times -0.5}{2}9+$$

$$\frac{0.5\times 0.5\times -1.5}{6}\times -25+\frac{0.5\times -0.5\times -1.5\times -2.5\times 37}{12\times 2}$$

$\therefore f(45)=47.8672$ (approx)

Thus no. of students who obtain less than 45 marks is approximately 48.

7) Estimate $\sin 45^\circ$ from the following data

x	:	0°	30°	60°	90°
$\sin x$	:	0	0.5	0.8660	1

Ans- Step 1 -

First we construct forward difference table.

x	$f(x)=\sin x$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
0	0			
		0.5		
30	0.5		-0.1340	
		0.3660		-0.098
60	0.8660		-0.232	
		0.1340		
90	1			

Step 2 - calculations

From forward difference table,

$h=30$, $a=0$, $f(a)=0$, $\Delta f(a)=0.5$

$\Delta^2 f(a) = -0.1340$, $\Delta^3 f(a) = -0.098$

$$m = \frac{x-a}{h} = \frac{45-0}{30} = \frac{45}{30} = 1.5$$

By using Newton's forward interpolation formula,

$$f(x) = f(a) + m\Delta f(a) + \frac{m(m-1)}{2!}\Delta^2 f(a) + \frac{m(m-1)(m-2)}{3!}\Delta^3 f(a)$$

$$\therefore f(45^\circ) = 0 + 1.5\times 0.5 + \frac{1.5\times 0.5}{2}\times -0.1340 + \frac{1.5\times 0.5\times -0.5}{6}\times -0.098)$$

$\therefore f(45) = 0.7059$ (approx.)

$\therefore \sin 45^\circ = 0.7059$

8) The population of a country for 4 decades is given below.

Year :	1960	1970	1980	1990
popⁿ. : (Million)	10.2	10.4	10.6	10.9

Estimate the popⁿ. for the year 1964.

Solⁿ- Step 1 - Table

First we construct forward difference table.

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
1960	10.2			
		0.2		
1970	10.4		0	
		0.2		0.1
1980	10.6		0.1	
		0.3		
1990	10.9			

Step 2 - Calculation

From forward difference table,

$a = 1960$, $f(a) = 10.2$, $\Delta f(a) = 0.2$

$\Delta^2 f(a) = 0$, $\Delta^3 f(a) = 0.1$

$$m = \frac{x-a}{h} = \frac{1964-1960}{10} = \frac{4}{10} = 0.4$$

∴ By using newton's forward interpolation formula,

$$f(x) = f(a) + m\Delta f(a) + \frac{m(m-1)}{2!}\Delta^2 f(a) + \frac{m(m-1)(m-2)}{3!}\Delta^3 f(a)$$

$$\therefore f(1964) = 10.2 + 0.4\times 0.2 + \frac{0.4\times 0.6}{2}\times 0 + \frac{0.4\times -0.6\times -1.6}{6}\times 0.1$$

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$\therefore f(1964) = 10.2864$ (approx.)

Q) Find the no. of persons getting wages less than rupees 25 from the following data

wages (₹)	0-20	20-40	40-60	60-80	80-100
no. of person	11	30	26	23	10

Soln- First we construct forward difference table

	x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
below	20	11				
			30			
"	40	41		-4		
			26		1	
	60	67		-3		-11
			23		-10	
	80	90		-13		
			10			
	100	100				

From above table,

$h = 20$, $a = 20$, $f(a) = 11$, $\Delta f(a) = 30$,
$\Delta^2 f(a) = -4$, $\Delta^3 f(a) = 1$, $\Delta^4 f(a) = -11$

$$m = \frac{x-a}{h} = \frac{25-20}{20} = 0.25$$

By using Newton's forward formula,

$$f(x) = f(a) + m\Delta f(a) + \frac{m(m-1)}{2!}\Delta^2 f(a) + \frac{m(m-1)(m-2)}{3!}\Delta^3 f(a) + \frac{m(m-1)(m-2)(m-3)}{4!}\Delta^4 f(a)$$

$$\therefore f(25) = 11 + 0.25 \times 30 + \frac{0.25 \times -0.75}{2} \times -4 + \frac{0.25 \times -0.75 \times -1.75}{3 \times 2} + \frac{0.25 \times -0.75 \times -1.75 \times -2.75}{4 \times 3 \times 2} \times -11$$

$\therefore f(25) = 19.3483$ (approx.)

Thus no. of person who have getting less than 25 ₹ wages is 19 (approx.)

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* Newton's backward interpolation formula

Let, $f(a-nh)$, $f(a-(n-1)h)$, ..., $f(a)$ be the values of $f(x)$ at equidistant points $a-nh$, $a-(n-1)h$, ..., a. Thus, by defn. of backward differences we have

$$\nabla f(a) = f(a) - f(a-h)$$

$$\Rightarrow f(a-h) = f(a) - \nabla f(a)$$

$$\Rightarrow f(a-h) = (I - \nabla) f(a) \quad \ldots (1)$$

Also

$$f(a-2h) = f(a-h) - \nabla f(a-h)$$

$$\Rightarrow f(a-2h) = (I-\nabla) f(a-h)$$

$$= (I-\nabla)^2 f(a) \quad \ldots \text{by eq}^n (1)$$

and so on.

$$f(a-nh) = (I-\nabla)^n . f(a)$$

and so on.

Now,

$$f(a-nh) = (I-\nabla)^n . f(a)$$

$$= \left[I - n\nabla + \frac{n(n-1)}{2!}\nabla^2 - \frac{n(n-1)(n-2)}{3!}\nabla^3 + \cdots \right] f(a)$$

$$= f(a) - n\nabla f(a) + \frac{n(n-1)}{2!}\nabla^2 f(a) - \frac{n(n-1)(n-2)}{3!}\nabla^3 f(a) + \cdots$$

This is called 'Newton's backward interpolation formula.

* Another form of Newton's backward interpolation formula-

Let $y=f(x)$ assume the values $f(a)$,

$f(a+h), f(a+2h), \dots, f(a+nh)$ at equidistant point $a, a+h, a+2h, \dots, a+nh$.

suppose we want to estimate,

$$x = (a+nh) + mh$$

$$\Rightarrow \quad m = \frac{x-(a+nh)}{h}$$

$$f(x) = f[(a+nh)+mh]$$

$$= f(a+nh) + m\nabla f(a+nh) + \frac{m(m+1)}{2!}\nabla^2 f(a+nh) + \frac{m(m+1)(m+2)}{3!}\nabla^3 f(a+nh) + \dots$$

1) Estimate $f(7)$ from the following data

x	:	2	4	6	8
$f(x)$	:	4	7	11	18

Soln. First we prepare backward difference table

x	$f(x)$	$\nabla f(x)$	$\nabla^2 f(x)$	$\nabla^3 f(x)$
2	4			
		3		
4	7		1	
		4		2
6	11		3	
		7		
8	18			

Type 1 -

Here, $a = 8$, $f(a) = 18$, $\nabla f(a) = 7$, $\nabla^2 f(a) = 3$, $\nabla^3 f(a) = 2$

$h = 2$

$a - nh = 7$

$\Rightarrow 8 - 2n = 7$

$\Rightarrow n = \frac{1}{2}$

By using Newton's backward interpol-ation formula,

$$f(a-nh) = f(a) - n\nabla f(a) + \frac{n(n-1)}{2!}\nabla^2 f(a) - \frac{n(n-1)(n-2)}{3!}\nabla^3 f(a) + \cdots$$

$$f(7) = 18 - (0.5)7 + \frac{0.5 \times -0.5}{2} 3 + \frac{0.5 \times -0.5 \times -1.5}{6} 2 + \cdots$$

$\therefore f(7) = 14$

OR

Type II -

From backward difference table,

$a + nh = 8$

$f(a+nh) = 18$

$\nabla f(a+nh) = 7$, $\nabla^2 f(a+nh) = 3$, $\nabla^3 f(a+nh) = 2$

$h = 2$, $x = 7$

$$m = \frac{x - (a+nh)}{h} = \frac{7-8}{2} = \frac{-1}{2} = -0.5$$

Thus, by using Newton's backward interpolation formula,

$$f(x) = f(a+nh) + m\nabla f(a+nh) + \frac{m(m+1)}{2!}\nabla^2 f(a+nh) + \frac{m(m+1)(m+2)}{3!}\nabla^3 f(a+nh) + \cdots$$

$$f(x) = 18 + (-0.5 \times 7) + \frac{-0.5 \times 0.5}{2} 3 + \frac{-0.5 \times 0.5 \times 1.5}{6} \times 2$$

$+ \cdots$

$f(7) = 14$

2) For the data

x	:	0.2	0.4	0.6	0.8	1.0
f	:	3.2	3.6	2.8	3	2.4

Generate backward difference table & hence using Newton's backward interpo-lation formula & interpolate $f(0.95)$.

Soln. First we construct backward difference table.

x	f	$\nabla f(x)$	$\nabla^2 f(x)$	$\nabla^3 f(x)$	$\nabla^4 f(x)$
0.2	3.2				
		0.4			
0.4	3.6		-1.2		
		-0.8		2.2	
0.6	2.8		+1.0		-4
		0.2		-1.8	
0.8	3		-0.8		
		-0.6			
1.0	2.4				

Here,

$a = 1$, $f(a) = 2.4$, $\nabla f(a) = -0.6$, $\nabla^2 f(a) = -0.8$

$\nabla^3 f(a) = -1.8$, $\nabla^4 f(a) = -4$

and $h = 0.2$

$a - nh = 0.95$

$1 - n(0.2) = 0.95$

$\therefore n = 0.25$

By using backward interpolation formula

$$\cancel{f(a-nh) = f(a) - n\nabla f(a) + \frac{n(n-1)}{2!}\nabla^2 f(a)}$$

$$- \frac{n(n-1)(n-2)}{3!}\nabla^3 f(a) + \frac{n(n-1)(n-2)(n-3)}{4!}\nabla^4 f(a)$$

$= 2.4 - (0.25)(-0.6) + \frac{0.25 \times -0.75 \times -0.8}{2}$

$- \frac{0.25 \times -0.75 \times -1.75}{6} (-1.8)$

$+ \frac{0.25 \times -0.75 \times -1.75 \times -2.75}{24} \times -4$

$f(0.95) = 2.8738$ (approx)

3)* Using Newton's backward interpolation formula, find $f(9)$ or $f'(9)$ if $f(0) = 0$, $f(2) = 3$, $f(4) = 8$, $f(6) = 15$, $f(8) = 24$, $f(10) = 35$

Soln – First we construct backward difference table:

x	$f(x)$	$\nabla f(x)$	$\nabla^2 f(x)$	$\nabla^3 f(x)$	$\nabla^4 f(x)$	$\nabla^5 f(x)$
0	0					
		3				
2	3		2			
		5		0		
4	8		2		0	
		7		0		2
6	15		2		-2	
		9		+0		
8	24		2			
		11				
10	35					

Here,

$a = 10$, $f(a) = 35$, $\nabla f(a) = 11$, $\nabla^2 f(a) = 2$,

$\nabla^3 f(a) = 0$

Consider, $a - nh = x$ where $h = 2$

$\Rightarrow n = \frac{10 - x}{2}$

By using Newton's backward

$80 - 10x - 8x + x^2$

interpolation formula,

$$f(a-nh) = f(a) - n\nabla f(a) + \frac{n(n-1)}{2!}\nabla^2 f(a) - \frac{n(n-1)(n-2)}{3!}\nabla^3 f(a)$$

$$f(x) = 85 - \left(\frac{10-x}{2}\right)(1) + \frac{\left(\frac{10-x}{2}\right)\left(\frac{8-x}{2}\right)(2)}{2!} - \frac{\left(\frac{10-x}{2}\right)\left(\frac{8-x}{2}\right)\left(\frac{6-x}{2}\right)(0)}{3!}$$

$$= 35 - \frac{(110-11x)}{2} + \frac{80-18x+x^2}{4}$$

$$4f(x) = \frac{140-(220-22x)+80-18x+x^2}{4}$$

$$4f(x) = x^2+4x$$

$$\Rightarrow f(x) = \frac{x^2+4x}{4} \quad \text{--- (i)}$$

from (i),

$$f'(x) = \frac{x}{2}+1 \quad \text{--- (ii)}$$

By eqn. (i),

$$f(9) = \frac{81+36}{4} = 29.25$$

By eqn. (ii),

$$f'(9) = \frac{9}{2}+1 = 5.5$$

4) construct a table of values of function $f(x) = x^3$ for $x = 1, 2, \dots, 10$. Use Newton's backward interpolation formula to find $f(7.5)$, $f'(7.5)$.

soln- First we construct backward interpola-tion formula difference table,

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x	$f(x)$	$\nabla f(x)$	$\nabla^2 f(x)$	$\nabla^3 f(x)$
1	1			
		7		
2	8		12	
		19		6
3	27		18	
		37		6
4	64		24	
		61		6
5	125		30	
		91		6
6	216		36	
		127		6
7	343		42	
		169		6
8	512		48	
		217		6
9	729		54	
		271		
10	1000			

From backward difference table,

$a = 10$, $f(a) = 1000$, $\nabla f(a) = 271$, $\nabla^2 f(a) = 54$

$\nabla^3 f(a) = 6$

$a - nh = 7.5 \Rightarrow 10 - n \times 1 = 7.5 \Rightarrow n = 2.5$

By using Newton's backward formula,

$$f(a-nh) = f(a) - n\nabla f(a) + \frac{n(n-1)}{2!}\nabla^2 f(a) - \frac{n(n-1)(n-2)}{3!}\nabla^3 f(a)$$

$$\therefore f(7.5) = 1000 - 2.5 \times 271 + \frac{2.5 \times 1.5}{2} \times 54 - \frac{2.5 \times 1.5 \times 0.5}{6} \times 6$$

$\therefore f(7.5) = 421.875$

Now, we have

$f(x) = x^3$

$\therefore f'(x) = 3x^2$

$\therefore f'(7.5) = 3 \times (7.5)^2$

$\therefore f'(7.5) = 168.75$

5) From the following data

Temp. °C	140	150	160	170
pressure kgf/cm²	3.685	4.854	6.302	8.076

Using Newton's backward formula find the pressure for temp. 142 °C.

Son Step I) First we construct backward difference table,

Here, we suppose temp. in °C is x & $f(x)$ = pressure in kgf/cm².

x	$f(x)$	$\nabla f(x)$	$\nabla^2 f(x)$	$\nabla^3 f(x)$
140	3.685			
		1.169		
150	4.854		0.279	
		1.448		0.047
160	6.302		0.326	
		1.774		
170	8.076			

Step II) - calculation

Here, $a = 170$, $f(a) = 8.076$, $\nabla f(a) = 1.774$

$\nabla^2 f(a) = 0.326$, $\nabla^3 f(a) = 0.047$

By using Newton's backward formula,

$a - nh = 142$

$170 - n \times 10 = 142$

$\therefore n = 2.8$

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$$f(a-nh) = f(a) - n\nabla f(a) + \frac{n(n-1)}{2!}\nabla^2 f(a) - \frac{n(n-1)(n-2)}{3!}\nabla^3 f(a)$$

$$f(142) = 8.076 - (2.8\times 1.774) + \frac{2.8\times 1.8}{2}\times 0.326 - \frac{2.8\times 1.8\times 0.8}{6}\times 0.047$$

$\therefore$ $f(142) = 3.8987$ (approx)

If temp is $142\,^\circ C$ then pressure is 3.8987 kgf/cm².

6) Using the method of seperations of symbols S.T. $\Delta^n u_{x-n} = u_x - n u_{x-1} + \frac{n(n-1)}{2} u_{x-2} + \cdots + (-1)^n u_{x-n}$

soln- consider,

$f(x+h)$

$$R.H.S = u_x - n u_{x-1} + \frac{n(n-1)}{2} u_{x-2} + \cdots + (-1)^n u_{x-n}$$

$f(x) = f(x-h)$

$$= u_x - n E^{-1} u_x + \frac{n(n-1)}{2} E^{-2} u_x + \cdots + (-1)^n E^{-n} u_x$$

$$= \left[1 - nE^{-1} + \frac{n(n-1)}{2}E^{-2} + \cdots + (-1)^n E^{-n}\right] u_x$$

$$= (1-E^{-1})^n u_x$$

$$= \left(1-\frac{1}{E}\right)^n u_x$$

$$= \left(\frac{E-1}{E}\right)^n u_x$$

$$= \left(\frac{\Delta^n}{E^n}\right) u_x$$

$$= \Delta^n E^{-n} u_x$$

$$= \Delta^n u_{x-n}$$

$$= LHS$$

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7) Show that,

$$e^x\left(u_0 + x\Delta u_0 + \frac{x^2}{2!}\Delta^2 u_0 + \cdots\right) = u_0 + u_1 x + u_2\frac{x^2}{2!} + \cdots$$

soln. consider,

$$L.H.S = e^x\left(u_0 + x\Delta u_0 + \frac{x^2}{2!}\Delta^2 u_0 + \cdots\right)$$

$$= e^x\left[1 + x\Delta + \frac{x^2\Delta^2}{2!} + \cdots\right]u_0$$

$$= e^x\left[\left(e^{x\Delta}\right)u_0\right.$$

$$= e^{x + x\Delta} \cdot u_0$$

$$= e^{x(1+\Delta)} \cdot u_0$$

$$= e^{xE}\, u_0$$

$$= \left[1 + x.E + \frac{x^2E^2}{2!} + \cdots\right]u_0$$

$$= u_0 + xEu_0 + \frac{x^2}{2!}E^2u_0 + \cdots$$

$$= u_0 + xu_1 + \frac{x^2}{2!}u_2 + \cdots$$

$$= R.H.S$$

UNIT - II
INTERPOLATION : UNEQUAL INTERVALS

* **Lagrange's Interpolation Formula** -

This formula is useful when the values of x are not equally spaced.

Suppose that the values of a function $f(x)$ are known for $n+1$ values of x say $x_0, x_1, x_2, \dots, x_n$ (Not necessarlily equidistant).

Then we assume $f(x)$ to be a polynomial of degree n in x and ~~from~~ ~~th~~ for this polynomial.

In Lagrange's method we approximate the function $f(x)$ by a polynomial. By writting $f(x)$ in special form the method to determine $f(x)$.

Case I) when $n=3$

Let the values of x be x_0, x_1, x_2. Then the corresponding values of $f(x_0)$, $f(x_1)$, $f(x_2)$. We write

$$f(x) = A_0(x-x_1)(x-x_2) + A_1(x-x_0)(x-x_2) + A_2(x-x_0)(x-x_1) \quad \dots (1)$$

In eqn. (1) put $x = x_0$ we get

$$f(x_0) = A_0(x_0-x_1)(x_0-x_2)$$

$$\Rightarrow A_0 = \frac{f(x_0)}{(x_0-x_1)(x_0-x_2)}$$

Similarly, we find

$$A_1 = \frac{f(x_1)}{(x_1-x_0)(x_1-x_2)} \quad \& \quad A_2 = \frac{f(x_2)}{(x_2-x_0)(x_2-x_1)}$$

Thus, eqn. (1) becomes,

$$f(x) = f(x_0)\frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} + f(x_1)\frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)}$$
$$+ f(x_2)\frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}$$

$$f(x) = f(x_0).l_0(x) + f(x_1).l_1(x) + f(x_2).l_2(x)$$

This is called 'Lagrange's interpolating polynomial'.

case II) when $n=4$

Let th- be values of x x_0, x_1, x_2, x_3 the corresponding values of $f(x)$ be $f(x_0)$, $f(x_1)$, $f(x_2)$, $f(x_3)$. We write

$$f(x) = A_0(x-x_1)(x-x_2)(x-x_3) + A_1(x-x_0)(x-x_2)(x-x_3) + A_2(x-x_0)(x-x_1)(x-x_3)$$
$$+ A_3(x-x_0)(x-x_1)(x-x_2) \quad \text{--- (ii)}$$

put $x = x_0$ in eqn. (ii), we get

$$f(x_0) = A_0(x_0-x_1)(x_0-x_2)(x_0-x_3)$$

$$\Rightarrow A_0 = \frac{f(x_0)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}$$

similarly, we find,

$$A_1 = \frac{f(x_1)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} \;\&\; A_2 = \frac{f(x_2)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}$$

$$\& \; A_3 = \frac{f(x_3)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}$$

Thus, eqn. (ii) becomes,

$$f(x) = f(x_0)\frac{(x_0-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}$$
$$+ f(x_1)\frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} + f(x_2)\frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}$$

$$f(x) = f(x_0)l_0(x) + f(x_1)l_1(x) + f(x_2)l_2(x) + f(x_3)l_3(x)$$

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which is Lagrange's interpolating polynomial

In general,

$$f(x) = \sum_{i=0}^{n} \cancel{f(x_i) \cdot l_i(x)}$$

which is Lagrange's interpolating polynomial of degree n.

Ex 1) Find interpolating polynomial in Lagrange's form for the data,

$x = -2, -1, 1, 3$

$f = -15, -4, 0, 20$

Hence, interpolate $f(0)$.

Soln- We note that,

$x_0 = -2, \quad x_1 = -1, \quad x_2 = 1, \quad x_3 = 3$

$\Rightarrow f(x_0) = -15, \quad f(x_1) = -4, \quad f(x_2) = 0, \quad f(x_3) = 20$

We have

$$f(x) = A_0 (x - x_1)(x - x_2)(x - x_3) + A_1 (x - x_0)(x - x_2)(x - x_3) + A_2 (x - x_0)(x - x_1)(x - x_3) + A_3 (x - x_0)(x - x_1)(x - x_2) \quad \ldots (1)$$

Here,

$$A_0 = \frac{f(x_0)}{(x_0 - x_1)(x_0 - x_2)(x_0 - x_3)} = \frac{-15}{-15} = 1$$

II'ly,

$$A_1 = \frac{f(x_1)}{(x_1 - x_0)(x_1 - x_2)(x_1 - x_3)} = \frac{-4}{-2 \times -4} = -1/2$$

$$A_2 = \frac{f(x_2)}{(x_2 - x_0)(x_2 - x_1)(x_2 - x_3)} = \frac{0}{3 \times 2 \times -2} = 0$$

$$A_3 = \frac{f(x_3)}{(x_3 - x_0)(x_3 - x_1)(x_3 - x_2)} = \frac{20}{5 \times 4 \times 2} = \frac{1}{2}$$

Thus eqn. (1) becomes,

$f(x) = (x+1)(x-1)(x-3) - \frac{1}{2}(x+2)(x-1)(x-3)$
$+\frac{1}{2}(x+2)(x+1)(x-1)$
which is required interpolating polynomial

$\therefore f(0) = (0+1)(0-1)(0-3) - \frac{1}{2}(2\times -1\times -3)$
$+\frac{1}{2}(2\times 1\times -1)$

$\therefore f(0) = -1$.

2) Find the interpolating polynomial in Lagrange's form for the given data & hence interpolate $f'(-1)$

$x = -2, 0, 1, 3$

$f = 7, 3, 1, 27$

Ans- We note that,

$x_0 = -2, \; x_1 = 0, \; x_2 = 1, \; x_3 = 3$

$\Rightarrow f(x_0) = 7, \; f(x_1) = 3, \; f(x_2) = 1, \; f(x_3) = 27$

We have,

$f(x) = A_0(x-x_1)(x-x_2)(x-x_3) + A_1(x-x_0)(x-x_2)$
$(x-x_3) + A_2(x-x_0)(x-x_1)(x-x_3)$
$+ A_3(x-x_0)(x-x_1)(x-x_2)$ --- ①

Here,

$$A_0 = \frac{f(x_0)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} = \frac{7}{-2\times -3\times -5} = \frac{-7}{30}$$

$$A_1 = \frac{f(x_1)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} = \frac{3}{2\times -1\times -3} = \frac{1}{2}$$

$$A_2 = \frac{f(x_2)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} = \frac{1}{3\times 1\times -2} = \frac{-1}{6}$$

$$A_3 = \frac{f(x_3)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} = \frac{27}{5\times 3\times 2} = \frac{\cancel{27}^{9}}{\cancel{30}_{10}} = \frac{9}{10}$$

Thus eqn. ① becomes,

Find the Lagrangian form of interpolating polynomial of the data
x : -1, 2, 4, 5 hence interpolate $f(0)$.
: 5, 3, 8, 5

$$f(x) = \frac{-7}{30}(x)(x-1)(x-3) + \frac{1}{2}(x+2)(x-1)(x-3)$$
$$-\frac{1}{6}(x+2)(x)(x-3) + \frac{9}{10}(x+2)(x)(x-1)$$

$$f(x) = \frac{-7}{30}(x^3-4x^2+3x) + \frac{1}{2}(x+2)\left[x^2-3x-x+3\right]$$
$$-\frac{1}{6}(x+2)\left[x^2-3x+2x-6\right] + \frac{9}{10}x\left[x^2-x+2x-2\right]$$

$$f(x) = \frac{-7}{30}(x^3-4x^2+3x) + \frac{1}{2}(x^3-2x^2-5x+6)$$
$$-\frac{1}{6}(x^3-x^2-6x) + \frac{9}{10}(x^3+x^2-2x)$$

$$f(x) = x^3\left(\frac{-7}{30} + \frac{1}{2} - \frac{1}{6} + \frac{9}{10}\right) + x^2\left(\frac{28}{30} - 1 + \frac{1}{6} + \frac{9}{10}\right)$$
$$+ x\left(\frac{-7}{10} - \frac{5}{2} + 1 - \frac{9}{5}\right) + 3$$

$$f(x) = x^3 + x^2 - 4x + 3$$
$$\therefore \quad f'(x) = 3x^2 + 2x - 4$$

Now
$$f'(-1) = 3 - 2 - 4$$
$$= -3$$

3) Find the Lagrangian interpolating polynomi[al] for the data

x = 2, -1, 4

$f(x)$ = 6, 3, 5

Hence interpolate $f(1)$ & $f'(1)$.

Ans: We note that,

$x_0 = 2$, $x_1 = -1$, $x_2 = 4$

$\Rightarrow$ $f(x_0) = 6$, $f(x_1) = 3$, $f(x_2) = 5$

We have,

$$f(x) = A_0(x-x_1)(x-x_2) + A_1(x-x_0)(x-x_2) + A_2(x-x_0)(x-x_1) \quad \cdots (1)$$

Here,

$$A_0 = \frac{f(x_0)}{(x_0-x_1)(x_0-x_2)} = \frac{6}{3\times -2} = \frac{6}{-6} = -1$$

$$A_1 = \frac{f(x_1)}{(x_1-x_0)(x_1-x_2)} = \frac{3}{-3\times -5} = \frac{3}{15} = \frac{1}{5}$$

$$A_2 = \frac{f(x_2)}{(x_2-x_0)(x_2-x_1)} = \frac{5}{2\times 5} = \frac{5}{10} = \frac{1}{2}$$

Thus eqn. (1) becomes,

$$f(x) = (-1)(x+1)(x-4) + \tfrac{1}{5}(x-2)(x-4) + \tfrac{1}{2}(x-2)(x+1)$$

$$f(x) = (-x^2+3x+4) + \tfrac{1}{5}(x^2-6x+8) + \tfrac{1}{2}(x^2-x-2)$$

$$= x^2\left(-1+\tfrac{1}{5}+\tfrac{1}{2}\right) + x\left(3-\tfrac{6}{5}-\tfrac{1}{2}\right) + \left(4+\tfrac{8}{5}-1\right)$$

$$f(x) = x^2\left(\frac{-3}{10}\right) + x\left(\frac{13}{10}\right) + \frac{23}{5}$$

$$\therefore f'(x) = \frac{-3}{10}(2x) + \frac{13}{10}$$

Now,

$$f'(1) = \frac{-6}{10} + \frac{13}{10} = \frac{7}{10}$$

4) Find the Lagrangian form of interpolating polynomial of the da

$x = -1, 2, 4, 5$

$f(x) = 5, 3, 8, 5$

hence interpolate $f(0)$.

Ans- We note that,

$x_0 = -1,\quad x_1 = 2,\quad x_2 = 4,\quad x_3 = 5$

$\Rightarrow f(x_0) = 5,\quad f(x_1) = 3,\quad f(x_2) = 8,\quad f(x_3) = 5$

We have,

$$f(x) = A_0(x-x_1)(x-x_2)(x-x_3) + A_1(x-x_0)(x-x_2)(x-x_3) + A_2(x-x_0)(x-x_1)(x-x_3) + A_3(x-x_0)(x-x_1)(x-x_2) \quad \text{.(1)}$$

Here,

$$A_0 = \frac{f(x_0)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} = \frac{5}{-3\times-5\times-6} = \frac{-1}{18}$$

$$A_1 = \frac{f(x_1)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} = \frac{3}{3\times-2\times-3} = \frac{1}{6}$$

$$A_2 = \frac{f(x_2)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} = \frac{8}{5\times2\times-1} = \frac{-4}{5}$$

and

$$A_3 = \frac{f(x_3)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} = \frac{5}{6\times3\times1} = \frac{5}{18}$$

Thus eqn. (1) becomes,

$$f(x) = \frac{-1}{18}(x-2)(x-4)(x-5) + \frac{1}{6}(x+1)(x-4)(x-5) - \frac{4}{5}(x+1)(x-2)(x-5) + \frac{5}{18}(x+1)(x-2)(x-4)$$

which is required interpolating polynomial.

$$\therefore f(0) = \frac{-1}{18}(-2\times-4\times-5) + \frac{1}{6}(1\times-4\times-5) - \frac{4}{5}(1\times-2\times-5) + \frac{5}{18}(1\times-2\times-4)$$

$$\therefore f(0) = \frac{-2}{9}$$

5) Find Lagrange's interpolating polynomial for the following data,

x	1	2	3	4
$f(x)$	1	4	9	16

Find $f'(2.5)$.

Ans- We note that,

$x_0 = 1, \quad x_1 = 2, \quad x_2 = 3, \quad x_3 = 4$

$\Rightarrow f(x_0) = 1, \quad f(x_1) = 4, \quad f(x_2) = 9, \quad f(x_3) = 16$

We have,

$$f(x) = A_0(x-x_1)(x-x_2)(x-x_3) + A_1(x-x_0)(x-x_2)(x-x_3) + A_2(x-x_0)(x-x_1)(x-x_3) + A_3(x-x_0)(x-x_1)(x-x_2) \quad \cdots (1)$$

Here,

$$A_0 = \frac{f(x_0)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} = \frac{1}{-1\times-2\times-3} = \frac{-1}{6}$$

$$A_1 = \frac{f(x_1)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} = \frac{4}{1\times-1\times-2} = 2$$

$$A_2 = \frac{f(x_2)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} = \frac{9}{2\times1\times-1} = \frac{-9}{2}$$

$$A_3 = \frac{f(x_3)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} = \frac{16}{3\times2\times1} = \frac{8}{3}$$

Thus eqn. (1) becomes,

$$f(x) = -\frac{1}{6}(x-2)(x-3)(x-4) + 2(x-1)(x-3)(x-4) - \frac{9}{2}(x-1)(x-2)(x-4) + \frac{8}{3}(x-1)(x-2)(x-3)$$

$$= \left\{-\frac{1}{6}(x^2-5x+6) + 2(x^2-4x+3) - \frac{9}{2}(x^2-3x+2)\right\}(x-4) + \frac{8}{3}(x^2-3x+2)(x-3)$$

$$= \left\{x^2\left(-\frac{1}{6}+2-\frac{9}{2}\right) + x\left(\frac{5}{6}-8+\frac{27}{2}\right) - 4\right\}(x-4) + \frac{8}{3}(x^3-3x^2+2x-3x^2+9x-6)$$

$$= \left\{x^2\left(\frac{-1}{6}+2-\frac{9}{2}\right)+x\left(\frac{5}{6}-8+\frac{27}{2}\right)-4\right\}(x-4) + \frac{8}{3}(x^3-6x^2+11x-6)$$

$$= \left(\frac{-8}{3}x^2+\frac{19}{3}x-4\right)(x-4)+\frac{8}{3}(x^3-6x^2+11x-6)$$

$$= \frac{-8}{3}x^3+17x^2+\frac{55}{3}x+16+\frac{8}{3}(x^3-6x^2+11x-6)$$

$f(x) = x^2$

$\therefore \quad f'(x) = 2x$

Now

$f'(2.5) = 2 \times 2.5 = 5.$

* Newton's Divided Differences.

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Let, $y = f(x)$ be a function which takes values $f_0, f_1, f_2, \dots, f_n$ with corre-sponding values of x i.e $x_0, x_1, x_2, \dots, x_n$ then for the points $(x_0, f_0), (x_1, f_1), (x_2, f_2), \dots, (x_n, f_n)$, the divided differ-nces is the defined as follows-

I) Newton's Divided Difference of zero 0^{th} order

We define,

$$f[x_i] = f_i$$

II) Newton's Divided Difference of first order.

For the values of x as x_0, x_1 the first divided difference is defined by reln.

$$[x_0, x_1] = f[x_0, x_1] = \frac{f_1 - f_0}{x_1 - x_0}$$

Similarly, for next values of x as

$$[x_1, x_2] = f[x_1, x_2] = \frac{f_2 - f_1}{x_2 - x_1}$$

In general,

$$[x_i, x_{i+1}] = f[x_i, x_{i+1}] = \frac{f_{i+1} - f_i}{x_{i+1} - x_i}$$

III) Newton's divided difference of 2^{nd} order -

For the values of x as x_0, x_1, x_2 the second divided difference is defined by relation,

$$[x_0, x_1, x_2] = f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0}$$

On same line for the next values of x as, x_1, x_2, x_3

$$[x_1, x_2, x_3] = f[x_1, x_2, x_3] = \frac{f[x_2, x_3] - f[x_1, x_2]}{x_3 - x_1}$$

In general, ~~x_i~~

$$[x_i, x_{i+1}, x_{i+2}] = f[x_i, x_{i+1}, x_{i+2}] = \frac{f[x_{i+1}, x_{i+2}] - f[x_i, x_{i+1}]}{x_{i+2} - x_i}$$

IV) Newton's divided difference of 3^{rd} order -

For the values of x as x_0, x_1, x_2, x_3 the third order divided ~~defined~~ difference is ~~divide~~ defined by relation,

$$f[x_0, x_1, x_2, x_3] = \frac{f[x_1, x_2, x_3] - f[x_0, x_1, x_2]}{x_3 - x_0}$$

In general,

$$f[x_i, x_{i+1}, x_{i+2}, x_{i+3}] = \frac{f[x_{i+1}, x_{i+2}, x_{i+3}] - f[x_i, x_{i+1}, x_{i+2}]}{x_{i+3} - x_i}$$

Note -

n^{th} divided difference is defined as,

$$f[x_0, x_1, x_2, \ldots, x_n] = \frac{f[x_1, x_2, \ldots x_n] - f[x_0, x_1, \ldots, x_{n-1}]}{x_n - x_0}$$

* Newton's Divided Difference Table

x	$f(x)$	1st order	2nd order	3rd order
x_0	f_0			
		$f[x_0,x_1] = \frac{f_1 - f_0}{x_1 - x_0}$		
			$f[x_0,x_1,x_2] = \frac{f[x_1,x_2] - f[x_0,x_1]}{x_2 - x_0}$	
x_1	f_1			$f[x_0,x_1,x_2,x_3] = \frac{f[x_1,x_2,x_3] - f[x_0,x_1,x_2]}{x_3 - x_0}$
		$f[x_1,x_2] = \frac{f_2 - f_1}{x_2 - x_1}$		
			$f[x_1,x_2,x_3] = \frac{f[x_2,x_3] - f[x_1,x_2]}{x_3 - x_1}$	
x_2	f_2			
		$f[x_2,x_3] = \frac{f_3 - f_2}{x_3 - x_2}$		
x_3	f_3			

Note -

i) We know that,

$$f[x_i, x_{i+1}] = \frac{f_{i+1} - f_i}{x_{i+1} - x_i}$$
$$= \frac{-[f_i - f_{i+1}]}{-[x_i - x_{i+1}]}$$
$$= \frac{f_i - f_{i+1}}{x_i - x_{i+1}}$$
$$= f[x_{i+1}, x_i]$$

ii) The divided difference are symmetrical in all their arguements i.e the value of any divided difference is independent of the order of arguments.

* Newton's Divided Difference Formula -

2mP/2M

Let, $y = f(x)$ be any function with values of x are $x_0, x_1, x_2, \cdots, x_n$ and corresponding values of $f(x)$ be $y_0, y_1, y_2, \cdots, y_n$

We estimate y at x.

By defn. of first divided difference we have,

$$f[x, x_0] = \frac{y - y_0}{x - x_0}$$

$$\Rightarrow y = (x - x_0) f[x, x_0] + y_0$$

$$\Rightarrow y = y_0 + (x - x_0) f[x, x_0] \quad \cdots (i)$$

Again, by defn. of second divided difference we have,

$$f[x, x_0, x_1] = \frac{f[x, x_0] - f[x_0, x_1]}{x - x_1}$$

$$\Rightarrow f[x, x_0] = f[x_0, x_1] + (x - x_1) f[x, x_0, x_1] \quad (ii)$$

By (i) and (ii), we get

$$y = y_0 + (x - x_0)\{f[x_0, x_1] + (x - x_1) f[x, x_0, x_1]\}$$

$$= y_0 + (x-x_0) f[x_0, x_1] + (x-x_0)(x-x_1) \cdot f[x, x_0, x_1] \quad \text{(iii)}$$

Again, by using defn. of 3rd divided difference,

we have,

$$f[x, x_0, x_1, x_2] = \frac{f[x, x_0, x_1] - f[x_0, x_1, x_2]}{x - x_2}$$

$$\Rightarrow f[x, x_0, x_1] = (x-x_2) f[x, x_0, x_1, x_2] + f[x_0, x_1, x_2] \quad \text{(iv)}$$

By (iii) and (iv)

$$\therefore y = y_0 + (x-x_0) f[x_0, x_1] + (x-x_0)(x-x_1) \cdot \{(x-x_2) f[x, x_0, x_1, x_2] + f[x_0, x_1, x_2]\}$$

$$\Rightarrow y = y_0 + (x-x_0) f[x_0, x_1] + (x-x_0)(x-x_1) \cdot f[x_0, x_1, x_2] + (x-x_0)(x-x_1)(x-x_2) \cdot f[x, x_0, x_1, x_2]$$

proceeding in this way we obtain,

$$y = y_0 + (x-x_0) f[x_0, x_1] + (x-x_0)(x-x_1) \cdot f[x_0, x_1, x_2] + (x-x_0)(x-x_1)(x-x_2) \cdot f[x_0 x_1, x_2, x_3] + \cdots + (x-x_0)(x-x_1)\cdots(x-x_n) f[x, x_0, x_1, \ldots, x_n]$$

This formula is called 'Newton's Divided Difference Formula'. with divided difference and the last term be the remainder after $(n+1)$ steps.

Ex. prepare divided difference table for the following data and Interpolate $f(3)$

x : -3 $\quad -2$ $\quad -1$ $\quad 0$ $\quad 2$

$f(x)$	-239	-29	1	1	31

Soln. First we prepare divided difference table

x	$f(x)$	1st order divided diff.	2nd order divided diff.	3rd	4th
-3	-239				
		210			
-2	-29		-90		
		30		25	
-1	1		-15		-4
		0		5	
0	1		5		
		15			
2	31				

From divided diff. table, we have

$f[x_0, x_1] = 210$, $f[x_0, x_1, x_2] = -90$,

$f[x_0, x_1, x_2, x_3] = 25$, $f[x_0, x_1, x_2, x_3, x_4] = -4$

$y_0 = -239$, $x = 3$.

Thus, by using Newton's divided diff. formula, we have,

$$f(x) = y_0 + (x-x_0) f[x_0, x_1] + (x-x_0)(x-x_1) f[x_0, x_1, x_2] + (x-x_0)(x-x_1)(x-x_2) \cdot f[x_0, x_1, x_2, x_3] + (x-x_0)(x-x_1)(x-x_2)(x-x_3) \cdot f[x_0, x_1, x_2, x_3, x_4] + \cdots$$

$$\Rightarrow f(3) = -239 + 6\times210 + 6\times5\times-90 + 6\times5\times4\times25 + 6\times5\times4\times3\times-4$$

$$= -119$$

2) Find $f'(x)$ at $x = 3.5$ by using Newton's divided diff. formula for the foll. data

x	0	1	2	3
$f(x)$	0	1	8	27

Soln- First we prepare Newton's divided diff. table

x	$f(x)$	1st	2nd
0	0		
1	1	1	3
		$\cancel{3}$ 7	
2	8		6
		19	
3	27		

From divided diff table,

x^3 $x_0 = 0,\ x_1 = \cancel{2}1,\ x_2 = \cancel{3}2,\ x_3 = 3$

$y_0 = 0,\ y_1 =$

$f[x_0, x_1] = 1,\quad f[x_0, x_1, x_2] = 7\,5$

By using Newton's divided diff. formula

$y = y_0 + (x - x_0) f[x_0, x_1] + (x - x_0)(x - x_1)$
$f[x_0, x_1, x_2]$

$= 1 -$

2) Using follo. table find $f(x)$ as a polynomial in x

x	:	-1	0	3	6	7
$f(x)$		3	-6	39	822	1611

$x^4 - 5x^3 + 5x^2 - 6$

find $f'(2)$

Soln- First we prepare Newton's divided diff. table.

From divided diff. table

$x_0 = -1,\ x_1 = 0,\ x_2 = 3,\ x_3 = 6,\ x_4 = 7$

$y_0 = 3$

$f[x_0, x_1] = -9,\ f[x_0, x_1, x_2] = 6,$

$f[x_0, x_1, x_2, x_3] = 5,\ f[x_0, x_1, x_2, x_3, x_4] = 1$

x	f(x)	1st	2nd	3rd	4th
-1	3				
		-9			
0	-6		6		
		15		5	
3	39		41		1
		261		13	
6	822		132		
		789			
7	1611				

By using Newton's divided diff. formula

$$f(x)\; y = y_0 + (x-x_0) f[x_0,x_1] + (x-x_0)(x-x_1) f[x_0,x_1,x_2] + (x-x_0)(x-x_1)(x-x_2) f[x_0,x_1,x_2,x_3] + (x-x_0)(x-x_1)(x-x_2)(x-x_3) f[x_0,x_1,x_2,x_3,x_4]$$

$$= 3 + (x+1)(-9) + (x+1)(x)(6) + (x+1)(x)(x-3)(5) + (x+1)(x)(x-3)(x-6)$$

$$= 3 - 9(x+1) + 6(x^2+x) + 5(x^3-2x^2-3x) + (x^4-8x^3+9x^2+18x)$$

$$= x^4 + (-8+5)x^3 + (6-10+9)x^2 + (-9-15+18+6)x - 9 + 3$$

$$\therefore f(x) = x^4 - 3x^3 + 5x^2 - 6$$

Now,

$$f'(x) = 4x^3 - 9x^2 + 10x$$

$$\therefore f'(2) = 16$$

3) Find $f'(x)$ at $x = 3.5$ by using Newton's divided diff. formula for the foll. data.

x	0	1	2	3
$f(x)$	0	1	8	27

soln. First we prepare Newton's divided diff. table.

x	$f(x)$	1st	2nd	3rd
0	0			
		1		
			3	
1	1			1
		7		
			6	
2	8			
		19		
3	27			

From divided diff. table,

$x_0 = 0$, $x_1 = 1$, $x_2 = 2$, $x_3 = 3$, $y_0 = 0$

$f[x_0, x_1] = 1$, $f[x_0, x_1, x_2] = 3$,

$f[x_0, x_1, x_2, x_3] = 1$

$\therefore$ By using Newton's divided diff. formula

$$f(x) = y_0 + (x - x_0) f[x_0, x_1] + (x - x_0)(x - x_1) f[x_0, x_1, x_2] + (x - x_0)(x - x_1)(x - x_2) f[x_0, x_1, x_2, x_3]$$

$$= 0 + (x-0).1 + (x)(x-1)(3) + (x)(x-1)(x-2).1$$

$$= x + 3(x^2 - x) + (x^3 - x^2 - 2x^2 + 2x)$$

$$= x^3 + (3-3)x^2 + (1-3+2)x$$

$\therefore f(x) = x^3$

Now

$f'(x) = 3x^2$

$\therefore f'(3.5) = 3 \times (3.5)^2 = 36.75$

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4) Find $y(1.3)$ using Newton's divided diff. formula. Given

x :	0	2	3	4	6
y	1	13	34	73	229

soln. First we construct divided diff. table

x	y	1st	2nd	3rd	4th
0	1				
		6			
2	13		5		
		21		1	
3	34		9		0
		39		1	
4	73		13		
		78			
6	229				

From divided diff. table

$x_0 = 0$, $x_1 = 2$, $x_2 = 3$, $x_3 = 4$, $x_4 = 6$, $y_0 = 1$

$f[x_0, x_1] = 6$, $f[x_0, x_1, x_2] = 5$, $f[x_0, x_1, x_2, x_3] = 1$

$\therefore$ By using Newton's divided diff. formula

$$y = y_0 + (x - x_0).f[x_0, x_1] + (x - x_0)(x - x_1) f[x_0, x_1, x_2]$$
$$+ (x - x_0)(x - x_1)(x - x_2) f[x_0, x_1, x_2, x_3]$$

$$\therefore y(1.3) = 1 + (1.3).(6) + (1.3)(1.3 - 2)(5)$$
$$+ (1.3)(1.3 - 2)(1.3 - 3)$$

$$= 5.797 \text{ (approx.)}$$

5) From a given tabulated values of x and $y = \cosh x$ Find by any suitable method the y at $x = 0.27$. Also compair the numerical value with actual value.

x	0.1	0.2	0.3	0.4
y	1.005	1.020	1.015	1.081

soln- First we construct divided diff. table,

Step I)

x	y	1st	2nd	3rd
0.1	1.005			
		0.15		
0.2	1.020		-1	
		-0.05		15.1667
0.3	1.015		3.55	
		0.66		
0.4	1.081			

From divided diff. table,

$x_0 = 0.1$, $x_1 = 0.2$, $x_2 = 0.3$, $x_4 = 0.4$, $y_0 = 1.005$

$f[x_0, x_1] = 0.15$, $f[x_0, x_1, x_2] = -1$,

$f[x_0, x_1, x_2, x_3] = 15.1667$

$\therefore$ By using Newton's divided diff. formula,

$$y = y_0 + (x - x_0) f[x_0, x_1] + (x - x_0)(x - x_1) f[x_0, x_1, x_2] + (x - x_0)(x - x_1)(x - x_2) f[x_0, x_1, x_2, x_3]$$

$$y(0.27) = 1.005 + (0.27 - 0.1)(0.15) + (0.27 - 0.1)(0.27 - 0.2)(-1) + (0.27 - 0.1)(0.27 - 0.2)(0.27 - 0.3)(15.1667)$$

$$= 1.0132 \text{ (approx)}$$

i.e $\cosh(0.27) = 1.0132$ ... (I)

and ~~actual~~ accurate value,

$\cosh(0.27) = 1.03678$ ~~(app~~ (... (II)

$\therefore$ by (I) & (II)

Error $= 0.0235$

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6) Derive Newton's divided diff. formula and use it to find f(2) if

x	1	4	6
$f(x)$	0	1.386294	1.791759

Soln. First we construct divided diff. table

x	$f(x)$	1st	2nd
1	0		
		0.462098	
4	1.386294		-0.05187
		0.202733	
6	1.791759		

From divided diff. table,

$x_0 = 1$, $x_1 = 4$, $x_2 = 6$, $y_0 = 0$

$f[x_0, x_1] = 0.462098$, $f[x_0, x_1, x_2] = -0.05187$

$\therefore$ By using Newton's divided diff. formula

$$f(x) = y_0 + (x - x_0) f[x_0, x_1] + (x - x_0)(x - x_1) f[x_0, x_1, x_2]$$

$$f(2) = 0 + (1)(0.462098) + (-2)(-0.05187)$$

$\therefore f(2) \simeq 0.5658442$ (approx)

7) ~~Find~~ using Newton's divided diff. formula find $\log_{10}(301)$ given that,

i.e. f(301)

x	300	304	305	307
y	2.4771	2.4829	2.4843	2.4871

Soln. First we prepare divided difference table.

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x	y	1st	2nd	3rd
300	2.4771	0.00145		
		0.0232	0	
			-0.00436	0
304	2.4829			0.0006229
		0.0014		
			0	
305	2.4843			
		0.0014		
307	2.4871			

From divided diff. table,

$x_0 = 300$, $x_1 = 304$, $x_2 = 305$, $x_3 = 307$, $y_0 = 2.4771$

$f[x_0, x_1] = 0.232$, $f[x_0, x_1, x_2] = -0.00436$,

$f[x_0, x_1, x_2, x_3] = 0.0006229$

$\therefore$ By using Newton's divided diff. formula

$$y = y_0 + (x - x_0) f[x_0, x_1] + (x - x_0)(x - x_1) f[x_0, x_1, x_2] + (x - x_0)(x - x_1)(x - x_2) f[x_0, x_1, x_2, x_3]$$

$$\log_{10}(301) = 2.4771 + 0.232 + (-3)(-0.00436) + (-3)(-5)(0.0006229)$$

$$= 2.732 \quad \text{(approx)}.$$

* Lemma -

With usual notations show that,

$$f[x_0, x_1, \ldots, x_K] = \frac{\Delta^K f_0}{K!\, h^K}.$$

proof - Here, operator Δ is used on one side.

Hence, it is harmless to consider values of x are equally spaced.

i.e $x_0, x_1, \ldots x_k$ are equally spaced with interval difference h.

Now, we can prove this result by mathematical induction.

For $k = 1$

$$\therefore f[x_0, x_1] = \frac{f_1 - f_0}{x_1 - x_0}$$

$$= \frac{\Delta f_0}{h}$$

$$= \frac{\Delta^1 f_0}{1!\, h^1}$$

statement is true for $k=1$.

Now, Assume the statement is true for $k = i$ i.e

$$f[x_0, x_1, \ldots, x_i] = \frac{\Delta^i f_0}{i!\, h^i}$$

Next we prove statement is true for $k = i+1$

so,

$$f[x_0, x_1, \ldots, x_{i+1}] = \frac{f[x_1, x_2, \ldots x_{i+1}] - f[x_0, x_1, \ldots x_i]}{x_{i+1} - x_0}$$

$$= \frac{\dfrac{\Delta^i f_1}{i!\, h^i} - \dfrac{\Delta^i f_0}{i!\, h^i}}{(i+1)\, h}$$

$$= \frac{\Delta^i f_1 - \Delta^i f_0}{(i+1)!\, h^{i+1}}$$

$$= \frac{\Delta^i (f_1 - f_0)}{(i+1)!\, h^{i+1}}$$

$$= \frac{\Delta^i (\Delta f_0)}{(i+1)!\, h^{i+1}}$$

$$= \frac{\Delta^{i+1} f_0}{(i+1)!\, h^{i+1}}$$

Hence, by induction it is true for all k.

Thus,

$$f[x_0, x_1, \cdots, x_k] = \frac{\Delta^k f_0}{k!\, h^k}$$

UNIT - III - Numerical Differentiation and Integration

* Numerical Differentiation.

I) Numerical Differentiation By Using Newton's Forward Interpolation formula -

We know that,

$$y = y_0 + m\Delta y_0 + \frac{m(m-1)}{2!}\Delta^2 y_0 + \frac{m(m-1)(m-2)}{3!}\Delta^3 y_0$$

$$+ \frac{m(m-1)(m-2)(m-3)}{4!}\Delta^4 y_0 + \cdots \quad \text{--- (I)}$$

We have,

$$m = \frac{x - x_0}{h} \quad \text{--- (II)}$$

By chane rule, we have,

$$\frac{dy}{dx} = \frac{dy}{dm} \cdot \frac{dm}{dx}$$

Differentiate eqn. (I) w.r.t m we get,

$$\frac{dy}{dm} = \Delta y_0 + \frac{(2m-1)}{2}\Delta^2 y_0 + \frac{3m^2 - 6m + 2}{6}\Delta^3 y_0$$

$$+ \frac{4m^3 - 18m^2 + 22m - 6}{24}\Delta^4 y_0 + \cdots$$

$$= \Delta y_0 + \frac{(2m-1)}{2}\Delta^2 y_0 + \frac{3m^2 - 6m + 2}{6}\Delta^3 y_0$$

$$+ \frac{2m^3 - 9m^2 + 11m - 3}{12}\Delta^4 y_0 + \cdots \quad \text{--- (III)}$$

Differentiate, eqn (II) w.r.t. x, we get,

$$\frac{dm}{dx} = \frac{1}{h}$$

Thus,

$$\frac{dy}{dx} = \frac{1}{h}\left[\Delta y_0 + \frac{(2m-1)}{2}\Delta^2 y_0 + \frac{3m^2-6m+2}{6}\Delta^3 y_0 + \frac{2m^3-9m^2+11m-3}{12}\Delta^4 y_0 + \cdots\right] \cdots \text{(IV)}$$

From eqn. (IV),

$$\left[\frac{dy}{dx}\right]_{x_0} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \cdots\right] \text{(V)} \left[\because m = 0 \text{ Bcoz } m = \frac{x-x_0}{h}\right]$$

similarly,

$$\left[\frac{dy}{dx}\right]_{x_1} = \frac{1}{h}\left[\Delta y_0 + \frac{1}{2}\Delta^2 y_0 - \frac{1}{6}\Delta^3 y_0 + \frac{1}{12}\Delta^4 y_0 + \cdots\right]$$

From eqn. (V)

$$\left[\frac{dy}{dx}\right]_{x_1} = \frac{1}{h}\left[\Delta y_1 - \frac{1}{2}\Delta^2 y_1 + \frac{1}{3}\Delta^3 y_1 - \frac{1}{4}\Delta^4 y_1 + \cdots\right]$$

diff. (IV) w.r.t. x, we get,

No theory

$$\frac{d^2y}{dx^2} = \frac{1}{h^2}\left[\Delta^2 y_0 + (m-1)\Delta^3 y_0 + \frac{6m^2-18m+11}{12}\Delta^4 y_0 + \cdots\right] \cdots \text{(VI)}$$

From eqn. (VI),

$$\left[\frac{d^2y}{dx^2}\right]_{x_0} = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 + \cdots\right]$$

similarly,

$$\left[\frac{d^2y}{dx^2}\right]_{x_1} = \frac{1}{h^2}\left[\Delta^2 y_0 - \frac{1}{12}\Delta^4 y_0 + \cdots\right]$$

Note -

1) $$\frac{dy}{dx} = \frac{1}{h}\left[\Delta y_0 + \frac{(2m-1)}{2}\Delta^2 y_0 + \frac{(3m^2-6m+2)}{6}\Delta^3 y_0 + \frac{(2m^3-9m^2+11m-3)}{12}\Delta^4 y_0 + \cdots\right]$$

2) $$\frac{d^2y}{dx^2} = \frac{1}{h^2}\left[\Delta^2 y_0 + (m-1)\Delta^3 y_0 + \frac{6m^2-18m+11}{12}\Delta^4 y_0 + \cdots\right]$$

1) From the following data obtain value of $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$ at $x = 5$

x	5	5.1	5.2	5.3
y	6.2146	6.2344	6.2538	6.2729

soln. step I) Table.

First we prepare forward diff. table.

x	y	Δ	Δ^2	Δ^3
5	6.2146			
		0.0198		
5.1	6.2344		-0.0004	
		0.0194		0.0001
5.2	6.2538		-0.0003	
		0.0191		
5.3	6.2729			

From forward diff.e table, we have,

$h = 0.1$, $\Delta y_0 = 0.0198$, $\Delta^2 y_0 = -0.0004$

$\Delta^3 y_0 = 0.0001$

Step II) calculation

$$\left[\frac{dy}{dx}\right]_{x_0} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 + \cdots\right]$$

$$\left[\frac{dy}{dx}\right]_{x_0} = \frac{1}{(0.1)}\left[0.0198 - \frac{1}{2} \times -0.0004 + \frac{1}{3} \times 0.0001\right]$$

$$= 0.2003$$

Now,

$$\left[\frac{d^2y}{dx^2}\right]_{x_0} = \frac{1}{h^2}\left[\Delta^2 y_0 - \cancel{\tfrac{1}{2}}\Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 + \ldots\right.$$

$$= \frac{1}{(0.1)^2}\left[-0.0004 - 0.0001\right]$$

$$= -0.05$$

2) Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ at $x = 2$ from foll. data.

x	1	3	5	7
y	2	6	14	24

Soln- Step I) Table

First we prepare forward diff. table,

x	y	Δ	Δ^2	Δ^3
1	2			
		4		
3	6		4	
		8		-2
5	14		2	
		10		
7	24			

From forward diff. table

$h = 2$, $\Delta y_0 = 4$, $\Delta^2 y_0 = 4$, $\Delta^3 y_0 = -2$

step II) calculation -

$$m = \frac{x - x_0}{h} = \frac{2-1}{2} = 0.5$$

Now,

$$\frac{dy}{dx} = \frac{1}{h}\left[\Delta y_0 + \frac{(2m-1)}{2}\Delta^2 y_0 + \frac{3m^2-6m+2}{6}\Delta^3 y_0 + \right]$$

$$= \frac{1}{2}\left[4 + \frac{(2\times0.5-1)}{2}\times 4 + \frac{3\times0.25-6\times0.5+2}{6} \times -2\right]$$

$$= 2.04167$$

Now,

$$\frac{d^2y}{dx^2} = \frac{1}{h^2}\left[\Delta^2 y_0 + (m-1)\Delta^3 y_0 + \cdots\right]$$

$$= \frac{1}{4}\left[4 + (0.5-1)(-2)\right]$$

$$= 1.25$$

II) Numerical Differentiation By using Newton's backward formula -

We have, know that,

$$y = y_n + m\nabla y_n + \frac{m(m+1)}{2!}\nabla^2 y_n + \frac{m(m+1)(m+2)}{3!}\nabla^3 y_n + \frac{m(m+1)(m+2)(m+3)}{4!}\nabla^4 y_n + \cdots$$

$$\Rightarrow y = y_n + m\nabla y_n + \frac{m^2+m}{2}\nabla^2 y_n + \frac{m^3+3m^2+2m}{6}\nabla^3 y_n + \frac{m^4+6m^3+11m^2+6m}{24}\nabla^4 y_n + \cdots \quad (I)$$

and $m = \dfrac{x - x_n}{h}$ $\quad \dfrac{x_{n-1} - x_n}{h}$ $\quad \cdots (II)$

Diff. eqn ① w.r.t. x, By using chane rule,

$$\frac{dy}{dx} = \frac{1}{h}\left[\nabla y_n + \frac{2m+1}{2}\nabla^2 y_n + \frac{3m^2+6m+2}{6}\nabla^3 y_n + \frac{4m^3+18m^2+22m+6}{24}\nabla^4 y_n + \cdots\right] \quad \text{--- (III)}$$

From eqn. (III),

$$\left[\frac{dy}{dx}\right]_{x_n} = \frac{1}{h}\left[\nabla y_n + \frac{1}{2}\nabla^2 y_n + \frac{1}{3}\nabla^3 y_n + \frac{1}{4}\nabla^4 y_n + \cdots\right] \quad \text{--- (IV)}$$

From eqn. (III),

$$\left[\frac{dy}{dx}\right]_{x_{n-1}} = \frac{1}{h}\left[\nabla y_n - \frac{1}{2}\nabla^2 y_n - \frac{1}{6}\nabla^3 y_n - \frac{1}{12}\nabla^4 y_n\right]$$

x_{n-1}

Diff. eqn. (III) w.r.t. x, we get,

$$\frac{d^2y}{dx^2} = \frac{1}{h^2}\left[\nabla^2 y_n + (m+1)\nabla^3 y_n + \frac{6m^2+18m+11}{12}\nabla^4 y_n + \cdots\right] \quad \text{--- (V)}$$

From eqn. (V),

$$\left[\frac{d^2y}{dx^2}\right]_{x_n} = \frac{1}{h^2}\left[\nabla^2 y_n + \nabla^3 y_n + \frac{11}{12}\nabla^4 y_n + \cdots\right]$$

similarly,

$$\left[\frac{d^2y}{dx^2}\right]_{x_{n-1}} = \frac{1}{h^2}\left[\nabla^2 y_n - \frac{1}{12}\nabla^4 y_n + \cdots\right]$$

Note -

1) $$\frac{dy}{dx} = \frac{1}{h}\left[\nabla y_n + \frac{2m+1}{2}\nabla^2 y_n + \frac{3m^2+6m+2}{6}\nabla^3 y_n + \frac{4m^3+18m^2+22m+6}{24}\nabla^4 y_n + \cdots\right]$$

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ii) $\dfrac{d^2y}{dx^2} = \dfrac{1}{h^2}\left[\nabla^2 y_n + (m+1)\nabla^3 y_n + \dfrac{6m^2+18m+11}{12}\nabla^4 y_n + \dots\right]$

Ex. following data is given for various values of x & it's function $y=f(x)$. Find the value of 1st derivative & 2nd derivative at $x=40$

x	0	10	20	30	40
$f(x)$	1	0.984	0.939	0.866	0.766

Soln. step I) - Table

First we prepare backward diff. table

x	$f(x)$	∇	∇^2	∇^3	∇^4
0	1				
		-0.016			
10	0.984		-0.029		
		-0.045		0.001	
20	0.939		-0.028		0
		-0.073		0.001	
30	0.866		-0.027		
		-0.1			
40	0.766				

From backward diff. table,

$h=10$, $\nabla y_n = -0.1$, $\nabla^2 y_n = -0.027$

$\nabla^3 y_n = 0.001$, $\nabla^4 y_n = 0$

$m = \dfrac{x - x_n}{h}$

step II) Calculation.

$$\left[\frac{dy}{dx}\right]_{x_n} = \frac{1}{h}\left[\nabla y_n + \frac{1}{2}\nabla^2 y_n + \frac{1}{3}\nabla^3 y_n + \frac{1}{4}\nabla^4 y_n + \dots\right]$$

$$= \frac{1}{10}\left[-0.1 + \frac{1}{2}(-0.027) + \frac{1}{3}(0.001) + \frac{1}{4}\times 0\right]$$

$$= -0.01132$$

similarly,

$$\left[\frac{d^2y}{dx^2}\right]_{x_n} = \frac{1}{h^2}\left[\nabla^2 y_n + \nabla^3 y_n + \frac{11}{12}\nabla^4 y_n + \cdots\right]$$

$$= \frac{1}{100}\left[-0.027 + 0.001 + \frac{11}{12}\times 0\right]$$

$$= -0.00026$$

III) Approximate Numerical Differentiation By Taylor's series -

Numerical differentiation is required when functions is given in data form as $x_0, x_1, x_2, \ldots x_n$

$f_0, f_1, f_2, \ldots f_n$

We have, Taylor's series of real function $f(x)$ of single variable x.

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \cdots$$

Now suppose, derivative is required at x_i when values of f are given with specing h i.e $x_i = x_0 + ih$, $i = 0, 1, 2, \ldots n$

From Taylor's series

$$f(x_i+h) = f(x_i) + hf'(x_i) + \frac{h^2}{2!}f''(x_i) + \cdots \quad \text{(I)}$$

$$\Rightarrow \frac{f(x_i+h) - f(x_i)}{h} = f'(x_i)$$

By neglecting higher order derivatives

i.e $$f_i' = \frac{f_{i+1} - f_i}{h} \quad \text{(II)}$$

is called forward differentiation.

similarly, we have,

$$f(x_i-h) = f(x_i) - hf'(x_i) + \frac{h^2}{2!} f''(x_i) + \cdots$$

$$\Rightarrow \frac{f(x_i-h) - f(x_i)}{-h} = f'(x_i)$$

By neglecting higher order derivatives

$$\Rightarrow \frac{f_i - f_{i-1}}{h} = f_i' \quad \cdots \text{(III)}$$

is called 'backward differentiation'

By adding eqn. (II) & eqn. (III) we get,

$$2f_i' = \frac{f_{i+1} - f_i}{h} + \frac{f_i - f_{i-1}}{h}$$

$$\Rightarrow 2f_i' = \frac{f_{i+1} - f_{i-1}}{h}$$

$$\Rightarrow f_i' = \frac{f_{i+1} - f_{i-1}}{2h} \quad \cdots \text{(IV)}$$

is called 'central differentiation'

similarly,

$$f(x_i+h) + f(x_i-h) = 2f(x_i) + \frac{2h^2}{2!} f''(x_i) + \cdots$$

$$\Rightarrow \frac{f(x_i+h) + f(x_i-h) - 2f(x_i)}{h^2} = f''(x_i)$$

neglecting higher order terms.

$$\Rightarrow f_i'' = \frac{f_{i+1} + f_{i-1} - 2f_i}{h^2} \quad \cdots \text{(V)}$$

is '2nd order central differentiation'.

Note -

1) $f_i' = \dfrac{f_{i+1} - f_i}{h}$

2) $f_i' = \dfrac{f_i - f_{i-1}}{h}$

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3) $f_i' = \frac{f_{i+1} - f_{i-1}}{2h}$

4) $f_i'' = \frac{f_{i+1} + f_{i-1} - 2f_i}{h^2}$

Ex- Find 1st order derivative of f(x) at x=0.1, 0.2, 0.3, where f(x) is given by,

x	0.1	0.2	0.3	0.4	0.5	0.6
f	0.425	0.475	0.4	0.450	0.525	0.575

soln- We have,

$f_i' = \frac{f_{i+1} - f_{i}}{h}$

Here, h = 0.1

$\therefore f'_{(0.1)} = \frac{0.475 - 0.425}{0.1}$

$= 0.5$ (approx.)

similarly,

$f'_{(0.2)} = \frac{0.4 - 0.475}{0.1}$

$= -0.75$ (approx.)

||ly,

$f'_{(0.3)} = \frac{0.450 - 0.4}{0.1}$

$= 0.5$ (approx.)

2) Find 2nd derivative at 0.3, 0.4, 0.5 for the function given by,

x	0.1	0.2	0.3	0.4	0.5	0.6
f	0.425	0.475	0.4	0.450	0.525	0.575

soln- We have,

$f_i'' = \frac{f_{i+1} + f_{i-1} - 2f_i}{h^2}$

Here $h = 0.1$

$$\therefore f''_{(0.3)} = \frac{0.450 + 0.475 - 2\times 0.4}{(0.1)^2}$$
$$= 12.5 \text{ (approx)}$$

similarly,

$$f''_{(0.4)} = \frac{0.525 + 0.4 - 2\times 0.450}{(0.1)^2}$$
$$= 2.5 \text{ (approx.)}$$

$$f''_{(0.5)} = \frac{0.575 + 0.450 - 2\times 0.525}{(0.1)^2}$$
$$= -2.5 \text{ (approx)}$$

Note -

If function is unequal intervals then we find polynomial for given data by using Newton's divided difference formula or By using Lagrange's interpolation formula and then we find $f'(x), f''(x), \ldots$

* Numerical Integration -

Newton - cote's formula -

No Theory

To evaluate the integral $\int_a^b y\,dx$ — y is replaced by an interpolating polynomial $f(x)$ which interpolates y at some points $x_0, x_1, \ldots, x_n$ in the interval $[a, b]$.

We write, $x_0 = a$ and $x_n = b$ this method are called 'closed Newton-cote's formula' otherwise this method is known as 'open Newton - cote's formula'.

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$$\therefore \int_a^b y\,dx = \int_a^b f(x)\,dx$$

$$= \int_a^b [f_0 l_0(x) + f_1 l_1(x) + \cdots + f_n l_n(x)]\,dx$$

$$= A_0 f_0 + A_1 f_1 + \cdots + A_n f_n$$

where, $\{l_i(x)\}_{i=0}^{n}$ are called 'Lagrange's polynomials for the points $x_0, x_1, \dots, x_n$ resp. and $A_i = \int_a^b l_i(x)\,dx \quad , i=0,1,\dots,n$

Here, A_i's are called 'weights'.

Imp * Basic Trapezoidal Rule -

Let, $\int_a^b f\,dx$ is to be evaluated.

f is replaced by a polynomial of degree one which interpolates f at a & b.

Taking $x_0 = a$ and $x_1 = b$ with $h = b-a$

$\therefore$ The interpolating polynomial of degree one is,

$$f(x) = f_0 + m\Delta f_0$$

where, $m = \dfrac{x-a}{h}$

$$\therefore \int_a^b f(x)\,dx = \int_{x_0}^{x_1} [f_0 + m\Delta f_0]\,dx$$

$$= f_0 [x]_{x_0}^{x_1} + \int_{x_0}^{x_1} m\Delta f_0\,dx$$

$$= f_0 (x_1 - x_0) + \int_{x_0}^{x_1} m\Delta f_0\,dx$$

$$= hf_0 + \int_{x_0}^{x_1} m\Delta f_0\, dx \quad \text{---} ①$$

To solve, $\int_{x_0}^{x_1} m\Delta f_0\, dx$

We have,

$$x = mh + x_0$$
$$\Rightarrow dx = h\,dm$$

Also,

$$x = x_0 \Rightarrow m = 0$$

and $x = x_1 \Rightarrow m = 1$,

$$\therefore \int_{x_0}^{x_1} m\Delta f_0\, dx = \int_0^1 m\Delta f_0\, h\, dm$$

$$= h\Delta f_0 \left[\frac{m^2}{2}\right]_0^1$$

$$= h\Delta f_0 \left(\frac{1}{2}\right)$$

Thus, eqn. ① becomes,

$$\int_a^b f(x)\,dx = hf_0 + \frac{h}{2}\Delta f_0$$

$$= hf_0 + \frac{h}{2}f_1 - \frac{h}{2}f_0 \quad [\because \Delta f_0 = f_1 - f_0]$$

$$= \frac{h}{2}f_0 + \frac{h}{2}f_1$$

$$= \frac{h}{2}[f_0 + f_1]$$

MCQ → $\therefore \int_a^b f(x)\,dx \cong \frac{(b-a)}{2}[f(a) + f(b)]$

This is known as 'Basic Trapezoidal Rule'.

Composite Trapezoidal Rule-

Suppose, $\int_a^b f(x)\,dx$ is to be evaluated

Here, interval $[a,b]$ is subdivided into n equal subinterval seperated by h.

i.e $a = x_0, x_1, x_2, \ldots, x_n = b$.

with $\frac{b-a}{n} = h$

For composite trapezoidal rule,

consider,

$$\int_a^b f(x)\,dx = \int_{x_0}^{x_1} f(x)\,dx + \int_{x_1}^{x_2} f(x)\,dx + \int_{x_2}^{x_3} f(x)\,dx + \cdots + \int_{x_{n-1}}^{x_n} f(x)\,dx$$

$$= \frac{h}{2}[f_0+f_1] + \frac{h}{2}[f_1+f_2] + \frac{h}{2}[f_2+f_3] + \cdots + \frac{h}{2}[f_{n-1}+f_n]$$

[$\because$ By Basic trapezoidal rule]

$$= \frac{h}{2}[f_0 + 2f_1 + 2f_2 + \cdots + 2f_{n-1} + f_n]$$

$$= \frac{h}{2}[f_0 + 2(f_1 + f_2 + \cdots + f_{n-1}) + f_n]$$

This is known as composite trapezoidal rule. i.e. $\int_a^b f(x)\,dx \simeq \frac{h}{2}[f_0 + 2(f_1+f_2+\cdots+f_{n-1}) + f_n]$

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Ex1) Find, $\int_0^1 \frac{\cos x - x}{1+x}\,dx$ using trapezoidal rule with 6 sub-interval.

Soln- Given that,

$$f(x) = \frac{\cos x - x}{1+x}, \quad n = 6$$

$h = \frac{b-a}{n}$ $\therefore h = \frac{1-0}{6} = \frac{1}{6} = 0.1666$

$\therefore$ [0, 0.166, 0.3332, 0.4998, 0.6664, 0.8330, 1]

$$\therefore f_0 = \frac{\cos(0) - 0}{1+0} = 1$$

$$f_1 = \frac{\cos(0.1666) - 0.1666}{1+0.1666} = 0.7025$$

$$f_2 = \frac{\cos(0.3332) - 0.3332}{1+0.3332} = 0.4589$$

$$f_3 = \frac{\cos(0.4998) - 0.4998}{1+0.4998} = 0.2520$$

$$f_4 = \frac{\cos(0.6664) - 0.6664}{1+0.6664} = 0.0718$$

$$f_5 = \frac{\cos(0.8330) - 0.8330}{1+0.8330} = -0.0875$$

$$f_6 = \frac{\cos(1) - 1}{1+1} = \cancel{+0.5} - 0.2298$$

By Trapezoidal rule,
We have,

$$\int_a^b f(x)\,dx = \frac{h}{2}\left[f_0 + 2(f_1 + f_2 + \cdots + f_{n-1}) + f_n\right]$$

$$\therefore \int_0^1 \frac{\cos x - x}{1+x} \simeq \frac{1}{12}\left[1 + 2(0.7025 + 0.4589 + 0.2520 + 0.0718 - 0.0875) - 0.2298\right]$$

with 4 points are divided by 3 n

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$= 0.2971$

2) Find $\int_0^2 e^x \sin x \, dx$ By Trapezoidal rule using 4 divisions.

Soln- Given,

$f(x) = e^x \sin x \, dx$, $n = 4$

$\therefore \frac{b-a}{n} = \frac{2-0}{4} = 0.5$

$\therefore$ Interval is subdivided into,

0 0.5 0.5,1 1.5 2

$f_0 = e^{0.0} \sin(0) = 0$

$f_1 = e^{0.5} \cdot \sin(0.5) = 0.7904$

$f_2 = e^1 \sin(1) = 2.2874$

$f_3 = 4.4705$

$f_4 = 6.7189$

By Trapezoidal rule, we have,

$$\int_0^2 e^x \sin x \, dx \simeq \frac{1}{4}[0 + 2(0.7904 + 2.2874 + 4.4705) + 6.7189]$$

$$= 5.4539$$

3) Evaluate $\int_0^1 \sqrt{\sin x + \cos x} \, dx$ using trapezoidal rule with 8 divisions.

Soln- Given that,

$f(x) = \int_0^1 \sqrt{\sin x + \cos x} \, dx$, $n = 8$

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$\therefore h = \dfrac{b-a}{n} = \dfrac{1-0}{8} = 0.125$

0.125 0.25 0.375 0.5 0.625 0.75 0.875 1

Now,

$f_0 = \sqrt{\sin 0 + \cos 0} = 1$

$f_1 = 1.05682$, $f_2 = 1.1029$

$f_3 = 1.1388$, $f_4 = 1.1649$

$f_5 = 1.1815$, $f_6 = 1.1888$

$f_7 = 1.1868$, $f_8 = 1.1765$

By Trapezoidal Rule,

$$\int_0^1 \sqrt{\sin x + \cos x}\, dx \cong \frac{1}{16}\Big[1 + 2(1.05682 + 1.1029 + 1.1388 + 1.1649 + 1.1815 + 1.1888 + 1.1868) + 1.1755\Big]$$

$$= 1.1385$$

*4) Evaluate, $\displaystyle\int_{\pi/2}^{\pi} \frac{x\cos x}{\sin x}\, dx$ By Trapezoidal rule with 5 points

Soln Given that,

$f(x) = \dfrac{x\cos x}{\sin x}$, $n = 4$

$\therefore h = \dfrac{b-a}{n} = \dfrac{\pi/2}{4} = \dfrac{\pi}{8} = 0.3927$

1.5708 1.9635 2.3562 2.7489 3.1416

Now,

$f_0 = \dfrac{(1.5708)\cos(1.5708)}{\sin(1.5708)} = 0$

$f_1 = -0.8133$

$f_2 = -2.3562$

$f_3 = -6.6366$

$f_4 = 427637.483$

By Trapezoidal Rule,

$$\int_a^b f(x)dx \cong \frac{h}{2}\left[f_0 + 2(f_1 + f_2 + \cdots + f_{n-1}) + f_n\right]$$

$$\therefore \int_{1.5708}^{3.1416} \frac{x\cos x}{\sin x} dx \cong \frac{0.3927}{2}\left[0 + 2(-0.8133 - 2.3562 - 6.6366) + 427637.483\right]$$

$$= 83962.75911$$

5) A function f(x) is described by foll. data

x	1	1.1	1.2	1.4	1.6	1.9	2.2
f(x)	3.123	4.247	5.635	9.299	14.303	24.759	39.319

Find numerical integration of the function in limits 1 to 2.2 using trap. rule.

Soln- From given information we divide the integral in 3 parts i.e

$$\int_1^{2.2} f(x)dx = \int_1^{1.2} f(x)dx + \int_{1.2}^{1.6} f(x)dx + \int_{1.6}^{2.2} f(x)dx \quad \text{---}(1)$$

Here,

formula 2

$$\int_1^{1.2} f(x)dx \cong \frac{0.1}{2}\left[3.123 + 2\times 4.247 + 5.635\right]$$

$$= 0.8626$$

similarly,

$$\int_{1.2}^{1.6} f(x)dx \cong \frac{0.2}{2}\left[5.635 + 2\times 9.299 + 14.303\right]$$

$$= 3.8536$$

similarly,

$$\int_{1.6}^{2.2} f(x)\,dx \simeq \frac{0.3}{2}\left[14.303 + 2\times 24.759 + 39.319\right]$$

$$= 15.471$$

Thus eqn. ① becomes,

$$\therefore \int_{1}^{2.2} f(x)\,dx \simeq 0.8626 + 3.8636 + 15.471$$

$$\simeq 20.1872$$

6) Use composite trapezoidal rule to evaluate the integral $\int_{0.1}^{0.6} f(x)\,dx$ of the function $f(x)$ given by

x	0.1	0.2	0.3	0.4	0.5	0.6
$f(x)$	0.425	0.475	0.4	0.450	0.575	0.675

Soln- ~~From~~ By composite Trap. rule,

$$\int_{0.1}^{0.6} f(x)\,dx \simeq \frac{0.1}{2}\left[0.425 + 2(0.475 + 0.4 + 0.450 + 0.575) + 0.675\right]$$

$$= 0.245$$

7) Evaluate the integral $\int_{1}^{2} \frac{e^{2x}}{1+x^2}\,dx$ using Trap. rule with 6 functional values.

Soln- Given, $f(x) = \frac{e^{2x}}{1+x^2}$, $n = 5$

$$\therefore h = \frac{b-a}{n} = \frac{2-1}{5} = \frac{1}{5} = 0.2$$

1 1.2 1.4 1.6 1.8 2

Now,

$$f_0 = \frac{e^{2(0)}}{1+(0)} = 1$$

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$f_1 = 4.5177$, $f_2 = 5.5556$, $f_3 = 6.8912$, $f_4 = 8.6317$, $f_5 = 10.9196$

By Trap. Rule,

$$\int_a^b f(x)\,dx \cong \frac{h}{2}\left[f_0 + 2(f_1 + f_2 + \cdots + f_{n-1}) + f_n\right]$$

$$\therefore \int_1^2 \frac{e^{2x}}{1+x^2}\,dx \cong \frac{0.2}{2}\left[1 + 2(4.5177 + 5.5556 + 6.8912 + 8.6317) - 10.9196\right]$$

$= \cancel{4.1273}$

$= 6.5807$

* Basic Simpson's $\left(\frac{1}{3}\right)^{rd}$ Rule -

consider, $\int_a^b f(x)\,dx$ is to be evaluated

The interval $[a, b]$ is divided into two equal subintervals with $h = \frac{b-a}{2}$

suppose, $x_0 = a$, $x_1 = \frac{a+b}{2}$, $x_2 = b$

Now, we interpolate polynomial $f(x)$ by using Newton's forward interpolation

$$\therefore f(x) = f_0 + m\Delta f_0 + \frac{m(m-1)}{2!}\Delta^2 f_0$$

where, $m = \frac{x-a}{h} = \frac{x-x_0}{h}$

$$\therefore \int_a^b f(x)\,dx = \int_{x_0}^{x_2}\left[f_0 + m\Delta f_0 + \frac{m(m-1)}{2}\Delta^2 f_0\right]dx$$

$$= f_0[x]_{x_0}^{x_2} + \int_{x_0}^{x_2} m\Delta f_0\,dx + \int_{x_0}^{x_2}\frac{m(m-1)}{2}\Delta^2 f_0\,dx$$

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$$= 2hf_0 + \int_{x_0}^{x_2} m\Delta f_0\, dx + \int_{x_0}^{x_2} \left(\frac{m^2-m}{2}\Delta^2 f_0\right) dx \quad \text{--- (I)}$$

To evaluate, $\int_{x_0}^{x_2} (m\Delta f_0)\, dx$

put, $m = \frac{x - x_0}{h}$

$$\Rightarrow \quad dm = \frac{dx}{h}$$

$$\Rightarrow \quad dx = h\,dm$$

As $x = x_0 \Rightarrow m = 0$

also $x = x_2 \Rightarrow m = 2$

$$\therefore \int_{x_0}^{x_2} m\Delta f_0\, dx = \int_0^2 m\,\Delta f_0\, h\, dm$$

$$= h\Delta f_0 \int_0^2 m\, dm$$

$$= h\Delta f_0 \left[\frac{m^2}{2}\right]_0^2$$

$$= h\Delta f_0 (2)$$

$$= 2h\Delta f_0$$

$$= 2h[f_1 - f_0] \quad \text{--- (II)}$$

To evaluate, $\int_{x_0}^{x_2} \left(\frac{m^2-m}{2}\Delta^2 f_0\right) dx$

put, $m = \frac{x - x_0}{h}$

$$\Rightarrow \quad dx = h\,dm$$

As $x = x_0 \Rightarrow m = 0$

& $x = x_1 \Rightarrow m = 2$

$$\therefore \int_{x_0}^{x_2} \left(\frac{m^2-m}{2}\right)\Delta^2 f_0\, dx = \int_0 \frac{\Delta^2 f_0 h}{2} \int_0^2 (m^2 - m)\, dxm$$

$$= \frac{h\Delta^2 f_0}{2}\left[\frac{m^3}{3} - \frac{m^2}{2}\right]_0^2$$

$$= \frac{h\Delta^2 f_0}{2}\left(\frac{2}{3}\right)$$

$$= \frac{h\Delta^2 f_0}{3}$$

$(\Delta^2 f_0 = \Delta(\Delta f_0) = \Delta(f_1 - f_0) = \Delta f_1 - \Delta f_0 = f_2 - f_1 - (f_1 - f_0))$

$$= \frac{h}{3}[f_2 - 2f_1 + f_0] \quad \text{(III)}$$

By (I), (II) and (III),

$$\int_a^b f(x)\,dx = 2hf_0 + 2h[f_1 - f_0] + \frac{h}{3}[f_2 - 2f_1 + f_0]$$

$$= 2h\cancel{f_0} + 2hf_1 - 2h\cancel{f_0} + \frac{h}{3}f_2 - \frac{2hf_1}{3} + \frac{h}{3}f_0$$

$$= \left(\frac{2}{3}\right)2hf_1 + \frac{h}{3}f_0 + \frac{h}{3}f_2$$

$$= \frac{h}{3}[4f_1 + f_0 + f_2]$$

$$= \frac{h}{3}[f_0 + 4f_1 + f_2]$$

$$\therefore \int_a^b f(x)\,dx \cong \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]$$

This is called 'Basic simpson's $\left(\frac{1}{3}\right)^{rd}$ Rule'.

* Composite simpson's $\left(\frac{1}{3}\right)^{rd}$ Rule.

consider, $\int_a^b f(x)\,dx$ is to be evaluated.

If the interval $[a,b]$ is very large then we divided $[a,b]$ into $2n$ equal subintervals and denote points of sub intervals by,

$a = x_0, x_1, x_2, \ldots, x_{2n} = b$ with $h = \frac{b-a}{2n}$

$$\therefore \int_a^b f(x)\,dx = \int_{x_0}^{x_2} f(x)\,dx + \int_{x_2}^{x_4} f(x)\,dx + \int_{x_4}^{x_6} f(x)\,dx$$

$$+ \cdots + \int_{x_{2n-2}}^{x_{2n}} f(x)\,dx$$

By Basic simpson's $\left(\frac{1}{3}\right)^{rd}$ Rule,

$$\int_a^b f(x)\,dx = \frac{h}{3}\left[f_0 + 4f_1 + f_2\right] + \frac{h}{3}\left[f_2 + 4f_3 + f_4\right]$$

$$+ \frac{h}{3}\left[f_4 + 4f_5 + f_6\right] + \cdots$$

$$+ \frac{h}{3}\left[f_{2n-2} + 4f_{2n-1} + f_{2n}\right]$$

$$= \frac{h}{3}\Big\{f_0 + 4(f_1 + f_3 + f_5 + \cdots + f_{2n-1}) + 2(f_2 + f_4 + f_6 + \cdots + f_{2n-2}) + f_{2n}\Big\}$$

$$\therefore \int_a^b f(x)\,dx \cong \frac{h}{3}\Big\{f_0 + 4(f_1 + f_3 + f_5 + \cdots + f_{2n-1}) + 2(f_2 + f_4 + f_6 + \cdots + f_{2n-2}) + f_{2n}\Big\}$$

This is known as 'composite simpson's $\left(\frac{1}{3}\right)^{rd}$ Rule'.

Note -

$$\int_a^b f(x)\,dx \cong \frac{h}{3}\Big[f_0 + 4(f_1 + f_3 + f_5 + \cdots + f_{m-1}) + 2(f_2 + f_4 + f_6 + \cdots + f_{m-2}) + f_m\Big]$$

where m is even integer.

Ex- Evaluate the $\int_{-1}^{1} x^2 e^{-x}\,dx$ by simpson's $(\frac{1}{3})^{rd}$ rule with spacing $h=0.25$.

Soln- Given that,

$\int_{-1}^{1} x^2 e^{-x}\,dx$ and $h=0.25$

Here, $f(x) = x^2 e^{-x}$

Here,

$[-1 \quad -0.75 \quad -0.5 \quad -0.25 \quad 0 \quad 0.25 \quad 0.5 \quad 0.75 \quad 1]$

Now, we calculate,

$f_0 = f(-1) = (-1)^2 e^{-(-1)} = e = 2.7183$

$f_1 = f(-0.75) = (-0.75)^2 e^{-(-0.75)} = 1.1908$

$f_2 = f(-0.5) = (-0.5)^2 e^{(0.5)} = 0.4122$

$f_3 = f(-0.25) = (-0.25)^2 e^{0.25} = 0.08025$

$f_4 = f(0) = 0$

$f_5 = f(0.25) =$ ~~0.08025~~ 0.04868

$f_6 = f(0.5) = 0.1516$

$f_7 = f(0.75) = 0.2657$

$f_8 = f(1) = 0.3679$

By simpson's $(1/3)^{rd}$ rule,

$$\int_{-1}^{1} x^2 e^{-x}\,dx \simeq \frac{h}{3}\left[f_0 + 4(f_1+f_3+f_5+f_7) + 2(f_2+f_4+f_6) + \cancel{f(8)} + f_8\right]$$

$$= \frac{0.25}{3}\left[2.7183 + 4(1.1908 + 0.08025 + 0.04868 + 0.2657) + 2(0.4122 + 0 + 0.1516) + 0.3679\right]$$

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$$= 0.8797$$

2) Evaluate the $\int_0^2 \frac{e^{2x}}{1+x^2}\,dx$ using composite simpson's $(1/3)^{rd}$ rule with nine functional values.

Soln. Given that,

$$\int_0^2 \frac{e^{2x}}{1+x^2}\,dx$$

Here, $f(x) = \frac{e^{2x}}{1+x^2}$, $n = 8$

$$h = \frac{2-0}{8} = 0.25$$

$[0, 0.25, 0.5, 0.75, 1, 1.25, 1.5, 1.75, 2]$

Now we calculate,

$$f_0 = \frac{e^{2(0)}}{1+0} = 1$$

$$f_1 = \frac{e^{2(0.25)}}{1+(0.25)^2} = 1.5517$$

$f_2 = f(0.5) = 2.1746$

$f_3 = f(0.75) = 2.8683$

$f_4 = f(1) = 3.6945$

$f_5 = f(1.25) = 4.7541$

$f_6 = f(1.5) = 6.1802$

$f_7 = f(1.75) = 8.1575$

$f_8 = f(2) = 10.9196$

By simpson's $(1/3)^{rd}$ rule,

$$\int_0^2 \frac{e^{2x}}{1+x^2}\,dx \simeq \frac{h}{3}\left[f_0 + 4(f_1+f_3+f_5+f_7) + 2(f_2+f_4+f_6) + f_8\right]$$

Using composite simpson's $(\frac{1}{3})^{rd}$ rule evaluate the $\int_{-1}^{3} f(x)\,dx$ from foll. data

x :	-1	-0.5	0	0.5	1	1.5	2	2.5	3
$f(x)$:	7	5	3.5	4	5.5	6	6.5	5	4.5

$$= \frac{0.25}{3}\left[1 + 4(1.5517 + 2.8683 + 4.7541 + 8.1515) + 2(2.1746 + 3.6945 + 6.1802) + 10.9196\right]$$

$$= 8.7767$$

3) Find the Int. of $4x+2$ in the limits 1 to 4 by simpson's $(1/3)^{rd}$ rule using 6 strips.

Given that,

$$f(x) = 4x + 2 \quad , \quad n = 6$$

$$\therefore h = \frac{4-1}{6} = \frac{3}{6} = 0.5$$

Here, [1, 1.5, 2, 2.5, 3, 3.5, 4]

Now,

$$f_0 = f(1) = 4(1) + 2 = 6$$
$$f_1 = f(1.5) = 8$$
$$f_2 = f(2) = 10$$
$$f_3 = f(2.5) = 12$$
$$f_4 = f(3) = 14$$
$$f_5 = f(3.5) = 16$$
$$f_6 = f(4) = 18$$

By simpson's $(\frac{1}{3})^{rd}$ rule,

$$\int_a^b f(x)\,dx \simeq \frac{h}{3}\left[f_0 + 4(f_1 + f_3 + f_5) + 2(f_2 + f_4) + f_6\right]$$

$$\therefore \int_1^4 (4x+2)\,dx \simeq \frac{0.5}{3}\left[6 + 4(8 + 12 + 16) + 2(10 + 14) + 18\right]$$

$$\simeq 36$$

Find the $\int_0^{\pi/2} e^{\sin x}\,dx$ using simpson's $\left(\frac{1}{3}\right)^{rd}$ rule with 4 divisions,

4) Using composite simpson's $\left(\frac{1}{3}\right)^{rd}$ rule evaluate the $\int_{-1}^{3} f(x)\,dx$ from following data

x :	-1	-0.5	0	0.5	1	1.5	2	2.5	3
$f(x)$:	7	5	3.5	4	5.5	6	6.5	5	4.5

Soln. Here,

$f_0 = 7$, $f_1 = 5$, $f_2 = 3.5$, $f_3 = 4$, $f_4 = 5.5$, $f_5 = 6$, $f_6 = 6.5$, $f_7 = 5$, $f_8 = 4.5$

Here, $n = 8$

$$\therefore h = \frac{b-a}{n} = \frac{3-(-1)}{8} = \frac{4}{8} = 0.5$$

Now, By simpson's $\left(\frac{1}{3}\right)^{rd}$ rule,

$$\int_{-1}^{3} f(x)\,dx \cong \frac{h}{3}\left[f_0 + 4(f_1 + f_3 + f_5 + f_7) + 2(f_2 + f_4 + f_6) + f_8\right]$$

$$= \frac{0.5}{3}\left[7 + 4(5+4+6+5) + 2(3.5 + 5.5 + 6.5) + 4.5\right]$$

$$= 20.4167$$

5) Find the $\int_0^{\pi/2} e^{\sin x}\,dx$ using simpson's $\left(\frac{1}{3}\right)^{rd}$ rule with 4 divisions.

Soln. Given, $f(x) = e^{\sin x}$, $n = 4$

$$\therefore h = \frac{b-a}{n} = \frac{\pi/2 - 0}{4} = 0.3927$$

$\therefore$ 0, 0.3927, 0.7854, 1.1781, 1.5708

Now,

$f_0 = f(0) = e^{\sin(0)} = 1$

$f_1 = f(0.3927) = 1.4662$

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$f_2 = f(0.7854) = 2.02812$

$f_3 = f(1.1781) = 2.5190$

$f_4 = f(1.5708) = 2.7183$

By simpson's $(\frac{1}{3})^{rd}$ rule,

$$\int_a^b f(x)\,dx \simeq \frac{h}{3}\left[f_0 + 4(f_1+f_3) + 2(f_2) + f_4\right]$$

$$= \frac{0.3927}{3}\left[1 + 4(1.4662 + 2.\cancel{02812}5190) + 2(2.02812) + 2.7183\right]$$

$$= 3.1043.$$

* Basic simpson's $(3/8)^{th}$ Rule -

Let, $\int_a^b f(x)\,dx$ is to be evaluated.

The interval $[a,b]$ is divided into three equal subintervals with $h = \frac{b-a}{3}$

i.e. $[a = x_0,\ x_1,\ x_2,\ b = x_3]$

∴ Therefore,

$$f(x) = f_0 + m\Delta f_0 + \frac{m(m-1)}{2!}\Delta^2 f_0 + \frac{m(m-1)(m-2)}{3!}\Delta^3 f_0$$

[By using Newton's forward interpolation]

Where, $m = \frac{x - x_0}{h}$

$$\therefore \int_a^b f(x)\,dx = \int_{x_0}^{x_3}\left[f_0 + m\Delta f_0 + \frac{m(m-1)}{2!}\Delta^2 f_0 + \frac{m(m-1)(m-2)}{3!}\Delta^3 f_0\right]dx$$

$$= 3hf_0 + \Delta f_0 \int_{x_0}^{x_3} m\,dx + \frac{\Delta^2 f_0}{2}\int_{x_0}^{x_3}(m^2 - m)\,dx + \frac{\Delta^3 f_0}{6}\int_{x_0}^{x_3}(m^3 - 3m^2 + 2m)\,dx \quad (1)$$

To solve, $\int_{x_0}^{x_3} m\,dx$, $\int_{x_0}^{x_3}(m^2-m)\,dx$, $\int_{x_0}^{x_3}(m^3-3m^2+2m)\,dx$

put, $m = \dfrac{x-x_0}{h}$

$\Rightarrow dm = \dfrac{dx}{h}$

$\Rightarrow dx = h\,dm$

As $x = x_0 \Rightarrow m = 0$

and $x = x_3 \Rightarrow m = 3$

$$\therefore \int_{x_0}^{x_3} m\,dx = \int_0^3 m\,h\,dm$$

$$= h\left[\frac{m^2}{2}\right]_0^3$$

$$= \frac{9h}{2} \quad \text{--- (ii)}$$

Simillarly,

$$\int_{x_0}^{x_3}(m^2-m)\,dx = \int_0^3 (m^2-m)\,h\,dm$$

$$= h\left[\frac{m^3}{3} - \frac{m^2}{2}\right]_0^3$$

$$= h\left[\frac{27}{3} - \frac{9}{2}\right]$$

$$= \frac{9h}{2} \quad \text{--- (iii)}$$

Now,

$$\int_{x_0}^{x_3}(m^3-3m^2+2m)\,dx = \int_0^3 (m^3-3m^2+2m)\,h\,dm$$

$$= h\left[\frac{m^4}{4} - 3\frac{m^3}{3} + 2\frac{m^2}{2}\right]_0^3$$

$$= h\left[\frac{81}{4} - 27 + 9\right]$$

$$= \frac{9h}{4} \quad \text{--- (IV)}$$

By eqns. (I) to (IV)

$$\int_a^b f(x)\,dx = 3hf_0 + \Delta f_0\left(\frac{9h}{2}\right) + \frac{\Delta^2 f_0}{2}\left(\frac{9h}{2}\right) + \frac{\Delta^3 f_0}{6}\left(\frac{9h}{4}\right)$$

$$= 3hf_0 + \frac{9h}{2}f_1 - \frac{9h}{2}f_0 + \frac{9h}{4}[f_2 - 2f_1 + f_0]$$

$$+ \frac{9h}{24}[f_3 - 3f_2 + 3f_1 - f_0]$$

$$= 3hf_0 + \frac{9h}{2}f_1 - \frac{9h}{2}f_0 + \frac{9h}{4}f_2 - \frac{9h}{2}f_1$$

$$+ \frac{9h}{4}f_0 + \frac{9h}{24}f_3 - \frac{9h}{8}f_2 + \frac{9h}{8}f_1 - \frac{9h}{24}f_0$$

$$= 3h\left[f_0 - \frac{3}{2}f_0 + \frac{3}{4}f_0 - \frac{1}{8}f_0 + \frac{3}{8}f_1\right.$$

$$\left. + \frac{3}{4}f_2 - \frac{3}{8}f_2 + \frac{1}{8}f_3\right]$$

$$= 3h\left[\frac{1}{8}f_0 + \frac{3}{8}f_1 + \frac{3}{8}f_2 + \frac{1}{8}f_3\right]$$

$$= \frac{3h}{8}[f_0 + 3f_1 + 3f_2 + f_3]$$

$$= \frac{3h}{8}[f_0 + 3(f_1 + f_2) + f_3]$$

$$\therefore \int_a^b f(x)\,dx \cong \frac{3h}{8}[f_0 + 3(f_1 + f_2) + f_3]$$

This is known as Basic simpson's $\left(\frac{3}{8}\right)^{th}$ Rule.

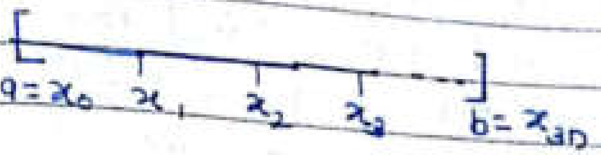
* composite simpson's $\left(\frac{3}{8}\right)^{th}$ Rule-

Let $\int_a^b f(x)\,dx$ is to be evaluated.

Ans = 40 13

consider the interval $[a,b]$ is divided into $3n$ equal subintervals. i.e. interval space, $h = \frac{b-a}{3n}$

i.e.



$$\therefore \int_a^b f(x)\,dx = \int_{x_0}^{x_{3n}} f(x)\,dx$$

$$= \int_{x_0}^{x_3} f(x)\,dx + \int_{x_3}^{x_6} f(x)\,dx + \int_{x_6}^{x_9} f(x)\,dx$$

$$+ \cdots \int_{x_{3n-3}}^{x_{3n}} f(x)\,dx$$

[Now, by Basic simpson's $\left(\frac{3}{8}\right)^{th}$ Rule].

$$= \frac{3h}{8}\left[f_0 + 3f_1 + 3f_2 + f_3\right] + \frac{3h}{8}\left[f_3 + 3f_4 + 3f_5 + f_6\right] + \frac{3h}{8}\left[f_6 + 3f_7 + 3f_8 + f_9\right]$$

$$+ \cdots + \frac{3h}{8}\left[f_{3n-3} + 3f_{3n-2} + 3f_{3n-1} + f_{3n}\right]$$

$$= \frac{3h}{8}\left[f_0 + 3(f_1 + f_2 + f_4 + f_5 + f_7 + f_8 + \cdots + f_{3n-2} + f_{3n-1}) + 2(f_3 + f_6 + f_9 + \cdots + f_{3n-3}) + f_{3n}\right].$$

$$\therefore \int_a^b f(x)\,dx \cong \frac{3h}{8}\left[f_0 + 3(f_1 + f_2 + f_4 + f_5 + \cdots + f_{3n-2} + f_{3n-1}) + 2(f_3 + f_6 + f_9 + \cdots + f_{3n-3}) + f_{3n}\right]$$

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This is known as 'composite Simpson's $\left(\frac{3}{8}\right)^{th}$ Rule'.

Ex-1) A rocket is launched given below a table of accln. and time. Find the velocity after 90 sec. using Simpson's $\left(\frac{3}{8}\right)^{th}$ Rule.

t :	0	10	20	30	40	50	60	70	80	90
a :	30	35	40	50	60	75	90	95	105	120

Where, t = time in sec.

a = accln in m/sec.

Soln- Step I) We have,

accln = rate of change of vel. w.r.t. time

$$\Rightarrow a = \frac{dv}{dt}$$

on integrating w.r.t. t,

$$\Rightarrow v = \int a\,dt$$

$\therefore$ required velocity,

$$v = \int_0^{90} a\,dt$$

$$\simeq \frac{3h}{8}\left[a_0 + 3(a_1 + a_2 + a_4 + a_5 + a_7 + a_8) + 2(a_3 + a_6) + a_9\right]$$

$$= \frac{3 \times 10}{8}\left[30 + 3(35 + 40 + 60 + 75 + 95 + 105 + 2(50 + 90) + 120\right]$$

$= 6225$ m/s. after 90 sec.

2) Evaluate the integration $\int_0^3 \frac{x^3}{1+x^2}\,dx$ using

ln base e

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Simpson's $\left(\frac{3}{8}\right)^{th}$ rule with 6 strips (subdivisions). ~~Hence~~ ~~find~~ $\cancel{\ln(28)^{1/3}}$.

Soln. Given, $f(x) = \frac{x^3}{1+x^2}$ with 6 strips.

$$h = \frac{3-0}{6} = 0.5$$

Here, $[0,3]$ is divided into 6 sub-intervals as,

$$[0 \quad 0.5 \quad 1.0 \quad 1.5 \quad 2.0 \quad 2.5 \quad 3]$$

$$f_0 = f(0) = \frac{0^3}{1+0^2} = 0$$

$f_1 = f(0.5) =$ ~~0.25~~ 0.1

$f_2 = f(1.0) = 0.5$

$f_3 = f(1.5) = 1.0385$

$f_4 = f(2.0) = 1.6$

$f_5 = f(2.5) = 2.1552$

$f_6 = f(3.0) = 2.7$

$\therefore$ By composite simpson's $\left(\frac{3}{8}\right)^{th}$ Rule,

$$\int_0^3 \frac{x^3}{1+x^2}\,dx \simeq \frac{3h}{8}\left[f_0 + 3(f_1+f_2+f_4+f_5) + 2(f_3) + f_6\right]$$

$$= \frac{3\times0.5}{8}\left[0+3(0.1+0.5+1.6+2.1552)+2\times1.0385+2.7\right]$$

$$= 3.3455$$

$$\therefore \int_0^3 \frac{x^3}{1+x^2}\,dx \simeq 3.3455 \quad \text{---(1)}$$

Now, we have,

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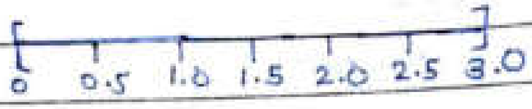
$$\int_0^3 \cancel{\frac{x^3}{1+x^2}}\, dx =$$

3) Evaluate, $\int_0^3 \frac{x^2}{1+x^3}\, dx$ using simpson's $\left(\frac{3}{8}\right)^{th}$ rule with 6 strips & hence find $\ln(28)^{1/3}$

Given, $f(x) = \frac{x^2}{1+x^3}$ with 6 strips.

$$h = \frac{3-0}{6} = 0.5$$

Here, $[0,3]$ is divided into 6 sub-intervals as,



$\therefore f_0 = f(0) = \frac{0^2}{1+0^3} = 0$

$f_1 = f(0.5) = 0.2222$

$f_2 = f(1.0) = 0.5$

$f_3 = f(1.5) = 0.5143$

$f_4 = f(2.0) = 0.4449$

$f_5 = f(2.5) = 0.3759$

$f_6 = f(3.0) = 0.3214$

$\therefore$ By using simpson's $\left(\frac{3}{8}\right)^{th}$ rule,

$$\int_0^3 \frac{x^2}{1+x^3}\, dx \cong \int \frac{3h}{8}\left[f_0 + 3(f_1+f_2+f_4+f_5) + 2f_3 + f_6\right]$$

$$= \frac{3\times 0.5}{8}\left[0 + 3(0.2222 + 0.5 + 0.4449 + 0.3759) + 2\times 0.5143 + 0.3214\right]$$

$$\cong 1.1208 \quad \cdots (1)$$

Now,

$$\int_0^3 \frac{x^2}{1+x^3}\,dx = \frac{1}{3}\int_0^3 \frac{3x^2}{1+x^3}\,dx$$

$$= \frac{1}{3}\left[\log(1+x^3)\right]_0^3$$

$$= \frac{1}{3}\left[\log 28 - \log 1\right]$$

$$= \frac{1}{3}\log(28)$$

$$= \frac{1}{3}\ln(28)$$

$$= \ln(28)^{1/3} \quad \cdots (ii)$$

By (i) & (ii),

$$\ln(28)^{1/3} \cong 1.1208$$

4) Use composite simpson's $\left(\frac{3}{8}\right)^{th}$ rule used at 7 equidistant points to evaluate the int. $\int_0^{\pi} \frac{\sin x}{x}\,dx$

Soln. Given, $f(x) = \frac{\sin x}{x}$ with 7 equidistant points

$$\therefore h = \frac{\pi - 0}{6} = \frac{\pi}{6} = 0.5233$$

Thus, $[0,\pi]$ is sub-divided into 6 parts as,

$$[\,0,\ \pi/6,\ \pi/3,\ \pi/2,\ 2\pi/6,\ 5\pi/6,\ \pi\,]$$

$f_0 = f(0) = 0$

$f_1 = f(\pi/6) = 0.9549$

$f_2 = 0.8269$

$f_3 = 0.6366$

$f_4 = 0.4135$

$f_5 = 0.1909$

$f_6 = 0$

By composite simpson's $\left(\frac{3}{8}\right)^{th}$ rule,

$$\int_0^{\pi} \frac{\sin x}{x}\,dx \cong \frac{3h}{8}\left[f_0 + 3(f_1+f_2+f_4+f_5)+2f_3+f_6\right]$$

$$= \frac{3(0.5233)}{8}\left[1+3(0.9549+0.8269+0.4135+0.1909)+2\times 0.6365\right]$$

$$\cong 1.8520.$$

1.6546

5) Evaluate the integral $\int_0^1 \frac{dx}{1+x}$ by simpson's $\left(\frac{3}{8}\right)^{th}$ rule with 9 divisions & hence find $\ln(2)$ Also compair it with accurate value.

Given that,

$f(x) = \frac{1}{1+x}$ with 9 divisions.

$\therefore \quad h = \frac{1}{9} = 0.1111$

Thus $[0,1]$ is sub divided into 9 parts as

[0, 0.1111, 0.2222, 0.3333, 0.4444, 0.5555, 0.6666, 0.7777, 0.8888, 0.9999 ≅ 1]

so

$f_0 = f(0) = 1$

$f_1 = 0.9$

$f_2 = 0.8182$

$f_3 = 0.75$

$f_4 = 0.6923$

$f_5 = 0.6429$

$f_6 = 0.6$

$f_7 = 0.5625$

$f_8 = 0.5294$

$f_9 = 0.5$

By simpson's $\left(\frac{3}{8}\right)^{th}$ rule,

$$\int_0^1 \frac{dx}{1+x} \simeq \frac{3h}{8}\left[f_0+3(f_1+f_2+f_4+f_5+f_7+f_8) + 2(f_3+f_6)+f_9\right]$$

$$= \frac{3(0.1111)}{8}\left[1+3(0.9+0.8182+0.6923+0.6429+0.5625+0.5294)+2(0.75+0.6)+0.5\right]$$

$$\cong 0.6930 \quad \cdots (I)$$

Also we have

$$\int_0^1 \frac{dx}{1+x} = \left[\log_e(1+x)\right]_0^1$$

$$= \log_e 2 - \log_e 1$$

$$= \log_e 2 \quad \cdots (II)$$

$$\cong 0.6930$$

By (I) & (II)

Soln Given that,

$f(x) = \frac{1}{1+x}$ with 9 divisions.

$\ln(2) \cong 0.6930 \quad \cdots (III)$

Also $\ln(2) = 0.6931$

$\therefore$ Error $= |\text{Appr. value of } \ln(2) - \ln(2)|$

$$= |0.6930 - 0.6931|$$

$$= 0.0001.$$

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6) Using simpson's $\left(\frac{3}{8}\right)^{th}$ rule calculate the value of $\int_4^{5.2} \log_e x \, dx$ with 6 strips compair it with exact value.

Soln Given that,

$$f(x) = \log_e x.$$

$$\therefore h = \frac{5.2-4}{6} = 0.2$$

4, 4.2, 4.4, 4.6, 4.8, 5.0, 5.2

$\therefore$ $f_0 = f(4) = 1.3863$

$f_1 = f(4.2) = 1.4351$

$f_2 = f(4.4) = 1.4816$

$f_3 = f(4.6) = 1.5261$

$f_4 = f(4.8) = 1.5686$

$f_5 = f(5) = 1.6094$

$f_6 = f(5.2) = 1.6487$

By using simpson's $\left(\frac{3}{8}\right)^{th}$ rule,

$$\int_4^{5.2} \log_e x \, dx \simeq \frac{3h}{8}\left[f_0 + 3(f_1 + f_2 + f_4 + f_5) + 2(f_3) + f_6\right]$$

$$= \frac{3 \times 0.2}{8}\left[1.3863 + 3(1.4351 + 1.4816 + 1.5686 + 1.6094) + 2 \times 1.5261 + 1.6487\right]$$

3.3730

1.5452

$$\simeq 1.8278$$

Also we have,

$$\int_4^{5.2} \log_e x \, dx = \left[x \log_e x - x\right]_4^{5.2}$$

$$= 1.8278$$

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$\therefore$ Error $= |A_p$

$= |1.8278 - 1.8278|$

$= 0$

7) Evaluate $\int_0^1 \frac{dx}{1+x^2}$ using simpson's $(\frac{1}{3})^{rd}$ & $(\frac{3}{8})^{th}$ rule hence obtain the approximate value of π in each case.

Given, $f(x) = \frac{1}{1+x^2}$, $h = \frac{1-0}{6} = 0.1667$

Here,

0, 0.1667, 0.3334, 0.5001, 0.6668, 0.8335, 1.0002

$\therefore f_0 = f(0) = 1$

$f_1 = 0.9730$

$f_2 = 0.9000$

$f_3 = 0.7999$

$f_4 = 0.6922$

$f_5 = 0.5901$

$f_6 = 0.5$

Now By using simpson's $(\frac{3}{8})^{th}$ rule,

$$\int_0^1 \frac{dx}{1+x^2} \simeq \frac{3h}{8}[f_0 + 3(f_1 + f_2 + f_4 + f_5) + 2f_3 + f_6]$$

$$= \frac{3 \times 0.1667}{8}[1 + 3(0.9730 + 0.9000 + 0.6922 + 0.5901) + 2 \times 0.7999 + 0.5]$$

$\simeq 0.7855$ .... (1)

Also we have,

$$\int_0^1 \frac{1}{1+x^2} dx = [\tan^{-1} x]_0^1$$

$$= [\tan^{-1}(1) - \tan^{-1}(0)]$$

$$= \pi/4 - 0$$
$$= \pi/4 \quad \cdots (II)$$
$\therefore$ from (I) & (II),
$$0.7855 = \frac{\pi}{4}$$
$$\therefore \boxed{\pi = 3.142}$$

Now,
By using simpson's $(\frac{1}{3})^{rd}$ rule,
$$\int_0^1 \frac{dx}{1+x^2} \cong \frac{h}{3}\left[f_0 + 4(f_1+f_3+f_5) + 2(f_2+f_4) + f_6\right]$$
$$= \frac{0.1667}{3}\left[1 + 4(0.9730 + 0.7999 + 0.5901) + 2(0.9000 + 0.6922) + 0.5\right]$$
$$\cong 0.7855 \quad \cdots (I)$$
Also we have,
$$\int_0^1 \frac{1}{1+x^2}\,dx = \left[\tan^{-1}x\right]_0^1$$
$$= \tan^{-1}(1) - \tan^{-1}(0)$$
$$= \pi/4 - 0$$
$$= \pi/4 \quad \cdots (II)$$
$\therefore$ from (I) & (II),
$$0.7855 = \pi/4$$
$$\therefore \boxed{\pi = 3.142}$$

Note -
i) Trapezoidal rule is used for any number of subintervals
ii) simpson's $(\frac{1}{3})^{rd}$ rule is used only when subintervals are multiple of 2.
iii) simpson's $(\frac{3}{8})^{th}$ rule is used only when subintervals are multiple of 3.

Ans = 40   15   13   B   9
C   12

Unit 4 - ORDINARY DIFFERENTIAL EQUATION

* Euler's Method -

Suppose we wish to solve differential eqn. of 1st order,

$$\frac{dy}{dx} = f(x,y) \quad \text{--- (1)}$$

For values of y at

$$x = x_0 + ih \qquad (i = 1, 2, 3, \ldots)$$

integrating eqn. (1) w.r.t. x, we obtained,

$$y_1 - y_0 = \int_{x_0}^{x_1} f(x,y)\,dx$$

$$\Rightarrow \quad y_1 = y_0 + \int_{x_0}^{x_1} f(x,y)\,dx$$

Assuming that, $f(x,y) = f(x_0, y_0)$

i.e $x_0 \le x \le x_1$

this gives Euler's formula

$$y_1 \cong y_0 + h f(x_0, y_0)$$

similarly for the range $x_1 \le x \le x_2$

We have,

$$y_2 = y_1 + \int_{x_1}^{x_2} f(x,y)\,dx$$

substituting, $f(x,y) = f(x_1, y_1)$

$$y_2 \cong y_1 + h f(x_1, y_1)$$

Proceeding in this way we obtained general formula,

$$y_{n+1} \cong y_n + h f(x_n, y_n) \quad , n = 0, 1, 2, \ldots$$

this is known as 'Euler's ~~theorem~~ form'.

1) solve the differential eqn. $\frac{dy}{dx} = \frac{x+y}{(y^2-\sqrt{xy})}$ using Euler's method under the boundary conditions $x=1.3$ & $y=2$ find y at $x=1.8$ in 5 steps.

Soln- Given that,

$$\frac{dy}{dx} = \frac{x+y}{y^2-\sqrt{xy}}$$

with initial conditions,

$$x_0 = 1.3 \quad \& \quad y_0 = 2$$

Here, $h = \frac{1.8-1.3}{5} = 0.1$

step I) By Euler's method we have,

$$y_{n+1} \cong y_n + h f(x_n, y_n) \qquad (n=0,1,2,\ldots)$$

$$\therefore y_1 \cong y_0 + h f(x_0, y_0)$$

$$= 2 + (0.1)\left[\frac{x_0+y_0}{y_0^2-\sqrt{x_0 y_0}}\right]$$

$$= 2 + (0.1)[1.3822]$$

$$\therefore y_1 \cong 2.1382 \quad \text{if} \quad x = 1.4$$

step II) We have,

$$y_2 \cong y_1 + h f(x_1, y_1)$$

Here, $y_1 = 2.1382$, $x_1 = x_0 + h = 1.4$

$$\therefore y_2 \cong 2.1382 + (0.1)[1.2451]$$

$$= 2.2627$$

$$\therefore y_2 \cong 2.2627 \quad \text{if} \quad x = 1.5$$

step III) Here,

$$y_2 = 2.2627, \quad x_2 = 1.5$$

We have

$y_3 \cong y_2 + h\,f(x_2, y_2)$

$= 2.2627 + (0.1)[1.1480]$

$= 2.3775$

$\therefore y_3 \cong 2.3775$ if $x = 1.5$ 1.6

Step IV)

Here, $y_3 = 2.3775$, $x_3 = 1.6$

Now we have,

$y_4 \cong y_3 + h\,f(x_3, y_3)$

$= 2.3775 + (0.1)(1.0744)$

$\therefore y_4 \cong 2.4849$ if $x = \cancel{1.6}$ 1.7

Step V)

Here, $y_4 = 2.4849$, $x_4 = 1.7$

$\therefore y_5 \cong y_4 + h\,f(x_4, y_4)$

$= 2.4849 + (0.1)(1.0159)$

$\therefore y_5 \cong 2.5865$ if $x. = \cancel{1.7}$ 1.8

Step VI)

Here, $y_5 = 2.5865$, $x_5 = 1.8$

$\therefore y_6 \cong y_5 + h\,f(x_5, y_5)$

$= 2.5865 + (0.1)(0.9678)$

$\therefore y_6 \cong 2.6833$ if $x = 1.8$

Thus, P.I particular soln. of diff. eqn. at $x = 1.8$ is ~~approx~~, $y \cong 2.6833$.

2) Solve the following diff. eqn. using Euler's method for given boundry condn. $x = 1$ & $y = 2.2$ find the value of y at $x = 1.2$ with 2 steps. $\dfrac{dy}{dx} = \sqrt{x+y}$

Given that,

$$\frac{dy}{dx} = \sqrt{x+y}$$

with initial conditions

$x_0 = 1$ & $y_0 = 2.2$

$\therefore h = \frac{1.2-1}{2} = 0.1$

Step I)

By Euler's method we have,

$y_{n+1} \cong y_n + h\, f(x_n, y_n)$ $(n=0,1,2,\ldots$

$\therefore y_1 \cong y_0 + h\, f(x_0, y_0)$

$= 2.2 + (0.1)\sqrt{1+2.2}$

$\therefore y_1 \cong 2.3789$ if $x = 1.1$

Step II)

We have,

$y_2 \cong y_1 + h\, f(x_1, y_1)$

Here, $y_1 = 2.3789$, $x_1 = 1.1$

$\therefore y_2 = 2.3789 + (0.1).\sqrt{1.1+2.3789}$

$\therefore y_2 \cong 2.5654$ if $x = 1.1$

Step III) We have

$y_3 \cong y_2 + h.f(x_2, y_2)$

Here, $y_2 = 2.5654$, $x_2 = 1.2$

$\therefore y_3 = 2.5654 + (0.1)\sqrt{1.2+2.5654}$

$\therefore y_3 \cong 2.7594$ if $x = 1.2$

Thus particular soln. of differential eqn. at $x = 1.2$ is $y \cong 2.7594$

3) Solve $\frac{dy}{dx} = \frac{x^2}{2y}$ ~~at~~ by Euler's method with $y(0) = 1.2$

take $h = 0.4$ & find y at $x = 0.8$.

Soln – Given that,

$$\frac{dy}{dx} = \frac{x^2}{2y}$$

with initial conditions

$x_0 = 0$, $y_0 = 1.2$

$\therefore$ ~~$h = \frac{0.8-0}{}$~~ $\therefore h = 0.4$

Step I) - By Euler's method we have,

$$y_{n+1} \cong y_n + h.f(x_n, y_n) \quad (n = 0, 1, 2, \ldots)$$

$$\therefore y_1 \cong y_0 + h\, f(x_0, y_0)$$

$$= 1.2 + (0.4)\left(\frac{0^2}{2 \times 1.2}\right)$$

$$= 1.2 + 0$$

$\therefore y_1 \cong 1.2$ if $x = 0.4$

Step II) We have

$$y_2 \cong y_1 + h\, f(x_1, y_1)$$

Here, $y_1 = 1.2$, $x_1 = 0.4$

$$\therefore y_2 = 1.2 + (0.4)\left(\frac{(0.4)^2}{2 \times 1.2}\right)$$

$\therefore y_2 \cong 1.2267$ if $x = 0.8$

Step III) We have,

$$y_3 \cong y_2 + h.f(x_2, y_2)$$

Here, $y_2 = 1.2267$, $x_2 = 0.8$

$$\therefore y_3 = 1.2267 + (0.4)\left(\frac{(0.8)^2}{2 \times 1.2267}\right)$$

$\therefore y_3 \cong 1.4354$ if $x = 0.8$

Thus particular soln. of differential eqn. at $x=0.8$ is $y \cong 1.4354$.

4) Solve the following diff. eqn. using Euler's method for the given boundry condn. find value

$$\frac{dy}{dx} = \frac{x}{4} + y \quad \text{with } y(1) = 0.1$$

of y at $x = 1.4$ take $h = 0.1$

soln- Given that,

$$\frac{dy}{dx} = \frac{x}{4} + y$$

with initial condn.

$x_0 = 1$ & $y_0 = 0.1$, $h = 0.1$

Step I) By Euler's method we have

$$y_{n+1} \cong y_n + h.f(x_n, y_n)$$

$$\therefore y_1 \cong y_0 + h.f(x_0, y_0)$$
$$= 0.1 + (0.1)(0.35)$$
$$\therefore y_1 \cong 0.135 \quad \text{if} \quad x = 1$$

Step II) We have,

$$y_2 \cong y_1 + hf(x_1, y_1)$$

Here, $y_1 = 0.135$, $x_1 = 1.1$

$$\therefore y_2 = 0.135 + (0.1)($$
$$\therefore y_2 \cong \cancel{0.41} \; 0.176 \quad \text{if} \quad x_i = 1.1$$

Step III) We have

$$y_3 \cong y_2 + h.f(x_2, y_2)$$

Here, $y_2 = \cancel{0.41} \; 0.176$, $x_2 = 1.2$

$$\therefore y_3 \cong 0.176 + (0.1)(0.476)$$

$\therefore y_3 \cong 0.2236$ if $x_2 = 1.2$

Step IV) We have,

$$y_4 \cong y_3 + h f(x_3, y_3)$$

Here, $y_3 = 0.2236$, $x_3 = 1.3$

$\therefore y_4 = 0.2236 + (0.1)(0.5486)$

$\therefore y_4 \cong 0.2785$ if $x_3 = 1.3$

Step V) We have,

$$y_5 \cong y_4 + h \cdot f(x_4, y_4)$$

Here, $y_4 = 0.2785$, $x_4 = 1.4$

$\therefore y_5 = 0.2785 + (0.1)(0.6285)$

$\therefore y_5 \cong 0.3414$ if $x = 1.4$

Thus particular soln. of diff. eqn. at $x = 1.4$ is $y \cong 0.3414$.

* Runge - Kutta Method

1) 2nd order R-K method

Formula -

Let $\frac{dy}{dx} = f(x, y)$

with initial condition $y(x_0) = y_0$ be solved at equispaced points $x_0, x_1, x_2, \ldots, x_n$ with spacing h.

soln. of above diff. eqn. is

$$y_{i+1} \cong y_i + \frac{1}{2}(k+l) \quad \text{for } i = 0, 1, 2, \ldots$$

where, $k = h f(x_i, y_i)$

and $l = h f(x_i + h, y_i + k)$

1) Two steps of 2nd order R-K method for the diff. eqn $\frac{dy}{dx} = xy + y^2$ with initial condn. $y(1) = 2$ take $h = 0.1$

Given that, $f(x, y) = \frac{dy}{dx} = xy + y^2$

with initial condn.

$x_0 = 1$, $y_0 = 2$, $h = 0.1$

Also $f(x, y) = xy + y^2$

By 2nd order R-K method we have

$$y_{i+1} \cong y_i + \frac{1}{2}(k + l)$$

where $k = hf(x_i, y_i)$

$l = hf(x_i + h, y_i + k)$

Step I)

$$y_1 \cong y_0 + \frac{1}{2}(k + l)$$

Here, $k = hf(x_0, y_0)$

$= h(x_0 y_0 + y_0^2)$

$= 0.1 (6)$

$= 0.6$

similarly,

$l = hf(x_0 + h, y_0 + k)$

$= (0.1) f(1.1, 2.6)$

$= (0.1)[(1.1)(2.6) + (2.6)^2]$

$= 0.962$

$$\therefore y_1 \cong 2 + \frac{1}{2}(0.6 + 0.962)$$

$\Rightarrow y_1 \cong 2.781$ at $x = 0$

Step II) $y_2 \cong y_1 + \frac{1}{2}(k + l)$

Here, $k = hf(x_1, y_1)$ $\quad \because x_1 = x_0 + h = 1.1$

$= (0.1)(x_1 y_1 + y_1^2)$

$= (0.1)[(1.1)(2.781) + (2.781)^2]$

$= 1.0793$

Now

$l = hf(x_1 + h, y_1 + k)$

$= (0.1) f(1.2, 3.860)$

$= (0.1)(1.2 \times 3.860, (3.860)^2)$

$= 1.9534$

$\therefore y_2 \cong 2.781 + \frac{1}{2}(1.0793 + 1.9534)$

$\Rightarrow y_2 \cong 4.2974$ at $x = 1.1$

$\therefore$ Particular soln. of given diff. eqn. at $x = 1.1$ is $y \cong 4.2974$.

2) compute approximation to $y(2.1)$ & $y(2.2)$ as soln. of differential eqn. $\frac{dy}{dx} = x^2 + y^2$ with $y(2) = 3$.

Soln.- Given that,

$$\frac{dy}{dx} = x^2 + y^2$$

with initial condition

$x_0 = 2, \quad y_0 = 3$

Here, $f(x, y) = x^2 + y^2$

We take, $h = 0.1$

By 2nd order R-K method we have,

$$y_{i+1} \cong y_i + \frac{1}{2}(k + l)$$

where, $k = h \cdot f(x_i, y_i)$

and $l = h \cdot f(x_i + h, y_i + k)$

step I)

$$y_1 \cong y_0 + \frac{1}{2}(k+l)$$

Here, $k = h f(x_0, y_0)$

$= (0.1)(13)$

$= 1.3$

||ly,

$l = h f(x_0 + h, y_0 + k)$

$= (0.1) f(2.1, 4.3)$

$= (0.1)(2.1^2 + 4.3^2)$

$= 2.29$

$$\therefore y_1 \cong 3 + \frac{1}{2}(1.3 + 2.29)$$

$$\Rightarrow y_1 \cong 4.795$$

step II)

$$y_2 \cong y_1 + \frac{1}{2}(k+l)$$

Here, $k = h f(x_1, y_1)$

$= (0.1) f(2.1, 4.795)$

$= 2.7402$

||ly,

$l = h f(x_1 + h, y_1 + k)$

$= (0.1)(2.1 + 0.1, 4.795 + 2.7402)$

$= (0.1)(2.2^2 + 7.5352^2)$

$= 6.1619$

$$\therefore y_2 \cong 4.795 + \frac{1}{2}(2.7402 + 6.1619)$$

$$\Rightarrow y_2 \cong 9.2461 \quad \text{at } x = 2.1$$

step III) $y_3 \cong y_2 + \frac{1}{2}(k+l)$

4) Solve $\frac{dy}{dx} = \frac{x^2}{2y}$ given $x_0 = 0$, $y_0 = 1.2$ find y at $x = 0.4$ $h = 0.1$

5) Solve $\frac{dy}{dx} = 2x^3 + 12x^2 - 20x + 8.5$ given $y(0) = 1$ find $y(0.5)$ by 2nd order R-K method $h = 0.1$

Here, $k = h f(x_2, y_2)$, $x_2 = x_1 + h = 2.2$

$= (0.1)(2.2$

$k = 9.0330$

$||^{ly}$,

$l = h f(x_2 + h, y_2 + k)$

$= (0.1) f(2.2 + 0.1, 9.2461 + 9.0330)$

$= (0.1)(2.3^2 + 18.2791^2)$

$= 33.9415$

$\therefore y_3 \cong 9.2461 + \frac{1}{2}(9.0330 + 33.9415)$

$\Rightarrow y_3 \cong 30.7334$ at $x = 2.2$.

$\therefore$

3) $\frac{dy}{dx} = x + y$ given that $y(0) = 1$ Take $h = 0.1$

find $y(0.3)$ by 2nd order R-K method.

Soln- Given that,

$$\frac{dy}{dx} = x + y$$

with initial condn.

$x_0 = 0$, $y_0 = 1$, $h = 0.1$

By 2nd order R-K method we have,

$$y_{i+1} \cong y_i + \frac{1}{2}(k + l)$$

where, $k = h.f(x_i, y_i)$

$l = h.f(x_i + h, y_i + k)$

Step I)

$$y_1 \cong y_0 + \frac{1}{2}(k + l)$$

Here, $k = h.f(x_0, y_0)$

$= (0.1)(0 + 1)$

$= 0.1$

$||^{ly}$. $l = h.f(x_0 + h, y_0 + k)$

$= h.f(0.1, 1.1)$

$= (0.1)(0.1 + 1.1)$

$l = 0.12$

$\therefore y_1 \cong 1 + \frac{1}{2}(0.1 + 0.12)$

$\Rightarrow y_1 \cong 1.11$ $x = 0.1$

Step II)

$y_2 \cong y_1 + \frac{1}{2}(k + l)$

Here, $k = h.f(x_1, y_1)$

$= (0.1)(0.1 + 1.11)$

$\therefore k = 0.121$

||'y. $l = h.f(x_1 + h, y_1 + k)$

$= (0.1)(0.2 + 1.231)$

$\therefore l = 0.1431$

$\therefore y_2 \cong 1.11 + \frac{1}{2}(0.121 + 0.1431)$

$\Rightarrow y_2 \cong 1.2421$ at $x = 0.1$

Step III)

$y_3 \cong y_2 + \frac{1}{2}(k + l)$

Here, $k = hf(x_2, y_2)$

$= (0.1)(0.2 + 1.2421)$

$\therefore k = 0.1442$

||'y. $l = h.f(x_2 + h, y_2 + k)$

$= (0.1)(0.3 + 1.3863)$

$\therefore l = 0.1686$

$\therefore y_3 \cong \cancel{y_2}\ 1.2421 + \frac{1}{2}(0.1442 + 0.1686)$

$\Rightarrow y_3 \cong 1.3985$ at $x = 0.2$

$h = 0.2$ 4) solve $\frac{dy}{dx} = \frac{x^2}{2y}$ given $x_0 = 0$, $y_0 = 1.2$ find y at $x = 0.4$.

1.2313

Sol^n. Given that,

$$\frac{dy}{dx} = \frac{x^2}{2y}$$

with initial condⁿ.

$x_0 = 0,\ y_0 = 1.2,\ h = 0.4$

step I)

$y_1 \cong y_0 + \frac{1}{2}(k+l)$

Here, $k = h.f(x_0, y_0)$

$= (0.4) f(0, 1.2)$

$= (0.4)(0)$

$\therefore k = 0$

ll'y. $l = h.f(x_0+h, y_0+k)$

$= (0.4) f(0.4, 1.2)$

$= (0.4)(0.06667)$

$\therefore l = 0.02667$

$\therefore y_1 \cong 1.2 + \frac{1}{2}(0 + 0.02667)$

$\Rightarrow y_1 \cong 1.2133$

5) solve $\frac{dy}{dx} = \cancel{2x^3 + 12x^2 - 20x + 8.5}$

given $y(0) = 1$ find $y(0.5)$ by 2nd order RK method

soln Given that, $\frac{dy}{dx} = \cancel{2x^3 + 12x^2 - 20x + 8.5}$

with initial condⁿ.

$x_0 = 0,\ y_0 = 1,\ h = 0.5$

step I) $y_1 \cong y_0 + \frac{1}{2}(k+l)$

Here, $k = hf(x_0, y_0)$ ll'y $l = h.f(x_0+h, y_0+k)$

$= (0.5)(\cancel{8.5})(0.1)$ $= (0.1) f(\cancel{0.1, 1.85})(0.1)$

$\therefore k = \cancel{0.85}\ 4.25$ $\therefore l = (0.1)(6.622) = 0.6622$

$\therefore y_1 \cong 1 + \frac{1}{2}(0.85 + 0.6622) \cong 1.7561$ at $x = 0$

step II) $y_2 \cong y_1 + \frac{1}{2}(k+l)$

Here, $k = hf(x_1, y_1)$

$= (0.1)(0.1, 1.7561)$

$\therefore k = 0.6622$

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step I) $y_1 \cong y_0 + \frac{1}{2}(k+l)$

$k = hf(x_0, y_0)$

$= (0.5) f(0,1)$

$= 4.25$

$l = hf(x_0+h, y_0+k)$

$= (0.5) f(0+0.5, 1+4.25)$

$= (0.5) f(0.5, 5.25)$

$l = 0.875$

$\therefore \ y_1 \cong 1 + \frac{1}{2}(4.25 + 0.875)$

$y_1 \cong 3.5625$ at $x = 0.5$

step II)

$y_2 \cong y_1 + \frac{1}{2}(k+l)$

$k = h f(x_1, y_1)$ $\quad x_1 = x_0 + h = 0.5$

$= (0.5) f(0.5, 3.5625)$

$\therefore k = 0.875$

$l = hf(x_1+h, y_1+k)$

$= (0.5) f(0.5+0.5, 3.5625+0.875)$

$= (0.5) f(1, 4.4375)$

$l = 1.25$

$y_2 \cong 3.5625 + \frac{1}{2}(0.875 + 1.25)$

$\cong 4.625$ at $x = \cancel{0.5}\,1$

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* 4^{th} order R-K Method -

Let $\frac{dy}{dx} = f(x, y)$

with initial condition $y(x_0) = y_0$

be solve at equispace points $x_0, x_1, \ldots, x_n$

with spacing h.

soln. of above differential eqn. is,

$$y_{i+1} \cong y_i + \frac{1}{6}[k + 2l + 2m + p]$$

for $i = 0, 1, 2, \ldots, n$.

where, $k = h f(x_i, y_i)$

$l = h \cdot f\left(x_i + \frac{h}{2}, y_i + \frac{k}{2}\right)$

$m = h f\left(x_i + \frac{h}{2}, y_i + \frac{l}{2}\right)$

$p = h f(x_i + h, y_i + m)$

Ex 1) solve the initial value problem $\frac{dy}{dx} = xy^2 + e^x$ with $y(1) = 4$ using 4^{th} order R-K method with spacing $h = 0.1$ do two steps.

Soln - Given that,

$$\frac{dy}{dx} = xy^2 + e^x$$

with initial condn.

$x_0 = 1$, $y_0 = 4$, $h = 0.1$

Here, $f(x, y) = xy^2 + e^x$

By 4^{th} order R-K method we have,

$$y_{i+1} \cong y_i + \frac{1}{6}[k + 2l + 2m + p]$$

where, $k = h f(x_i, y_i)$

$l = h f\left(x_i + \frac{h}{2}, y_i + \frac{k}{2}\right)$

$m = h f\left(x_i + \frac{h}{2}, y_i + \frac{l}{2}\right)$

$p = h f(x_i + h, y_i + m)$

Step I

$$y_1 \cong y_0 + \frac{1}{6}[k + 2l + 2m + p]$$

Here, $k = (0.1)[x_0 y_0^2 + e^{x_0}]$

$= (0.1)[16 + 2.71$

$= 1.8718$

$l = h(0.1)[1.05, 4.9859]$

$= (0.1)[(1.05)(4.9359)^2 + e^{(1.05)}]$

$= 2.8439$.

$m = (0.1)[1.05, 5.4220]$

$= (0.1)[(1.05)(5.4220)^2 + e^{(1.05)}]$

$= 3.3725$

$p = (0.1)[1.1, 7.3725]$

$= 6.2795$

$\therefore y_1 \cong 4 + \frac{1}{6}[1.8718 + 2\times 2.8439 + 2\times 3.3725 + 6.2795]$

$\therefore y_1 \cong 7.4307$ at $x = 1.1$

Step II) $y_2 \cong y_1 + \frac{1}{6}[k + 2l + 2m + p]$

Here, $k = h.f[x_1, y_1]$, $x_1 = x_0 + h$

$= (0.1)[(1.1)(7.4307)^2 + e^{(1.1)}]$

$\therefore k = 6.3741$

$l = h.f(x_1 + \frac{h}{2}, y_1 + \frac{k}{2})$

$= (0.1)[(1.15)(10.6178)^2 + e^{(1.15)}]$

$\therefore l = 13.\cancel{4180}$ 2807

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③

$m = (0.1)\,[(1.15)(14.0711)^2 + e^{(1.15)}]$
$= 23.0853$

$p = (0.1)\,[(1.2)(30.516)^2 + e^{(1.2)}]$
$= 112.0792$

$\therefore y_2 \simeq 7.4307 + \frac{1}{6}\,[6.3741 + 2\times 13.2807 + 2\times 23.0853) + 112.0792]$

$\therefore y_2 \simeq 39.2949$. at $x = 1.2$

2) solve $\frac{dy}{dx} = \sqrt{x+y}$ for the given boundry conditions at $x_0 = 1$, & $y_0 = 2.2$ find y at $x = 1.1$ $(h = 0.1)$ by 4th order R-K method.

3) solve $\frac{dy}{dx} = 10 + y^2$ for the given boundry conditions that $y_0 = 0$ $y(0) = 0$ find y at $x = 0.5$ $(h = 0.5)$ by 4th order R-K method.

4) solve, $\frac{dy}{dx} = \log(x+y)$ with initial condn. $y(1) = 2$ bind value of y at $x = 1.4$ $(h = 0.2)$ By 4th order R-K method.

5) solve $\frac{dy}{dx} = 4 + 3xy$ with $y(0) = 0$ bind value of y at $x = 0.2$ take $h = 0.1$ by 4th order R-K method.

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2) Given that,

$$\frac{dy}{dx} = \sqrt{x+y}$$

with initial condn.

$x_0 = 1,\ y_0 = 2.2,\ h = 0.1$

By 4th order R-K method we have

$$y_{i+1} \simeq y_i + \frac{1}{6}[k + 2l + 2m + p]$$

step I)

$$y_1 \simeq y_0 + \frac{1}{6}[k+2l+2m+p]$$

Here, $k = h f(x_0, y_0)$

$= (0.1)[\sqrt{1+2.2}]$

$\therefore k = 0.1789$

$l = h f \cancel{(x_0, y_0)} (x_0 + \frac{h}{2}, y_0 + \frac{k}{2})$

$= (0.1) f(1.05, 2.2895)$

$\therefore l = 0.1827$

$m = h \cdot f(x_0 + \frac{h}{2}, y_0 + \frac{l}{2})$

$= (0.1) f(1.05, 2.2914)$

$\therefore m = 0.1828$

$p = h f(x_0 + h, y_0 + m)$

$= (0.1) f(1.1, 2.3828)$

$\therefore p = 0.1866$

$\therefore y_1 \simeq 2.2 + \frac{1}{6}[0.1789 + 2\times 0.1827$

$+ 2\times 0.1828 + 0.1866]$

$\therefore y_1 \simeq 2.3828$ at $x = 1.1$

step II) $y_2 \simeq y_1 + \frac{1}{6}[k + 2l + 2m + p]$

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Here, $k = h\,f(x_1, y_1)$
$= (0.1)\,f(1.1,\ 2.3828)$
$\therefore k = 0.1866$

$l = h\,f\left(x_1 + \frac{h}{2},\ y_1 + \frac{k}{2}\right)$
$= (0.1)\,f(1.15,\ 2.4761)$
$\therefore l = 0.1904$

$m = h\,f\left(x_1 + \frac{h}{2},\ y_1 + \frac{l}{2}\right)$
$= (0.1)\,f(1.15,\ 2.478)$
$\therefore m = 0.1905$

$p = h\,f(x_1 + h,\ y_1 + m)$
$= (0.1)\,f(1.2,\ 2.5733)$
$\therefore p = 0.1942$

$\therefore\ y_2 \cong y_1 + \frac{1}{6}[k + 2l + 2m + p]$

$\therefore\ y_2 \cong 2.5732$ at $x = 1.1$

3) Given that,

$$\frac{dy}{dx} = 10 + y^2$$

with initial condition

$x_0 = 0$, $y_0 = 0$, $h = 0.5$

By 4th order R-K method we have,

$$y_{i+1} \cong y_i + \frac{1}{6}[k + 2l + 2m + p]$$

step I)

$$y_1 \cong y_0 + \frac{1}{6}[k + 2l + 2m + p]$$

Here, $k = h\,f(x_0, y_0)$
$= (0.5)\,f(0, 0)$
$\therefore k = 5$

⑥

$l = h.f(x_0 + \frac{h}{2}, y_0 + \frac{k}{2})$

$= (0.5).f(0 + \frac{0.5}{2}, 0 + \frac{5}{2})$

$\therefore l = 8.125$

$m = (0.5) f(\frac{0.5}{2}, \frac{8.125}{2})$

$\therefore m = 13.252$

$p = (0.5) f(0 + 0.5, 0 + 13.252)$

$\therefore p = 92.8078$

$\therefore y_1 \cong 0 + \frac{1}{6}[5 + 2(8.125) + 2(13.252) + 92.8078]$

$\Rightarrow y_1 \cong 23.427$ at $x = 0.5$

step II)

$y_2 \cong y_1 + \frac{1}{6}[k + 2m + 2l + p]$

$k = (0.5) f(x_1, y_1)$

$= (0.5) f(0.5, 23.427)$

$\therefore k = 279.4122$

$l = (0.5) f(0.75, 163.1331)$

$\therefore l = 13311.2042$

$m = (0.5) f(0.75, 6679.03)$

$\therefore m = 22304725.87$

$p = (0.5) f(1, 22304725.87)$

$\therefore p = 2.4875 \times 10^{14}$

$\therefore y_2 \cong (23.427) + \frac{1}{6}[279.4122 + 2 \times 13311.2042 + 2 \times 22304725.87 + 2.4875 \times 10^{14}]$

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$\Rightarrow y_2 \cong 4.1450 \times 10^{13}$.

4) Given that,

$$f(x) = \log_{10}(x+y)$$

$$x_0 = 1, \quad y_0 = 2, \quad h = 0.2$$

Step I)

$$y_1 = y_0 + \frac{1}{6}[k + 2l + 2m + p]$$

$k = h\,f(x_0, y_0)$
$= (0.2) f(1, 2)$
$= 0.0954$

$l = h.f(x_0 + h/2,\; y_0 + k/2)$
$= (0.2) f(1.1, 2.0477)$
$= 0.0996$

$m = h.f(x_0 + \frac{h}{2},\; y_0 + \frac{l}{2})$
$= (0.2) f(1.1, 2.0498)$
$= 0.0997$

$p = h f(x_0 + h,\; y_0 + m)$
$= (0.2) f(1.2, 2.0997)$
$= 0.1037$

$$y_1 \cong 2 + \frac{1}{6}[0.09521 + 2(0.0996) + 2(0.0997) + 0.1037]$$

$$\Rightarrow y_1 \cong 2.0996$$

at $x = 1.2$

Step II)

$$y_2 \cong y_1 + \frac{1}{6}[k + 2l + 2m + p]$$

$k = h.f(x_1, y_1)$
$= (0.2) f(1.2, 2.0996)$
$= 0.1037$

$l = h.f(x_1 + \frac{h}{2},\; y_1 + \frac{k}{2})$
$= (0.2) f(1.3, 2.1515)$
$= 0.1076$

$m = h f(x_1 + \frac{h}{2},\; y_1 + \frac{l}{2})$
$= (0.2) f(1.3, 2.1534)$
$= 0.1076$

$p = h f(x_1 + h,\; y_1 + m)$
$= (0.2) f(1.4, 2.2072)$
$= 0.1114$

$$y_2 \cong 2.0996 + \frac{1}{6}[0.1037 + (0.1076)2 + 2(0.1076) + 0.1114]$$

$$= 2.2072$$

at $x = 1.4$

f f (8)

Step III) $y_3 \simeq y_2 + \frac{1}{6}[k + 2l + 2m + p]$

$k = h \cdot f(x_2, y_2)$
$= (0.2) f(1.4, 2.2072)$
$= 0.1114$

$l = hf(x_2 + \frac{h}{2}, y_2 + \frac{k}{2})$
$= (0.2) f(1.5, 2.2629)$
$= 0.1151$

$m = hf(x_2 + h/2, y_2 + l/2)$
$= (0.2) f(1.5, 2.2648)$
$= 0.1151$

$p = h \cdot f(x_2 + h, y_2 + m)$
$= (0.2) f(1.6, 2.3178)$
$= 0.1186$

$y_3 \simeq 2.2072 + \frac{1}{6}[0.1114 + 2(0.1151) + 2(0.1151) + 0.1186]$

$= 2.3178$ at $x = 1.6$

5) Given that,

$\frac{dy}{dx} = 4 + 3xy$

$x_0 = 0, \; y_0 = 0, \; h = 0.1$

step 1) $y_1 = y_0 + \frac{1}{6}[k + 2l + 2m + p]$

$k = hf(x_0, y_0)$
$= (0.1) f(0, 0)$
$= 0.4$

$l = h \cdot f(x_0 + h/2, y_0 + k/2)$
$= (0.1) f(0.05, 0.2)$
$= 0.403$

$m = hf(x_0 + h/2, y_0 + l/2)$
$= (0.1) f(0.05, 0.2015)$
$= 0.4030$

$p = hf(x_0 + h, y_0 + m)$
$= (0.1) f(0.1, 0.4030)$
$= 0.4121$

$y_1 = 0 + \frac{1}{6}[0.4 + 2(0.403) + 2(0.4030) + 0.4121]$

$= 0.4040$ at $x = 0.1$

Step II) $y_2 \simeq y_1 + \frac{1}{6}[k + 2l + 2m + p]$

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$k = h.f(x_1, y_1)$
$= (0.1)\, f(0.1, 0.4040)$
$= 0.4121$

$l = hf(x_1 + h/2, y_1 + k/2)$
$= (0.1)\, f(0.15, 0.6107)$
$= 0.4275$

$m = hf(x_1 + h/2, y_1 + l/2)$
$= (0.1)\, f(0.15, 0.6178)$
$= 0.\cancel{2127}\,4278$

$p = hf(x_1 + h, y_1 + m)$
$= (0.1)\, f(0.2, 0.8318)$
$= 0.4499$

$$y_2 = 0.4040 + \tfrac{1}{6}[0.4121 + 2(0.4275) + 2(0.4278) + 0.4499]$$
$$= 0.8328$$

Step III) $y_3 \simeq y_2 + \frac{1}{6}[k + 2l + 2m + p]$

$k = h.f(x_2, y_2)$
$= (0.1)\, f(0.2, 0.8328)$
$= 0.4499$

$l = hf(x_2 + h/2, y_2 + k/2)$
$= (0.1)\, f(0.25, 1.0577)$
$= 0.4798$

$m = hf(x_2 + h/2, y_2 + l/2)$
$= (0.1)\, f(0.25, 1.0727)$
$= 0.4805$

$p = h.f(x_2 + h, y_2 + m)$
$= (0.1)\, f(0.3, 1.3133)$
$= 0.5182$

$$y_3 \simeq 0.8328 + \tfrac{1}{6}[0.4499 + 2(0.4798) + 2(0.48) + 0.5182]$$
$$= 1.3142$$

Q- Solve $\frac{dy}{dx} = x + y$ with initial condn. $y(0) = 1$ find $y(0.5)$ in 4 steps. By 2nd order R-K method.

Soln- Here, $n = 4$, i.e no. of steps $= 4$

$$h = \frac{0.5 - 0}{4} = 0.125$$