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Simplex Method

Basic Notations :-

Consider LPP,

$\max z = c^T x$

subject to $Ax = b$

$x \geqslant 0$

The first basic feasible sol^n is,

$x_1 = x_2 = \cdots = x_n = 0$

and $x_{n+1} = b_1, x_{n+2} = b_2, \cdots x_{n+m} = b_m$.

Here A is $m \times (n+m)$ ordered matrix.

i.e.

$A = [a_1 \; a_2 \ldots a_n , a_{n+1} , \ldots a_{n+m}]$

x_{n+1} … x_{n+m} slack / surplus variable.

We form $m \times m$ non-singular matrix B called basis matrix whose column vectors are m linearly indpt. columns selected from matrix A and renamed as $B_1, B_2, \ldots B_m$.

$\therefore B = [a_{n+1} \; a_{n+2} \; \ldots \; a_{n+m}]$.

For initial basic feasible sol^n.

$$B = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & & 0 \\ 0 & 0 & & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & & 1 \end{bmatrix} = I_m$$

We denote the basic variable $x_{n+1}, x_{n+2}, \ldots x_{n+m}$ by, $x_{B_1}, x_{B_2}, \ldots x_{B_m}$ resp.

$\therefore$ Initial basic feasible sol^n is,

$x_B = (b_1, b_2 \ldots b_m)$.

The coefficient of basic variables $x_{B_1}, x_{B_2}, \ldots x_{B_m}$ in the objective funⁿ z will denoted by $c_{B_1}, c_{B_2}, \ldots c_{B_m}$.

i.e.

$$c_B = (c_{B_1}, c_{B_2}, \ldots c_{B_m}).$$

For initial basic feasible solⁿ,

$$c_B = (0, 0, \ldots 0).$$

objective funⁿ for initial solⁿ,

$$z = c_1x_1 + c_2x_2 + \cdots + c_nx_n + 0x_{n+1} + \cdots + 0x_{n+m}$$

$$z = 0$$

Since B is an $m\times m$ non-singular matrix. Any vector in A can be expressed as a linear combination of vectors in B. The notation for such linear combination is given by,

$$a_j = x_{1j}B_1 + x_{2j}B_2 + \cdots + x_{mj}B_m$$

$$= (B_1\ B_2 \ldots B_m)\begin{pmatrix} x_{1j} \\ x_{2j} \\ \vdots \\ x_{mj} \end{pmatrix}$$

$$a_j = Bx_j$$

$$x_j = B^{-1}a_j$$

For initial solⁿ $a_j = I_m x_j$

$$\therefore \underline{\underline{a_j = x_j}}$$

We define new variable say z_j.

$$z_j = x_{1j}c_{B_1} + x_{2j}c_{B_2} + \cdots + x_{mj}c_{B_m}$$

$$z_j = (c_{B_1}, c_{B_2} \ldots c_{Bm}) \begin{pmatrix} x_{1j} \\ x_{2j} \\ \vdots \\ x_{mj} \end{pmatrix}$$

$$z_j = c_B x_j$$

The net evaluation is denoted as Δ_j.

$$\Delta_j = z_j - c_j$$

i.e. $\Delta_j = c_B x_j - c_j$

The simplex table is given as.

Basic Variable	c_B	X_B	x_1	x_2	-- x_n	x_{n+1} ..	x_{n+m}	Min Ratio
$x_{n+1} = S_1$	0	b_1	a_{11}	a_{12}	a_{1n}	1	0	
$x_{n+2} = S_2$	0	b_2	a_{21}	a_{22}	a_{2n}	0	0	
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	
$x_{n+m} = S_m$	0	b_m	a_{m1}	a_{m2}	a_{mn}	0	1	
		z	Δ_1	Δ_2	Δ_n	Δ_{n+1}	Δ_{n+m}	

Procedure of Simplex method :-

Step I]

Check whether the objective fun^n of the given LPP is to be maximized or minimized. If it is minimized then we convert it into a problem of maximaization.

Step II]

Check whether all bi's are non-negative. If any one of the bi is negative then multiply the corresponding eq^n of the constraint by $\underline{\underline{-1}}$ so as to get all b_i's are non-negative.

Step III]

convert all the ineq^n of the constraint into eq^ns by introducing slack or surplus variable in the constraint. Put the cost of the variables equal to zero.

Step IV]

obtain the initial basic feasible sol^n to the problem in the form.

$$x_B = B^{-1} b.$$

and put it in the simplex table.

Step V]

Compute the net ~~in~~volution evaluation $z_j - c_j$.

Optimality Test:-

i] If all $z_j - c_j \geq 0$ then the initial basic feasible sol^n x_B is an optimum basic feasible sol^n.

ii] If atleast one $z_j - c_j < 0$ then sol^n is not optimal then we proceed to improve the sol^n in the next step.

iii] If ~~the~~ corresponding to any negative Δ_j, all the elements of column x_j are negative or zero then sol^n under the test will be unbounded.

Step VI]

In order to improve the basic feasible solⁿ the vector entering the basis matrix and the vector to be remove from the basis matrix are determine by the following rules. Such vectors are usually named as incoming vector and outgoing vector.

i] Incoming vector:-

The incoming vector x_k is always selected according to the most negative value of Δj.

ii] outgoing vector:-

The outgoing vector b_r is selected corresponding to the minimum ratio of x_B by the corresponding positive elements of predetermined incoming vector of x_k. This rule is called the minimum ratio rule.

In mathematical form this rule can be written as,

$$\frac{x_{Br}}{x_{rk}} = \min \left\{ \frac{x_{Bi}}{x_{ik}}, \; x_{ik} > 0 \right\}$$

The common element which is in the r^{th} row & k^{th}

Step VII] column is known as the leading element of the table.

Step VIII]

Convert the leading element to unity by dividing its row by the leading element itself, and all other elements in in its column to zero.

Step VIII]

Go to step VI] and repeat the computational procedure until the optimal sol^n is obtain.

1] Solve the LPP.

$$\max z = 4x_1 + 3x_2$$

subject to, $x_1 + x_2 \le \cancel{18}\ 8$

$2x_1 + x_2 \le 10$

$x_1, x_2 \ge 0,$

By using simplex method.

→ Let,

The standard form of LPP is,

$$\max z = 4x_1 + 3x_2 + 0s_1 + 0s_2$$

subject to $x_1 + x_2 + s_1 = 8$

$2x_1 + x_2 + s_2 = 10$

$x_1, x_2 \ge 0$

The first simplex table is.

BV	C_B	X_B	x_1	x_2	s_1	s_2	min ratio
s_1	0	8	1	1	1	0	$8/1 = 8$
s_2	0	10	2	1	0	1	$10/2 = 5$
		$z = 0$	$\Delta_1 = -4$	$\Delta_2 = -3$	$\Delta_3 = 0$	$\Delta_4 = 0$	

We have,

$$\Delta j = z_j - c_j$$

$$= C_B x_j - c_j$$

$$\therefore \Delta_1 = (0 \quad 0)\begin{pmatrix}1\\2\end{pmatrix} - 4 = 4$$

$$\Delta_2 = (0,\ 0)\begin{pmatrix}1\\1\end{pmatrix} - 3 = -3$$

$$\Delta_3 = (0\ \ 0)\begin{pmatrix}1\\0\end{pmatrix} - 0 = 0$$

$$\Delta_4 = (0\ \ 0)\begin{pmatrix}1\\0\end{pmatrix} - 0 = 0$$

Here,

All the Δj are not positive.

$\therefore$ It is not optimal sol^n.

The most negative Δj is corresponding to column x_1. Therefore x_1 is incoming vector.

By minimum ratio rule the outgoing vector is s_2.

The leading element is 2, so in order to get second simplex table we calculate the intermediate coefficient matrices.

We divide 2^{nd} Row by 2.

	C_B	X_B	x_1	x_2	s_1	s_2
R_1	0	8	1	1	1	0
R_2	0	5	1	$1/2$	0	$1/2$
R_3	0	0	-4	-3	0	0

$R_1 \to R_1 - R_2$

$R_3 \to R_3 + 4R_2$

	C_B	X_B	x_1	x_2	s_1	s_2
R_1	0	3	0	$1/2$	1	$-1/2$
R_2	0	5	1	$1/2$	0	$1/2$
R_3	0	20	0	-1	0	2

The second simplex table is,

BV	C_B	X_B	x_1	x_2	s_1	s_2	min ratio
s_1	0	3	0	$1/2$	1	$-1/2$	$3/\frac{1}{2} = 6$
x_1	4	5	1	$1/2$	0	$1/2$	$5/\frac{1}{2} = 10$
		20	0	-1	0	2	6

Here Δ_2 is negative therefore it is not a optimal sol^n, and the incoming vector is x_2. By minimum ratio rule the outgoing vector is s_1 and leading element is $\frac{1}{2}$.

So to get next simplex table we first divide first row by $1/2$.

	C_B	X_B	x_1	x_2	s_1	s_2
R_1	0	6	0	1	2	-1
R_2	4	5	1	$1/2$	0	$1/2$
R_3		20	0	-1	0	2

$R_2 \rightarrow R_2 - \frac{1}{2} R_1$

$R_3 \rightarrow R_3 + R_1$

	C_B	X_B	x_1	x_2	s_1	s_2
R_1	0	6	0	1	2	-1
R_2	4	2	1	0	-1	1
R_3		26	0	0	2	1

BV	C_B	X_B	x_1	x_2	s_1	s_2	min ratio
x_2	3	6	0	1	2	-1	
x_1	4	2	1	0	-1	1	
		26	0	0	2	1	

Here all Δj's are non-negative. Therefore optimal solution is $\underline{z=26}$.

2) $\max z = 3x_1 + 5x_2 + 4x_3$

subject to $2x_1 + 3x_2 \le 8$

$2x_2 + 5x_3 \le 10$

$3x_1 + 2x_2 + 4x_3 \le 15$

$x_1, x_2, x_3 \geq 0$.

→ Let,

$\max z = 3x_1 + 5x_2 + 4x_3 + 0s_1 + 0s_2 + 0s_3$

subject to

$2x_1 + 3x_2 + s_1 = 8$

$2x_2 + 5x_3 + s_2 = 10$

$3x_1 + 2x_2 + 4x_3 + s_3 = 15$

$x_1, x_2, x_3, s_1, s_2, s_3 \geq 0$.

The first simplex table is,

B.V. basic variable

BV	C_B	X_B	x_1	x_2	x_3	s_1	s_2	s_3	min. ratio
s_1	0	8	2	3	0	1	0	0	$8/3 = 2.67$
s_2	0	10	0	2	5	0	1	0	$10/2 = 5$
s_3	0	15	3	2	4	0	0	1	$15/2 = 7.5$
		0	-3	-5	-4	0	0	0	

$\Delta j = z_j - c_j$

$\Delta_1 = z_1$

-3, -5, -4, 0, 0, 0

$\Delta j = z_j - c_j$

$= C_B X_j - c_j$

$\Delta_1 = (0,0,0)\begin{pmatrix}2\\0\\3\end{pmatrix} - 3 = -3$

$\Delta_2 = (0,0,0)\begin{pmatrix}3\\2\\2\end{pmatrix} - 5 = -5$

$\Delta_3 = (0,0,0)\begin{pmatrix}0\\5\\4\end{pmatrix} - 4 = -4$

$\Delta_4 = (0,0,0)\begin{pmatrix}1\\0\\0\end{pmatrix} - 0 = 0$

$\Delta_5 = (0,0,0)\begin{pmatrix}0\\1\\0\end{pmatrix} - 0 = 0$

$\Delta_6 = (0,0,0)\begin{pmatrix}0\\0\\1\end{pmatrix} - 0 = 0$

All the Δj are not positive.

$\therefore$ It is not optimal soln.

The most negative Δj is corresponding to column x_2. Therefore x_2 is incoming vector.

By minimum ratio the outgoing vector is s_1.

The leading element is $\underline{\underline{3}}$. So in order to get second simplex table we calculate the intermediate coefficient matrices.

We divide 1^{st} row by 3,

$R_2 - 2R_1$ 4

C_B	x_B	x_1	x_2	x_3	s_1	s_2	s_3
R_1	8/3	2/3	1	0	1/3	0	0
R_2	14/3	$\cancel{0}$-4/3	0	5	-2/3	1	0
R_3	29/3	3	2	4	0	0	1
		-3	-5	-4	0	0	0

$R_1 + 2R_1$ $R_3 - 2R_1$

	x_B	x_1	x_2	x_3	s_1	s_2	s_3
R_1	8/3	2/3	1	0	1/3	0	0
R_2	14/3	$\cancel{0}$-4/3	0	5	-2/3	1	0
R_3	29/3	5/3	0	4	-2/3	0	1
R_4		-3$\cancel{0}$	-5	-$\cancel{3}$4	+$\cancel{0}$	0	1

$R_4 + 5R_1$

C_B	x_B	x_1	x_2	x_3	s_1	s_2	s_3
R_1	8/3	2/3	1	0	1/3	0	0
R_2	14/3	-4/3	0	5	-2/3	1	0
R_3	29/3	5/3	0	4	-2/3	0	1
R_4	40/3	+1/3	0	-4	5/3	0	0

The second simplex table is,

B.V	C_B	X_B	x_1	x_2	x_3	s_1	s_2	s_3	Min Ratio
x_2	5	$8/3$	$2/3$	1	0	$1/3$	0	0	∞
$(s_2$	0	$14/3$	$-4/3$	0	5	$-2/3$	1	$0)$	$14/15 = 0.93$
s_3	0	$29/3$	$5/3$	0	4	$-2/3$	0	1	$29/12 = 2.41$
		$40/3$	$+1/3$	$+0$	-4	$5/3$	0	0	

All the Δj are not positive

$\therefore$ It is not optimal sol^n.

The most negative Δj is corresponding to column x_3. Therefore x_3 is incoming vector.

By minimum ratio outgoing vector is s_2.

The leading element is 5. So in order to get second simplex table we calculate the intermediate coefficient matrices.

We divide 2^{nd} row by 5.

	C_B	X_B	x_1	x_2	x_3	s_1	s_2	s_3
R_1	5	$8/3$	$2/3$	1	0	$1/3$	0	0
R_2	0	$14/15$	$-4/15$	0	1	$-2/15$	$1/5$	0
R_3	0	$29/3$	$5/3$	0	4	$-2/3$	0	1
R_4		$40/3$	$1/3$	0	-4	$5/3$	0	0

$R_3 - 4R_2$, $R_4 + 4R_2$

	C_B	X_B	x_1	x_2	x_3	s_1	s_2	s_3	min ratio.
x_2	5	$8/3$	$2/3$	1	0	$1/3$	0	0	$\frac{8/3}{2/3} = 4$
x_3	4	$14/15$	$-4/15$	0	1	$-2/15$	$1/5$	0	$\frac{14/15}{-4/15} = -3.5$
s_3	0	$89/15$	$\boxed{44/15}$	0	0	$-2/15$	$-4/5$	1	$\frac{89/15}{44/15} = 2.17$
		$z = \frac{256}{15}$	$\frac{-11}{15}$	0	0	$17/15$	$4/5$	0	

$\therefore z = \frac{256}{15}$

All the Δ_j are not positive.
$\therefore$ It is not optimal solⁿ.
The most negative Δ_j is corresponding to column x_1. Therefore x_1 is incoming vector.

$R_3/41/15$

C_B	X_B	x_1	x_2	x_3	s_1	s_2	s_3
	8/3	2/3	1	0	1/3	0	0
	14/15	-4/15	0	1	-2/15	1/5	0
	89/41	1	0	0	-2/41	-12/41	15/41
	z=256/15	-11/15	0	0	17/15	4/15	0

$R_1 - \frac{2}{3}R_3$, $R_2 + \frac{4}{15}R_3$, $R_4 + \frac{11}{15}R_3$

BV	C_B	X_B	x_1	x_2	x_3	s_1	s_2	s_3
x_2	5	50/41	0	1	0	15/41	8/41	19/41
x_3	4	62/41	0	0	1	-6/41	5/41	4/41
x_1	3	89/41	1	0	0	-2/41	-12/41	15/41
		z=765/41	0	0	0	45/41	24/41	11/41

$x_1 = 89/41$
$x_2 = 50/41$
$x_3 = 62/41$
$z = \frac{765}{41}$

All Δ_j's are non-negative. Hence optimal solⁿ is $z = \frac{765}{41}$.

3] $\min z = x_1 - 3x_2 + 2x_3$

subject to $3x_1 - x_2 + 2x_3 \le 7$

$-2x_1 + 4x_2 \le 12$

$-4x_1 + 3x_2 + 3x_3 \le 10$

$x_1, x_2, x_3 \ge 0$

$\rightarrow$ Let,

$\max(-z) = -x_1 + 3x_2 - 2x_3 + 0s_1 + 0s_2 + 0s_3$

subject to $3x_1 - x_2 + 2x_3 + s_1 = 7$

$-2x_1 + 4x_2 + s_2 = 12$

$-4x_1 + 3x_2 + 3x_3 + s_3 = 10$

$x_1, x_2, x_3, s_1, s_2, s_3 \ge 0$

BV	C_B	X_B	x_1	x_2 (Incom)	x_3	s_1	s_2	s_3	Min Ratio
s_1	0	7	3	-1	2	1	0	0	$7/-1 = -7$
outgo. s_2	0	12	-2	4	0	0	1	0	$12/4 = 3$
s_3	0	10	-4	3	3	0	0	1	$10/3 = 3.3$
		$-z=0$	1	-3	2	0	0	0	

All Δj are not positive.

It is not optimal solⁿ.

The most negative Δj is corresponding to column x_2. Therefore x_2 is incoming vector.

By minimum ratio outgoing vector is s_2.

The leading element is 4, so in order to get second simplex table we calculate the intermediate coefficient matrices.

We divide R_2 by 4.

	C_B	x_B	x_1	x_2	x_3	s_1	s_2	s_3
s_1	0	7	3	-1	2	1	0	0
R_2	0	3	$-1/2$	1	0	0	$1/4$	0
R_3	0	10	-4	3	3	0	0	1
Z_4		0	1	-3	2	0	0	0

$R_1 + R_2$, $R_3 - 3R_2$, $R_4 + 3R_2$

	C_B	x_B	x_1	x_2	x_3	s_1	s_2	s_3	min ratio
s_1	0	10	$5/2$	0	2	1	$1/4$	0	$10/(5/2) = 4$
x_2	3	3	$-1/2$	1	0	0	$1/4$	0	$= -6$
s_3	0	1	$-5/2$	0	3	0	$-3/4$	1	
	-	~~30~~	~~-11~~	~~0~~	~~11~~	~~0~~	~~0~~	~~3~~	
		9	$-1/2$	0	2	0	$3/4$	0	

$-4+\frac{5}{2}$

$1-\frac{5}{2}$

The 2^{nd} simplex table is,

BV	C_B	X_B	x_1	x_2	x_3	s_1	s_2	s_3	Min ratio
s_1	0	10	$5/2$	0	2	1	$1/4$	0	$10/5_2 = 4$
x_2	3	3	$-1/2$	1	0	0	$1/4$	0	$3/-1_2 = -6$
s_3	0	1	$-5/2$	0	3	0	$-3/4$	1	$1/-5_2 = -4$
		9	$-1/2$	0	2	0	$3/4$	0	

x_1 is incoming vector & s_1 is outgoing vector.

$R_1/5/2$

	C_B	X_B	x_1	x_2	x_3	s_1	s_2	s_3
$R_1\ s_1$	0	4	1	0	$4/5$	$2/5$	$1/10$	0
$R_2\ x_2$	3	3	$-1/2$	1	0	0	$1/4$	0
$R_3\ s_3$	0	1	$-5/2$	0	3	0	$-3/4$	1
R_4		9	$-1/2$	0	2	0	$3/4$	0

$R_2 + \frac{1}{2}R_1$, $R_3 + \frac{5}{2}R_1$, $R_4 + \frac{1}{2}R_1$

BV	C_B	X_B	x_1	x_2	x_3	s_1	s_2	s_3
$R_1\ x_1$	1	4	1	0	$4/5$	$2/5$	$1/10$	0
$R_2\ x_2$	3	5	0	1	$2/5$	$1/5$	$3/10$	0
$R_3\ s_3$	0	11	0	0	5	1	$-1/2$	1
R_4		$z=11$	0	0	$12/5$	$1/5$	$4/5$	0

$\therefore$ Here all the Δ_j's are non-negative. Therefore optimal sol^n is $z = 11$.

$x_3 = 0$

$x_1 = 4$
$x_2 = 5$
$x_3 = 0$

since x_3 is non-basic variable here

BV	C_B	X_B	x_1	x_2	x_3	s_1	s_2	s_3
x_1	01	4	1	0	$4/5$	$2/5$	$1/10$	0
x_2	3	3	$-1/2$	1	0	0	$1/4$	0
s_3	0	1	$-5/2$	0	3	0	$-3/4$	1
		9	$-1/2$	0	2	0	$3/4$	0

4) $\min z = +7x_1 + 5x_2$

subject to $-x_1 - 2x_2 \geq -6$

$4x_1 + 3x_2 \leq 12$

$x_1, x_2 \geq 0.$

→ Let,

$\max(+z) = +7x_1 + 5x_2 + 0s_1 + 0s_2$

subject to, $-x_1 - 2x_2 - s_1 = -6 \Rightarrow x_1 + 2x_2 + s_1 = 6$

$4x_1 + 3x_2 + s_2 = 12$

$x_1, x_2 \geq 0 \quad s_1, s_2 \geq 0$

BV	C_B	x_B	x_1	x_2	s_1	s_2	Min Ratio
s_1	0	6	1	2	1	0	$6/1 = 6$
s_2	0	12	4	3	0	1	$12/4 = 3$
		$z=0$	-7	-5	0	0	

$\therefore$ x_1 is incoming vector & s_2 is outgoing.

$R_2/4$

x_B	x_1	x_2	s_1	s_2
6	1	2	1	0
3	1	$3/4$	0	$1/4$
0	-7	-5	0	0

$R_1 - R_2 \quad , \quad R_3 + 7R_2$

x_B	x_1	x_2	s_1	s_2
3	0	$5/4$	1	$-1/4$
3	1	$3/4$	0	$1/4$
$\cancel{0}$	$\cancel{+7}$	$\cancel{+5}$	$\cancel{0}$	$\cancel{0}$
$z=21$	0	$1/4$	0	$7/4$

BV	C_B	X_B	x_1	x_2	S_1	S_2	Min Ratio
S_1	0	3	0	5/4	1	$-1/4$	
x_1	7	3	1	3/4	0	1/4	
		21	0	1/4	0	7/4	

$\therefore x_2$ is non-basic variable here.

$\therefore$ All the Δj's are non-negative. Hence optimal sol^n is $z = 21$, $x_1 = 3$, $x_2 = 0$.

5] $\max z = 5x_1 + 3x_2$

subject to $3x_1 + 5x_2 \leq 15$

$5x_1 + 2x_2 \leq 10$

$x_1, x_2 \geq 0$.

→ Let, The standard form of LPP is,

$\max z = 5x_1 + 3x_2 + 0s_1 + 0s_2$

subject to $3x_1 + 5x_2 + s_1 = 15$

$5x_1 + 2x_2 + s_2 = 10$

$x_1, x_2 \geq 0$

The first simplex table is,

BV	C_B	X_B	x_1	x_2	S_1	S_2	Min Ratio
S_1	0	15	3	5	1	0	$15/3 = 5$
S_2	0	10	5	2	0	1	$10/5 = 2$
		$z=0$	-5	-3	0	0	

All the Δj are not positive.

$\therefore$ It is not optimal sol^n.

The most negative Δj is corresponding to column x_1. Therefore x_1 is incoming vector.

$\therefore$ By minimum ratio rule the outgoing vector is S_2.

The leading element is 5, so in order to get second simplex table we calculate the intermediate coefficient matrices.

$\therefore\ R_2/5$

	x_B	x_1	x_2	s_1	s_2
R_1	15	3	5	1	0
R_2	2	1	2/5	0	1/5
R_3	0	-5	-3	0	0

$R_1 - 3R_2,\quad R_3 + 5R_2$

$\cancel{9}$	$\cancel{0}$	$\cancel{19/5}$	$\cancel{1}$	$\cancel{-3/5}$
$\cancel{9}$	$\cancel{0}$	$\cancel{19/5}$	$\cancel{1}$	$\cancel{-3/5}$
$\cancel{2}$	$\cancel{1}$	$\cancel{2/5}$	$\cancel{0}$	$\cancel{1/5}$
$\cancel{8}$	$\cancel{0}$	$\cancel{-1}$	$\cancel{0}$	

x_B	x_1	x_2	s_1	s_2
9	0	19/5	1	-3/5
2	1	2/5	0	1/5
10	0	-1	0	1

$\rightarrow$ 2nd Simplex Table.

	C_B	x_B	x_1	x_2	s_1	s_2	Min ratio
s_1	0	9	0	19/5	1	-3/5	$9/\tfrac{19}{5} = 2.36$
x_1	5	2	1	2/5	0	1/5	$2/\tfrac{2}{5} = 5$
		10	0	-1	0	1	

All the Δj are not positive.
$\therefore$ It is not optimal sol^n.

The most negative Δj is corresponding to column x_2. Therefore x_2 is incoming vector.
$\therefore$ By minimum ratio rule the outgoing vector is s_1.

The leading element is $19/5$, so in order to get third simplex table we calculate the intermediate coefficient matrices.

$R_1 / 19/5$

x_B	x_1	x_2	s_1	s_2
45/19	0	1	5/19	-3/19
2	1	2/5	0	1/5
10	0	-1	0	1

$R_2 - \frac{2}{5}R_1$, $R_3 + R_1$

x_B	x_1	x_2	s_1	s_2
45/19	0	1	5/19	-3/19
20/19	1	0	-2/19	5/19
235/19	0	0	5/19	16/19

$\therefore$

BV	C_B	x_B	x_1	x_2	s_1	s_2
x_2	3	45/19	0	1	5/19	-3/19
x_1	5	20/19	1	0	-2/19	5/19
		235/19	0	0	5/19	16/19

$\therefore$ All Δj's are non-negative. Hence optimal solution is $z = \frac{235}{19}$.

c] $\max z = 2x_1 + 4x_2$

subject to $2x_1 + 3x_2 \le 48$

$x_1 + 3x_2 \le 42$

$x_1 + x_2 \le 21$

$x_1, x_2 \ge 0$.

→ Let,

$\max z = 2x_1 + 4x_2 + 0s_1 + 0s_2 + 0s_3$

subject to, $2x_1 + 3x_2 + s_1 = 48$

$x_1 + 3x_2 + s_2 = 42$

$x_1 + x_2 + s_3 = 21$

BV	C_B	X_B	x_1	x_2	s_1	s_2	s_3	Min.
s_1	0	48	2	3	1	0	0	$48/3 = 16$
(s_2	0	42	1	3	0	1	0)	$42/3 = 14$
s_3	0	21	1	1	0	0	1	$21/1 = 21$
		$z=0$	-2	-4	0	0	0	

x_2 is incoming vector s_2 is outgoing.

$R_2/3$

	X_B	x_1	x_2	s_1	s_2	s_3
R_1	48	2	3	1	0	0
R_2	14	$1/3$	1	0	$1/3$	0
R_3	21	1	1	0	0	1
	$z=0$	-2	-4	0	0	0

$R_3 - R_2$, $R_1 - 3R_2$, $R_4 + 4R_2$

X_B	x_1	x_2	s_1	s_2	s_3
6	1	0	1	-1	0
14	$1/3$	1	0	$1/3$	0
7	$2/3$	0	0	$-1/3$	1
$z=56$	$-2/3$	0	0	$4/3$	0

7] $\max z = 3x_1 + 4x_2$

subject to $x_1 - x_2 \le 1$

$-x_1 + x_2 \le 2$

$x_1, x_2 \geq 0$.

→ Let,

$\max z = 3x_1 + 4x_2$

s. t. $x_1 - x_2 + s_1 = 1$

$-x_1 + x_2 + s_2 = 2$

$x_1, x_2, s_1, s_2 \geq 0$.

BV	C_B	X_B	x_1	x_2	S_1	S_2	Min ratio
S_1	0	1	1	-1	1	0	$1/-1 = -1$
S_2	0	2	-1	1	0	1	$2/1 = 2$
		$z = 0$	-3	-4	0	0	

R

X_B	x_1	x_2	S_1	S_2
1	1	-1	1	0
2	-1	1	0	1
0	-3	-4	0	0

$R_1 + R_2$, $R_3 + 4R_2$

X_B	x_1	x_2	S_1	S_2
3	0	0	1	1
2	-1	1	0	1
8	-7	0	0	4

BV	C_B	X_B	x_1	x_2	S_1	S_2	Min ratio
S_1	0	3	0	0	1	1	$3/0 = \infty$
x_2	4	2	-1	1	0	1	$2/-1 = -2$
		$z = 8$	-7	0	0	4	

by using step (ii)

$\therefore$ Given solⁿ is unbounded.

The element in the column x_1 are zero and negative hence given LPP has unbounded solution.

* 8) $\max z = 4x_1 + 10x_2$

subject to $2x_1 + x_2 \le 50$

$2x_1 + 5x_2 \le 100$

$2x_1 + 3x_2 \le 90$

$x_1, x_2 \ge 0.$

→ Let,

$\max z = 4x_1 + 10x_2 + 0s_1 + 0s_2 + 0s_3$

subject to $2x_1 + x_2 + s_1 = 50$

$2x_1 + 5x_2 + s_2 = 100$

$2x_1 + 3x_2 + s_3 = 90$

$x_1, x_2, s_1, s_2, s_3 \ge 0.$

BV	C_B	X_B	x_1	x_2	s_1	s_2	s_3	Min ratio
s_1	0	50	2	1	1	0	0	$50/1 = 50$
s_2	0	100	2	5	0	1	0	$100/5 = 20$
s_3	0	90	2	3	0	0	1	$90/3 = 30$
		$z=0$	-4	-10	0	0	0	

$R_2/5$

X_B	x_1	x_2	s_1	s_2	s_3
50	2	1	1	0	0
20	2/5	1	0	1/5	0
90	2	3	0	0	1
$z=0$	-4	-10	0	0	0

BV	C_B	$R_1 - R_2$		$R_3 - 3R_2$		$R_4 + 10R_2$		min ratio
s_1	0	30	8/5	0	1	-1/5	0	$30/\frac{8}{5} = 18.75$
x_2	10	20	2/5	1	0	1/5	0	$20/\frac{2}{5} = 50$
s_3	0	30	4/5	0	0	-3/5	1	$30/\frac{4}{5} = 37.5$
		$z=200$	0	0	0	2	0	

non-basic net evaluation is 0 the alternate sol exist.

$R_1/8/5$

x_B	x_1	x_2	s_1	s_2	s_3
18.75	1	0	5/8	-1/8	0
20	2/5	1	0	1/5	0
30	4/5	0	0	-3/5	1
z-200	0	0	0	2	0

$R_2 - \frac{2}{5}R_1, \quad R_3 - \frac{4}{5}R_1$

BV	C_B	x_B	x_1	x_2	s_1	s_2	s_3
x_1	4	18.75	1	0	5/8	-1/8	0
x_2	10	12.5	0	1	-1/4	1/4	0
s_3	0	15	0	0	-1/2	-1/2	1
		200	0	0	0	2	0

$x_1 = 18.75, \quad x_2 = 12.5$

The net evaluation value of non-basic variable is 0 this implies that it has alternate sol

9] $\max z = 4x_1 + 5x_2 + 9x_3 + 11x_4$

subject to $x_1 + x_2 + x_3 + x_4 \le 15$

$7x_1 + 5x_2 + 3x_3 + 2x_4 \le 120$

$3x_1 + 5x_2 + 10x_3 + 15x_4 \le 100$

→ Let,

$\max z = 4x_1 + 5x_2 + 9x_3 + 11x_4 + 0s_1 + 0s_2 + 0s_3$

subject to $x_1 + x_2 + x_3 + x_4 + s_1 = 15$

$7x_1 + 5x_2 + 3x_3 + 2x_4 + s_2 = 120$

$3x_1 + 5x_2 + 10x_3 + 15x_4 + s_3 = 100$

$x_1, x_2, x_3, s_1, s_2, s_3 \ge 0$

BV	C_B	X_B	x_1	x_2	x_3	x_4	S_1	S_2	S_3	Min ratio
S_1	0	15	1	1	1	1	1	0	0	$1/1 = 1$
S_2	0	120	7	5	3	2	0	1	0	$7/2 = 3.5$
S_3	0	100	3	5	10	15	0	0	1	$3/15 = 0.2$
		$z=0$	-4	-5	-9	-11	0	0	0	

$R_3/15$

X_B	x_1	x_2	x_3	x_4	S_1	S_2	S_3
15	1	1	1	1	1	0	0
120	7	5	3	2	0	1	0
$20/3$	$1/5$	$1/3$	$2/3$	1	0	0	$1/15$
$z=0$	-4	-5	-9	-11	0	0	0

$R_1 - R_3, \quad R_2 - 2R_3, \quad R_4 + 11R_3$

BV	C_B	X_B	x_1	x_2	x_3	x_4	S_1	S_2	S_3	min ratio
S_1	0	$25/3$	$4/5$	$2/3$	$1/3$	0	1	0	$-1/15$	$\frac{25/3}{4/5} = 10.41$
S_2	0	$320/3$	$33/5$	$13/3$	$5/3$	0	0	1	$-2/15$	$\frac{320/3}{33/5} = 16.16$
x_4	11	$20/3$	$1/5$	$1/3$	$2/3$	1	0	0	$1/15$	$\frac{20/3}{1/5} = 33.33$
		$z = 220/3$	$-9/5$	$-4/3$	$-5/3$	0	0	0	$11/15$	

$R_1 / 4/5$

X_B	x_1	x_2	x_3	x_4	S_1	S_2	S_3
$125/12$	1	$5/6$	$5/12$	0	$5/4$	0	$-1/12$
$320/3$	$33/5$	$13/3$	$5/3$	0	0	1	$-2/15$
$20/3$	$1/5$	$1/3$	$2/3$	1	0	0	$1/15$
$z = 220/3$	$-9/5$	$-4/3$	$-5/3$	0	0	0	$11/15$

$R_2 - \frac{33}{5}R_1$, $R_3 - \frac{1}{5}R_1$, $R_4 + \frac{9}{5}R_1$

	BV	C_B	X_B	x_1	x_2	x_3	x_4	s_1	s_2	s_3	Min ratio
①	x_1	4	$125/12$	1	$5/6$	$5/12$	0	$5/4$	0	$-1/12$	25
②	s_2	0	$455/12$	0	$-7/6$	$-13/12$	0	$-33/4$	1	$5/12$	-35
③	x_4	11	$55/12$	0	$1/6$	$7/12$	1	$-1/4$	0	$1/12$	7.8
④			$z = \frac{1105}{12}$	0	$1/6$	$-11/12$	0	$9/4$	0	$7/12$	

$R_3 / 7/12$

X_B	x_1	x_2	x_3	x_4	s_1	s_2	s_3
$125/12$	1	$5/6$	$5/12$	0	$5/4$	0	$-1/12$
$455/12$	0	$-7/6$	$-13/12$	0	$-33/4$	1	$5/12$
$55/7$	0	$2/7$	1	$12/7$	$-3/7$	0	$1/7$
$z = \frac{1105}{12}$	0	$1/6$	$-11/12$	0	$9/4$	0	$7/12$

$R_1 - \frac{5}{12}R_3$, $R_2 + \frac{13}{12}R_3$ $R_4 + \frac{11}{12}R_3$

BV	C_B	X_B	x_1	x_2	x_3	x_4	s_1	s_2	s_3
x_1	4	$50/7$	1	$5/7$	0	$-5/7$	$10/7$	0	$-1/7$
s_2	0	$325/7$	0	$-6/7$	0	$13/7$	$-61/7$	1	$4/7$
x_3	9	$55/7$	0	$2/7$	1	$12/7$	$-3/7$	0	$1/7$
		$z = \frac{695}{7}$	0	$3/7$	0	$11/7$	$13/7$	0	$5/7$

All Δj's are positive.

$\therefore$ Optimal sol^n is $z = \frac{695}{7}$

Introduction and use of artificial variables:-

When we solve an LPP using simplex method we start with a ready BFS with basis matrix as identity matrix. If the identity matrix is missing then we introduce an artificial variable a_i for ready BFS and start the simplex method.

Big M Method:-

We introduce artificial variables for ready identity as submatrix of coefficient matrix in LPP with high penalty / cost (M) in maximization problem and with penalty or cost (M) in minimization problem.

Ingeneral Big M can give following cases

case I] If no artificial variable remain in the last simplex table then system has a sol^n.

case II] If artificial variable remain in the last simplex table with positive value then system have no sol^n

case III] If artificial variable remain in the last simplex table with zero value then system has a sol^n

1] Solve by using Big M method the following LPP.

$\max(z) = -2x_1 - x_2$

subject to $3x_1 + x_2 = 3$, $4x_1 + 3x_2 \geq 6$, $x_1 + 2x_2 \leq 4$

$x_1, x_2 \geq 0$.

→ Let;

$$\max z = -2x_1 - x_2$$
$$3x_1 + x_2 = 3$$
$$4x_1 + 3x_2 \geq 6$$
$$x_1 + 2x_2 \leq 4$$
$$x_1, x_2 \geq 0$$

Write the LPP into standard form by introducing slack and surplus variable with cost zero and an artificial variable with cost $-M$.

$\therefore$ $\max z = -2x_1 - x_2 + 0s_1 + 0s_2 - Ma_1 - Ma_2$

subject to $3x_1 + x_2 + a_1 = 3$

$4x_1 + 3x_2 - s_1 + a_2 = 6$

$x_1 + 2x_2 + s_2 = 4$

$x_1, x_2, s_1, s_2, a_1, a_2, M > 0$.

BV	C_B	X_B	x_1	x_2	s_1	s_2	a_1	a_2	Min ratio
a_1	$-M$	3	3	1	0	0	1	0	$3/3 = 1$
a_2	$-M$	6	4	3	-1	0	0	1	$6/4 = \frac{3}{2}$
s_2	0	4	1	2	0	1	0	0	4
		$z = -9M$	$2-7M$	$-4M+1$	M	0	0	0	

$-5 \quad -3 \quad 1$

$R_1/3$, $R_2 - 4R_1$, $R_3 - R_1$, $R_4 - (2-7M) \cdot R_1$

1	1	$1/3$	0	0	$1/3$	0
2	0	$+5/3$	-1	0	$-4/3$	1
3	0	$5/3$	0	1	$-1/3$	0
$-2M-2$	0	$\frac{1-5M}{3}$	M	0	$\frac{7M-2}{3}$	0

BV	C_B	X_B	x_1	x_2	s_1	s_2	a_1	a_2	Min ratio.
R_1 x_1	-2	$\cancel{3}\,1$	1	$1/3$	0	0	$1/3$	0	$\frac{1}{1/3} = 3$
R_2 a_2	$-M$	$\cancel{6}\,2$	0	$5/3$	-1	0	$-4/3$	1	$\frac{2}{5/3} = \frac{6}{5} = 1.2$
R_3 s_2	0	3	0	$5/3$	0	1	$-1/3$	0	$9/5 = 1.8$
		$z = -2M - 2$	0	$\frac{1-5M}{3}$	M	0	$\frac{7M-2}{3}$	0	

$R_2 / 5/3$,

X_B	x_1	x_2	s_1	s_2	a_1	a_2
1	1	$1/3$	0	0	$1/3$	0
$6/5$	0	1	$-3/5$	0	$-4/5$	$3/5$
3	0	$5/3$	0	1	$-1/3$	0
$z = -2M-2$	0	$\frac{1-5M}{3}$	M	0	$\frac{7M-2}{3}$	0

$R_1 - \frac{1}{3}R_2$, $R_3 - \frac{5}{3}R_2$, $R_4 - \frac{(1-5M)}{3}R_2$

	C_B	X_B	x_1	x_2	s_1	s_2	a_1	a_2
x_1	-2	$3/5$	1	0	$1/5$	0	$1/5$	$-1/5$
x_2	-1	$6/5$	0	1	$-3/5$	0	$-4/5$	$3/5$
s_2	0	1	0	0	1	1	1	-1
		$-12/5$	0	0	$1/5$	0	$M-2/5$	$M-1/5$

$\therefore$ All Δ_j are positive.

$\therefore$ It's optimal soln is $z = \frac{-12}{5}$,

$x_1 = 3/5$, $x_2 = 6/5$, $s_2 = 1$.

1) $\max z = x_1 + 2x_2 + 3x_3 - x_4$

subject to, $x_1 + 2x_2 + 3x_3 = 15$

$2x_1 + x_2 + 5x_3 = 20$

$x_1 + 2x_2 + x_3 + x_4 = 10$

$x_1, x_2, x_3, x_4 \geq 0$.

→ Let

$\max z = x_1 + 2x_2 + 3x_3 - x_4$

subject to $x_1 + 2x_2 + 3x_3 = 15$

$2x_1 + x_2 + 5x_3 = 20$

$x_1 + 2x_2 + x_3 + x_4 = 10$

$x_1, x_2, x_3, x_4 \geq 0$

Write the LPP in standard form by Introducing slack and surplus variable with cost zero (if required) and artificial variable with cost -M.

$\therefore \max z = x_1 + 2x_2 + 3x_3 - x_4 - Ma_1 - Ma_2$

$x_1 + 2x_2 + 3x_3 + a_1 = 15$

$2x_1 + x_2 + 5x_3 + a_2 = 20$

$x_1 + 2x_2 + x_3 + x_4 = 10$

$M > 0$ $\quad x_1, x_2, x_3, x_4 \geq 0$

a_1	a_2	x_4	BV	CB	X_B	x_1	x_2	x_3	x_4	a_1	a_2	Min ratio
1	0	0	a_1	$-M$	15	1	2	3	0	1	0	$15/3 = 5$
0	1	0	a_2	$-M$	20	2	1	5	0	0	1	$20/5 = 4$
0	0	1	x_4	-1	10	1	2	1	1	0	0	$10/1 = 10$
					$z = -35M - 10$	$-3M-2$	$-3M-4$	$-8M-4$	0	+0	0	

$R_2/5$,

x_B	x_1	x_2	x_3	x_4	a_1	a_2
15	1	2	3	0	1	0
4	2/5	1/5	1	0	0	1/5
10	1	2	1	1	0	0
-3M-10	-3M-2	-3M-4	-8M-4	0	0	0

$R_1 - 3R_2, \quad R_3 - R_2, \quad R_4 - (-8M-4)R_2$

	x_B	x_1	x_2	x_3	x_4	a_1	a_2	
a_1	3	-1/5	7/5	0	0	1	-3/5	$\frac{15}{7}$
a_2	4	2/5	1/5	1	0	0	1/5	20
x_4	6	3/5	9/5	0	1	0	-1/5	$\frac{30}{9}$
	-3M+6	$\frac{M-2}{5}$	$\frac{-7M-16}{5}$	0	0	0	$\frac{8M+4}{5}$	

		x_B	x_1	x_2	x_3	x_4	a_1	a_2
x_2	2	3	-1/5	7/5	0	0	1	-3/5
a_2		4	2/5	1/5	1	0	0	1/5
x_4		6	3/5	9/5	0	1	0	-1/5
		-3M+6	$\frac{M-2}{5}$	$\frac{-7M-16}{5}$	0	0	0	$\frac{8M+4}{5}$

$R_1 / 7/5$

		x_B	x_1	x_2	x_3	x_4	a_1	a_2
R_1	x_2	15/7	-1/7	1	0	0	5/7	-3/7
R_2	a_2	4	2/5	1/5	1	0	0	1/5
R_3	x_4	6	3/5	9/5	0	1	0	-1/5
R_4		-3M+6	$\frac{M-2}{5}$	$\frac{-7M-16}{5}$	0	0	0	$\frac{8M+4}{5}$

$R_2 - \frac{1}{5}R_1, \quad R_3 - \frac{9}{5}R_1, \quad R_4 - \left(\frac{-7M-16}{5}\right) \cdot R_1$

		x_B	x_1	x_2	x_3	x_4	a_1	a_2	
x_2	2	15/7	-1/7	1	0	0	5/7	-3/7	
x_3	3	25/7	3/7	0	1	0	-1/7	2/7	$\frac{25}{3}$
x_4	-1	15/7	30/35	0	0	1	-9/7	3/5	$\frac{75}{30}$
		90/7	-6/7	0	0	0	0		

$\frac{3}{5} - \frac{9}{5} \times -\frac{1}{7} \quad \frac{21+9}{35} \quad \frac{9}{5} - \frac{9}{5} \times \ldots \quad 0 - \frac{9}{5} \times \frac{5}{7} \quad -\frac{1}{5} + \frac{9}{5} \cdot \frac{3}{7}$

	x_B	x_1	x_2	x_3	x_4	a_1	a_2
	15/7	-1/7	1	0	0	5/7	-3/7
	25/7	3/7	0	1	0	-1/7	2/7
	75/29	1/1	0	0	35/29	-45/29	21/29
	90/7	-6/7	0	0	0		

$R_4 + \frac{6}{7}R_3 \quad R_1 + \frac{1}{7}R_3, \quad R_2 - \frac{3}{7}R_3$

	x_B	x_1	x_2	x_3	x_4
x_2	635.36	0	1	0	5/29
x_3	2.463	0	0	1	-15/29
x_1	75/29	1	0	0	35/29
	$z = 15.07$	0	0	0	30/29

x_2

x_3

x_1

2) $\max z = -x_1 + x_2$

subject to $x_1 + x_2 \le 1$

$2x_1 + 3x_2 \ge 6$

$x_1, x_2 \ge 0.$

→ Let,

$\max z = -x_1 + x_2 + 0s_1 + 0s_2 - Ma_1$

s.t. $x_1 + x_2 + s_1 = 1$

$2x_1 + 3x_2 - s_2 + a_1 = 6$

$x_1, x_2 \ge 0$

BV	C_B	x_B	x_1	x_2	s_1	s_2	a_1	Min ratio
s_1	0	1	1	1	1	0	0	1/1 = 1
a_1	-M	6	2	3	0	-1	1	6/3 = 2
		$z = -6M$	$1-2M$	$-1-3M$	0	M	0	

$R_2 - 3R_1$ $\quad$ $R_3 - (-1-3M)R_1$

	X_B	x_1	x_2	s_1	s_2	a_1
	1	1	1	1	0	0
a_1	3	-1	0	-3	-1	1
	$z = 1-3M$	$2+4M$	0	$1+3M$	M	0

$a_1 = 3$, case II (Big M Method)

Given LPP has infeasible way sol^n

because artificial variable has positive value.

3] $\max z = -2x_1 - 2x_2$

s.t. $3x_1 + x_2 = 3$

$4x_1 + 3x_2 \geq 6$

$x_1 + 2x_2 \leq 3$.

→ Let,

$\max z = -2x_1 - 2x_2 - Ma_1 - Ma_2$

s.t. $3x_1 + x_2 + a_1 = 3$

$4x_1 + 3x_2 - s_1 + a_2 = 6$

$x_1 + 2x_2 + s_2 = 3$

maximization problem -M

BV	C_B	X_B	x_1	x_2	s_1	s_2	a_1	a_2	min ratio
a_1	-M	3	3	1	0	0	1	0	$3/3 = 1$
a_2	-M	6	4	3	-1	0	0	1	$6/4 = \frac{3}{2} = 1.5$
s_2	0	3	1	2	0	1	0	0	$3/1 = 3$
		$z = -9M$	$2-7M$	$2-4M$	M	0	0	0	

$R_1/3$, R_2

$(2-4M)-(2-7M)\frac{1}{3}$

$2-4M-\frac{2}{3}+\frac{7M}{3}$ $\quad 2-\frac{2}{3}$ $\quad \frac{4}{3}-\frac{5M}{3}$

		x_B	x_1	x_2	s_1	s_2	a_1	a_2
R_1	x_1	1	1	1/3	0	0	1/3	0
R_2	a_2	6	4	3	-1	0	0	1
R_3	S_2	3	1	2	0	1	0	0
R_4		$-9M$	$2-7M$	$2-4M$	M	0	0	0

$2-4M-\frac{2}{3}7M$

$R_2-4R_1, \quad R_3-R_1, \quad R_4-(2-7M)R_1$

	x_B	x_1	x_2	s_1	s_2	a_1	a_2	min ratio
x_1	1	1	1/3	0	0	1/3	0	1/⅓ 3
a_2	2	0	5/3	-1	0	-4/3	0	4/5=1.2
s_2	2	0	5/3	0	1	-1/3	0	4/5=1.2
	$-2-2M$	0	$\frac{4-5M}{3}$	M	0	$\frac{-2+7M}{3}$	0	

x_B $\quad R_3/5/3$

	x_B	x_1	x_2	s_1	s_2	a_1	a_2
x_1	1	1	1/3	0	0	1/3	0
a_2	2	0	5/3	-1	0	-4/3	0
x_2	6/5	0	1	0	3/5	-1/3	0
	$-2-2M$	0	$\frac{4-5M}{3}$	M	0	$\frac{7M-2}{3}$	0

$R_1-\frac{1}{3}R_3, \quad R_2-\frac{5}{3}R_3, \quad R_4-\frac{(4-5M)}{3}R_3$

x_B	x_1	x_2	s_1	s_2	a_1	a_2
3/5	1	0	0	-1/5	4/15	0
0	0	0	-1	-1	-1	0
6/5	0	1	0	3/5	-1/15	0

$2-\frac{4}{7}$

$-4M-2+7M$

$x=2$

$-2-7M$

$2-6M-\frac{2}{3}+\frac{7M}{3}$

$\frac{4}{3}-\frac{5M}{3}$

$-\frac{4}{3}$

$\frac{1}{3}\cdot\frac{6}{5}$

$\frac{1}{3}\cdot\frac{1}{3}$

$2-2$

$-2M-\frac{4}{3}+\frac{4M}{3}$

$=\frac{7}{5}$

$-2M-4M$

Two Phase Method:-

Procedure:-

Phase I:-

In this phase we introduce additional objective function in place of the given objective function, the new LPP formed is called auxillary LPP. The auxillary LPP is,

$$\max z^* = -a_1 - a_2 \ldots -a_r$$
$$\min z^* = a_1 + a_2 + \ldots + a_r$$

ie. we assign a cost $\underline{\underline{-1}}$ to each artificial variable and cost zero to all remaining variable.

LPP subject to the constraint of the original LPP.

Solve the above LPP with usual simplex algorithm, the following three cases may arise.

Case I] If $\max z^* < 0$ or $\min z^* > 0$ and artificial variable remain in the basis then LPP has no solution or have infeasible sol^n.

Case II] If $\max z^* = 0$ or $\min z^* = 0$ and no artificial variables remain in the basis then the LPP has feasible sol^n and go to phase II

Case III] If $\max z^* = 0$ or $\min z^* = 0$ and artificial variable remain in the basis with zero value then LPP has feasible sol^n and go to phase II.

Phase II :-

Assign the actual cost to the variables in the objective fun^n and zero cost to every artificial variable that appears in the basis.

This new objective fun^n is maximized or minimized by simplex method with last simplex table of phase (I) as starting simplex table with actual cost values.

1] $\max z = -x_1 + x_2$

subjec to $x_1 + x_2 \le 1$

$2x_1 + 3x_2 \geq 6$

$x_1, x_2 \geq 0.$

→ Let,

$\max z = -x_1 + x_2 + 0s_1 + 0s_2$

subject to $x_1 + x_2 + s_1 = 1$

$2x_1 + 3x_2 - s_2 = 6$

$x_1, x_2, s_1, s_2 \geq 0.$

$$\begin{array}{cccc} x_1 & x_2 & s_1 & s_2 \end{array}$$

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 2 & 3 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ s_1 \\ s_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \end{bmatrix}$$

adding $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Since coefficient matrix do not contain identity matrix then we have to solve this problem by two phase method by introducing artificial variable a_1.

Phase I]

Given LPP becomes,

$\max z^* = -a_1$

subject to $x_1 + x_2 + s_1 = 1$

$2x_1 + 3x_2 - s_2 + a_1 = 6$

$x_1, x_2, s_1, s_2 \geq 0.$

BV	C_B	x_B	x_1	x_2	s_1	s_2	a_1	min ratio
s_1	0	1	1	1	1	0	0	$1/1 = 1$
a_1	-1	6	2	3	0	-1	1	$6/3 = 2$
		$z = -6$	-2	-3	0	1	0	

$R_2 - 3R_1, \quad R_3 + 3R_1$

	C_B	x_B	x_1	x_2	s_1	s_2	a_1
x_2	0	1	1	1	1	0	0
a_1	-1	3	-1	0	-3	-1	1
		$z^* = -3$	1	0	3	1	0

$z^* = -3 < 0$

$\therefore \max z^* < 0 \quad \therefore$ It has infeasible sol^n.

2] $\min z = 2x_1 + 8x_2$

$5x_1 + 10x_2 = 150$

$x_1 \leq 20$

$x_2 \geq 14$

$x_1, x_2 \geq 0.$

→ Let

$\max(-z) = -2x_1 - 8x_2$

s.t. $5x_1 + 10x_2 = 150$

$x_1 + s_1 = 20$

$x_2 - s_2 = 14$

$$\begin{array}{cccc} x_1 & x_2 & s_1 & s_2 \end{array}$$
$$\begin{bmatrix} 5 & 10 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ s_1 \\ s_2 \end{bmatrix} = \begin{bmatrix} 150 \\ 20 \\ 14 \end{bmatrix}$$

Since coefficient matrix do not contain identity matrix then we have to solve by

Phase I]

Given LPP becomes,

$$\max z^* = -a_1 - a_2$$
$$5x_1 + 10x_2 + a_1 = 150$$
$$x_1 + s_1 = 20$$
$$x_2 - s_2 + a_2 = 14$$

BV	C_B	x_B	x_1	x_2	s_1	s_2	a_1	a_2	min ratio
a_1	-1	150	5	10	0	0	1	0	$150/10 = 15$
s_1	0	20	1	0	1	0	0	0	$20/0 = \infty$
a_2	-1	14	0	1	0	-1	0	1	$14/1 = 14$
		$z = -164$	-5	-11	0	1	0	0	

$R_1 - 10R_3$, $R_4 + 11R_3$

	C_B	x_B	x_1	x_2	s_1	s_2	a_1	a_2	min ratio
a_1	-1	10	5	0	0	-10	1	-10	$10/10 = 1$
s_1	0	20	1	0	1	0	0	0	$20/0 = \infty$
x_2	0	14	0	1	0	-1	0	1	$14/-1 = -14$
		$z = -10$	-5	0	0	-10	0	11	

$R_1/10$

BV	C_B	X_B	x_1	x_2	s_1	s_2	a_1	a_2
s_2	0	1	$1/2$	0	0	1	$1/10$	-1
s_1	0	20	1	0	1	0	0	0
x_2	0	14	0	1	0	-1	0	1
		$z=-10$	-5	0	0	-10	0	11

$R_3 + R_1$, $R_4 + 10R_1$

BV	C_B	X_B	x_1	x_2	s_1	s_2	a_1	a_2
s_2	0	1	$1/2$	0	0	1	$1/10$	-1
s_1	0	20	1	0	1	0	0	0
x_2	0	15	$1/2$	1	0	0	$1/10$	0
		$z^*=0$	0	0	0	0	1	1

$\max z^* = 0$, (no artificial variable)

$\therefore$ It has a feasible solⁿ.

Now we go to phase II.

$$\max(-z) = -2x_1 - 8x_2$$

$$\therefore \quad 5x_1 + 10x_2 = 150$$

$$x_1 + s_1 = 20$$

$$x_2 - s_2 = -14$$

$$x_1, x_2, s_1, s_2 \geq 0.$$

BV	C_B	X_B	x_1	x_2	s_1	s_2	Min ratio
s_2	0	1	$1/2$	0	0	1	$1/\tfrac{1}{2} = 2$
s_1	0	20	1	0	1	0	$20/1 = 20$
x_2	-8	15	$1/2$	1	0	0	$15/\tfrac{1}{2} = 30$
	$z=-1$	$-z=-120$	-2	0	0	0	

$R_1/\frac{1}{2}$, 8

		x_B	x_1	x_2	s_1	s_2
x_1	-2	2	$\frac{1}{2}$	0	0	2
s_1	0	20	1	0	~~0~~1	0
x_2	-8	15	$\frac{1}{2}$	1	0	0
		$-z = -120$	-2	0	0	0

$R_2 - R_1$, $R_3 - \frac{1}{2} R_1$, $R_4 + 2R_1$

		x_B	x_1	x_2	s_1	s_2
x_1	-2	2	1	0	0	2
s_1	0	18	0	0	1	-2
x_2	-8	14	0	1	0	-1
		$-z = -116$	0	0	0	4

-120 +4

$$-z = -116$$
$$z = 116$$
$$x_1 = 2 \ , \ x_2 = 14$$

3) $\max z = -2x_1 - x_2$

subject to $3x_1 + x_2 = 3$

$4x_1 + 3x_2 - x_3 = 6$

$x_1 + 2x_2 + x_4 = 3$

$x_i \geq 0.$

→ Let,

$\max z = -2x_1 - x_2$

subject to $3x_1 + x_2 = 3$

$4x_1 + 3x_2 - x_3 = 6$

$x_1 + 2x_2 + x_4 = 3$

$$\begin{bmatrix} 3 & 1 & 0 & 0 \\ 4 & 3 & -1 & 0 \\ 1 & 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \\ 3 \end{bmatrix}$$

Phase I

Given LPP becomes,

$$\max z^* = -a_1 - a_2$$

s.t. $3x_1 + x_2 + a_1 = 0$

$4x_1 + 3x_3 - x_3 + a_2 = 6$

$x_1 + 2x_2 + x_4 = 3$

B.V.	C_B	X_B	x_1	x_2	a_3	x_4	a_1	a_2	min ratio
a_1	-1	3	3	1	0	0	1	0	$3/3 = 1$
a_2	-1	6	4	3	-1	0	0	1	$6/4 = \frac{3}{2} = 1.5$
x_4	0	3	1	2	0	1	0	0	$3/1 = 3$
		$z^* = -9$	-7	-4	1	0	0	0	

$R_1/3$

		X_B	x_1	x_2	x_3	x_4	a_1	a_2
x_1	0	1	1	1/3	0	0	1/3	0
a_2	-1	6	4	3	-1	0	0	1
x_4	0	3	1	2	0	1	0	0
		$z^* = -9$	-7	-4	1	0	0	0

$R_2 - 4R_1 \quad R_3 - R_1 \quad R_4 + 7R_1$

BV	C_B	X_B	x_1	x_2	x_3	x_4	a_1	a_2	Min ratio
x_1	0	1	1	1/3	0	0	1/3	0	$\frac{1}{1/3} = 3$
a_2	-1	2	0	5/3	-1	0	-4/3	1	$\frac{2}{5/3} = \frac{6}{5}$
x_4	0	2	0	5/3	0	1	-1/3	0	$\frac{2}{5/3} = \frac{6}{5}$
		$z^* = -2$	0	-5/3	1	0	7/3	0	

$R_2 / 5/3$

X_B	x_1	x_2	x_3	x_4	a_1	a_2
1	1	1/3	0	0	1/3	0
6/5	0	1	-3/5	0	-4/5	3/5
2	0	5/3	0	1	-1/3	0
$z^* = -2$	0	-5/3	1	0	7/3	0

$R_1 - \frac{1}{3}R_2,\ R_3 - \frac{5}{3}R_2,\ R_4 + 5/3 R_2$

BV	C_B	x_B	x_1	x_2	x_3	x_4	a_1	a_2
x_1	0	3/5	1	0	1/5	0	3/5	-1/5
x_2	0	6/5	0	1	-3/5	0	-4/5	3/5
x_4	0	0	0	0	1	1	1	-1
		$z^* = 0$	0	0	0	0	1	1

Phase II]

BV	C_B	x_B	x_1	x_2	x_3	x_4
x_1	-2	3/5	1	0	1/5	0
x_2	-1	6/5	0	1	-3/5	0
x_4	0	0	0	0	1	1
		$z = -12/5$	0	0	1/5	0

All Δj's are positive. $\therefore$ LPP has optimal solution $z = -12/5$.

4] $\max z = 2x_1 + x_2$

subject to $x_1 - x_2 \le 10$

$2x_1 - x_2 \le 40$

$x_1, x_2 \geq 0$.

$\rightarrow$ Let, The standard form of LPP is,

$\max z = 2x_1 + x_2 + 0s_1 + 0s_2$

subject to $x_1 - x_2 + s_1 = 10$

$2x_1 - x_2 + s_2 = 40$

$x_1, x_2, s_1, s_2 \geq 0$.

BV	C_B	x_B	x_1	x_2	s_1	s_2	Min Ratio
s_1	0	10	1	-1	1	0	10/1 = 10
s_2	0	40	2	-1	0	1	40/2 = 20
		$z = 0$	-2	-1	0	0	

$R_2 - 2R_1$, $R_3 + 2R_1$

BV	C_B	x_B	x_1	x_2	s_1	s_2	Min ratio
x_1	2	10	1	-1	1	0	$10/-1 = -10$
s_2	0	20	0	1	-2	1	$20/1 = 20$
		z = 20	0	-3	2	0	

$R_1 + R_2$, $R_3 + 3R_2$

	C_B	x_B	x_1	x_2	s_1	s_2
x_1	2	30	1	0	-1	1
x_2	1	20	0	1	-2	1
		z = 80	0	0	-4	3

The elements in the column s_1 are all negative. Hence given LPP has unbounded solution.

5) $\max z = 107x_1 + x_2 + 2x_3$

subject to $14x_1 - x_2 - 6x_3 + 3x_4 = 7$

$16x_1 - \frac{1}{2}x_2 - 6x_3 \leq 5$

$3x_1 - x_2 - x_3 \leq 0$.

$x_1, x_2, x_3, x_4 \geq 0$.

→ Let,

The standard form of LPP is,

$\max z = 107x_1 + x_2 + 2x_3 + 0s_1 + 0s_2$

subject to $14x_1 - x_2 - 6x_3 + 3x_4 = 7$

$16x_1 - \frac{1}{2}x_2 - 6x_3 + s_1 = 5$

$3x_1 - x_2 - x_3 + s_2 = 0$

$x_1, x_2, x_3, x_4, s_1, s_2 \geq 0$

Now we get a matrix,

$$\begin{array}{cccccc} x_1 & x_2 & x_3 & x_4 & s_1 & s_2 \end{array}$$
$$\begin{bmatrix} 14 & -1 & -6 & 3 & 0 & 0 \\ 16 & -\frac{1}{2} & -6 & 0 & 1 & 0 \\ 3 & -1 & 1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ s_1 \\ s_2 \end{bmatrix} = \begin{bmatrix} 7 \\ 5 \\ 0 \end{bmatrix}$$

Since coefficient matrix do not contain identity matrix then we have to solve this problem by two phase method by introducing artificial variable, a_1.

Phase I]

Given LPP becomes,

$\max z^* = -a_1$

subject to $14x_1 - x_2 - 6x_3 + 3x_4 + a_1 = 7$

$16x_1 - \frac{1}{2}x_2 - 6x_3 + s_1 = 5$

$3x_1 - x_2 + x_3 + s_2 = 0$

BV	C_B	X_B	x_1	x_2	x_3	x_4	s_1	s_2	a_1	min ratio
a_1	-1	7	14	-1	-6	3	0	0	1	$7/14 = \frac{1}{2} = 0.5$
s_1	0	5	16	$-\frac{1}{2}$	-6	0	1	0	0	$5/16 = 0.3$
s_2	0	0	3	-1	1	0	0	1	0	$0/3 = 0$
		$z^* = -7$	-14	1	6	-3	0	0	0	

$R_3/3$

X_B	x_1	x_2	x_3	x_4	s_1	s_2	a_1
7	14	-1	-6	3	0	0	1
5	16	$-\frac{1}{2}$	-6	0	1	0	0
0	1	$-\frac{1}{3}$	$\frac{1}{3}$	0	0	$\frac{1}{3}$	0
-7	-14	1	6	-3	0	0	0

$R_1 - 14R_3$, $R_2 - 16R_3$, $R_4 + 14R_3$

x_B	x_1	x_2	x_3	x_4	s_1	s_2	a_1
7	0	$11/3$	$-32/3$	3	0	$-14/3$	1
5	0	$13/6$	$-34/3$	0	1	$-16/3$	0
0	1	$-1/3$	$1/3$	0	0	$1/3$	0
$z^* = -7$	0	$-14/3$	$32/3$	-3	0	$14/3$	0

BV	C_B	x_B	x_1	x_2	x_3	x_4	s_1	s_2	a_1	min. ratio
a_1	-1	7	0	$11/3$	$-32/3$	3	0	$-14/3$	1	$7/11/3 = 1.9$
s_1	0	5	0	$13/6$	$-34/3$	0	1	$-16/3$	0	$5/13/6 = 2.3$
x_1	0	0	1	$-1/3$	$1/3$	0	0	$1/3$	0	$0/-1/3 = -0$
		$z^* = -7$	0	$-14/3$	$32/3$	-3	0	$14/3$	0	

$R_1 / 11/3$

x_B	x_1	x_2	x_3	x_4	s_1	s_2	a_1
$21/11$	0	$\underline{1}$	$-32/11$	$9/11$	0	$-14/11$	$3/11$
5	0	$13/6$	$-34/3$	0	1	$-16/3$	0
0	1	$-1/3$	$1/3$	0	0	$1/3$	0
$z^* = -7$	0	$-14/3$	$32/3$	-3	0	$14/3$	0

$R_2 - \frac{13}{6}R_1$, $R_3 + \frac{1}{3}R_1$, $R_4 + \frac{14}{3}R_1$

x_B	x_1	x_2	x_3	x_4	s_1	s_2	a_1

Here $z^* = -7 < 0$ and artificial variable remain in the basis matrix. Hence give LPP has no solution ie. it has infeasible solution.

Problem of Degeneracy :-

At the stage of improving the sol^n during simplex procedure minimum ratio is determine, in the last column of simple table to find the leading element.

But sometimes the ratio is may not unique the value of one or more basic variables in the x_B column becomes equal to zero ~~is the~~ this case is the problem of degeneracy.

Tie in entering variable:-

To break tie in entering variable we choose min ratio j. Δj as a most negative

1] $\max\ z = 4x_1 + 4x_2 + 3x_3$

subject to $2x_1 + x_2 + x_3 \le 4$

$2x_1 + 3x_2 + 3x_3 \le 5$

$x_1 + x_2 - x_3 \le 3$

$x_1, x_2, x_3 \geq 0$

→ Let,

$\max\ z = 4x_1 + 4x_2 + 3x_3 + 0s_1 + 0s_2 + 0s_3$

subject to $2x_1 + x_2 + x_3 + s_1 = 4$

$2x_1 + 3x_2 + 3x_3 + s_2 = 5$

$x_1 + x_2 - x_3 + s_3 = 3$

$x_1, x_2, x_3 \geq 0.$

B.V.	C_B	X_B	x_1	x_2	x_3	s_1	s_2	s_3	Min. ratio x_1	Min. ratio x_2
s_1	0	4	2	1	1	1	0	0	$4/2 = 2$	$4/1 = 4$
s_2	0	5	2	3	3	0	1	0	$5/2 = 2.5$	$5/3 = 1.6$
s_3	0	3	1	1	-1	0	0	1	$3/1 = 3$	$3/1 = 3$
		$z = 0$	-4	-4	-3	0	0	0		

$2 \times -4 = -8$ & $1.6 \times -4 = -6.4$

2 is corresponding to x_1

$\therefore$ The min. ratio $j \cdot \Delta j$ for x_1 is -8 & for x_2 is -6.4.

$R_1/2$, $R_2 - 2R_1$,

X_B	x_1	x_2	x_3	s_1	s_2	s_3
2	1	$1/2$	$1/2$	$1/2$	0	0
5	2	3	3	0	1	0
3	1	1	-1	0	0	1
0	-4	-4	-3	0	0	0

$R_2 - 2R_1$, $R_3 - R_1$, $R_4 + 4R_1$

		X_B	x_1	x_2	x_3	s_1	s_2	s_3	min ratio
x_1	4	2	1	$1/2$	$1/2$	$1/2$	0	0	$2/\frac{1}{2} = 4$
s_2	0	1	0	2	2	-1	1	0	$1/2 = 0.5$
s_3	0	+1	+0	$1/2$	$-3/2$	$-1/2$	0	1	$1/\frac{1}{2} = 2$
		$z = 8$	0	-2	-1	2	0	0	

$R_2/2$

		X_B	x_1	x_2	x_3	s_1	s_2	s_3
x_1		2	1	$1/2$	$1/2$	$1/2$	0	0
x_2		$1/2$	0	1	1	$-1/2$	$1/2$	0
s_3		1	0	$1/2$	$-3/2$	$-1/2$	0	1
		8	0	-2	-1	2	0	0

$R_1 - \frac{1}{2}R_2$, $R_3 - \frac{1}{2}R_2$, $R_4 + 2R_2$

	x_B	x_1	x_2	x_3	s_1	s_2	s_3
x_1	7/4	1	0	0	3/4	-1/4	0
x_2	1/2	0	1	1	-1/2	1/2	0
s_3	3/4	0	0	-2	0	-1/4	1
	z = 9	0	0	1	1	1	0

not neccesary this procedure

* Tie in leaving variable:-

Method to resolve tie in minimum ratio:-

Step I]

First pick up the rows for which the minimum non-negative ratio is same.

Step II]

Now rearrange the columns of the usual simplex table so that the columns forming the original unit matrix come first in proper order.

Step III]

Then find the minimum of the ratio.

$$\frac{\text{Elements of first column of unit matrix}}{\text{corresponding element of key column}}$$

only for the rows for which minimum ratio is not unique.

i] If this minimum is attained for rows then this will determined the key element by intersecting the key column.

ii) If this minimum is not unique then go to next step.

Step IV]

Now compute the minimum ratio,

$$\frac{\text{elements of second column of unit matrix}}{\text{corresponding element of key column}}$$

Only for the rows for which minimum ratio was not unique is step III]. If this minimum ratio is unique for some rows then this row will determine key element by intersecting the key column. If this minimum is still not unique then we go to the next step.

1] $\max z = 2x_1 + x_2$

subject to
$$4x_1 + 3x_2 \le 12$$
$$4x_1 + x_2 \le 8$$
$$4x_1 - x_2 \le 8$$
$$x_1, x_2 \ge 0.$$

→ Let,

$$\max z = 2x_1 + x_2$$
$$4x_1 + 3x_2 + s_1 = 12$$
$$4x_1 + x_2 + s_2 = 8$$
$$4x_1 - x_2 + s_3 = 8$$
$$x_1, x_2 \ge 0.$$

BV	C_B	X_B	x_1	x_2	x_3	s_1	s_2	s_3	min
s_1	0	12	4	3		1	0	0	$12/4 = 3$
s_2	0	8	4	1		0	1	0	$8/4 = 2$
s_3	0	8	4	-1		0	0	1	$8/4$ 2
		$z = 0$	-2	-1		0	0	0	

min ratio is not unique

tie in s_2 & s_3, x_1 is key column / incoming also.

Now,

$$\left\{ \frac{0}{4}, \frac{0}{4} \right\} = \{0, 0\} \quad \text{not unique}$$

$$\left\{ \frac{1}{4}, \frac{0}{4} \right\} = \left\{ \frac{1}{4}, 0 \right\} \quad 0 \text{ is mini ratio corresponding to } s_3$$

$\therefore$ s_3 is outgoing / leaving vector.

$R_3/4$

BV	X_B	x_1	x_2	s_1	s_2	s_3
	12	4	3	1	0	0
	8	4	1	0	1	0
	2	1	$-\frac{1}{4}$	0	0	$\frac{1}{4}$
	0	-2	-1	0	0	0

$R_1 - 4R_3, \quad R_2 - 4R_3, \quad R_4 + 2R_3$

		X_B	x_1	x_2	s_1	s_2	s_3	min. ratio
s_1	0	4	0	4	1	0	-1	$4/4 = 1$
s_2	0	0	0	2	0	1	-1	$0/2 = 0$
x_1	2	2	1	$-1/4$	0	0	$1/4$	$2/(-1/4) = -$
		$z = 4$	0	$-3/2$	0	0	$1/2$	

$R_{2/2}$, $R_1 -$

	X_B	x_1	x_2	s_1	s_2	s_3
s_1	4	0	4	1	0	-1
x_2	0	0	1/4	0	1/2	-1/2
x_1	2	1	-1/4	0	0	1/4
	4	0	-3/2	0	0	1/2

$R_1 - 4R_2$, $R_3 + \frac{1}{4}R_2$, $R_4 + \frac{3}{2}R_2$

	X_B	x_1	x_2	s_1	s_2	s_3	min ratio
s_1	4	0	0	1	-2	1	4/1 = 4
x_2	0	0	1	0	1/2	-1/2	= 0
x_1	2	1	0	0	1/8	1/8	2/1/8 = 16
	z = 4	0	0	0	3/4	-1/4	

$R_2 + \frac{1}{2}R_1$, $R_3 - \frac{1}{8}R_1$, $R_4 + \frac{1}{4}R_1$

	C_B	X_B	x_1	x_2	s_1	s_2	s_3
s_3	0	4	0	0	1	-2	1
x_2	2	2	0	~~3/2~~ 1	1/2	-1/2	0
x_1	1	3/2	1	~~1/80~~	-1/8	3/8	0
		z = 5	0	~~1/40~~	1/4	1/4	0

2) $\max z = \frac{3}{4}x_1 - 150x_2 + \frac{1}{50}x_3 - x_4$

subject to $\frac{1}{4}x_1 - 60x_2 - \frac{1}{25}x_3 + 9x_4 \le 0$

$\frac{1}{2}x_1 - 90x_2 - \frac{1}{50}x_3 + 3x_4 \le 0$

$x_3 \le 1$

$x_i \ge 0$.

→ Let,

The standard form of LPP is,

$$\max z = \frac{3}{4}x_1 - 150x_2 + \frac{1}{50}x_3 - x_4 + 0s_1 + 0s_2 + 0s_3$$

subject to

$$\frac{1}{4}x_1 - 60x_2 - \frac{1}{25}x_3 + 9x_4 \cancel{\le 0} + s_1 = 0$$

$$\frac{1}{2}x_1 - 90x_2 - \frac{1}{50}x_3 + 3x_4 \cancel{\le 0} + s_2 = 0$$

$$x_3 + s_3 = 1$$

$$x_i \ge 0.$$

	BV	C_B	X_B	x_1	x_2	x_3	x_4	s_1	s_2	s_3	Min Ratio
$1/4$	s_1	0	0	$1/4$	-60	$-1/25$	9	1	0	0	$0/(1/4)=0$
$1/2$	s_2	0	0	$1/2$	-90	$-1/50$	3	0	1	0	$0/(1/2)=0$
0	s_3	0	1	0	0	1	0	0	0	1	$1/0=\infty$
			$z=0$	$-3/4$	150	$-1/50$	1	0	0	0	

min ratio is not unique

Tie is s_1 & s_2

Now,

$$\left\{\frac{1}{1/4}, \frac{0}{1/2}\right\} = \{4, \underline{0}\}$$, 0 is min ratio corresponding to s_2

$\therefore$ s_2 is outgoing vector.

$R_2/\frac{1}{2}$

X_B	x_1	x_2	x_3	x_4	s_1	s_2	s_3
0	$1/4$	-60	$-1/25$	9	1	0	0
0	$\underline{1}$	-180	$-1/25$	6	0	2	0
1	0	0	1	0	0	0	1
$z=0$	$-3/4$	150	$-1/50$	1	0	0	0

$R_1 - \frac{1}{4}R_2$, $R_4 + \frac{3}{4}R_2$

X_B	x_1	x_2	x_3	x_4	s_1	s_2	s_3
0	0	-15	$-3/100$	$15/2$	1	$-1/2$	0
0	1	-180	$-1/25$	6	0	2	0
1	0	0	1	0	0	0	1
$z=0$	0	15	$-1/20$	$11/2$	$\cancel{3/2}$ 0	$3/2$	0

BV	C_B	x_B	x_1	x_2	x_3	x_4	s_1	s_2	s_3	Min ratio
s_1	0	0	0	-15	$-3/100$	$15/2$	1	$-1/2$	0	$0/-3/100 = 0$
x_1	$3/4$	0	1	-180	$-1/25$	6	0	2	0	$0/-1/25 = 0$
s_3	0	1	0	0	1	0	0	0	1	$1/1 = 1$
	$z = 0$		0	15	$-1/20$	$11/2$	0	$3/2$	0	

Min ratio is not unique.

Tie in s_1 & x_1.

Now,

$$\left\{ \frac{1}{-3/100}, \frac{0}{-1/25} \right\} = \left\{ \frac{-100}{3}, 0 \right\}$$, 0 is min ratio corresponding to x_1

$\therefore$ x_1 is outgoing vector.

$R_2 / -1/25$

x_B	x_1	x_2	x_3	x_4	s_1	s_2	s_3
0	0	-15	$-3/100$	$15/2$	1	$-1/2$	0
0	-25	4500	1	-150	0	-50	0
1	0	0	1	0	0	0	1
$z = 0$	0	15	$-1/20$	$11/2$	0	$3/2$	0

$R_1 + \frac{3}{100} R_2$, $R_3 - R_1$, $R_4 + \frac{1}{20} R_2$

BV	C_B	x_B	x_1	x_2	x_3	x_4	s_1	s_2	s_3	Min ratio
s_1	0	0	$-3/4$	66	0	3	1	-2	0	$0/3 = 0$
x_3	$\frac{1}{50}$	0	-25	4500	1	-150	0	-50	0	$0/-150 = 0$
s_3	0	1	25		0	150	0	50	1	$1/150 =$
		0	$-5/4$	150	0	-2	0	-1	0	

$R_1 + \frac{3}{100} R_2$, $R_3 - R_2$, $R_4 + \frac{1}{20} R_2$

BV	C_B	x_B	x_1	x_2	x_3	x_4	s_1	s_2	s_3
s_1	0	0	$-3/4$	$-3/2$	0	3	1	-2	0
x_3	$1/50$	0	-25	4500	1	-150	0	-50	0
s_3	0	1	250	-4500	0	150	0	50	1
		0	$-5/4$	$75/2$ 240	0	-2	0	-1	0

BV	C_B	x_B	x_1	x_2	x_3	x_4	s_1	s_2	s_3	Min ratio
s_1	0	0	$-3/4$	$-3/2$	0	3	1	-2	0	$0/3 = 0$
x_3	$1/50$	0	-25	450	1	-150	0	-50	0	$0/-150 = 0$
s_3	0	1	25	-450	0	150	0	50	1	$1/150$
		$z = 0$	$-5/4$	$75/2$	0	-2	0	-1	0	

Min Ratio is not unique.
Tie in s_1 4 x_3

Now,

$\left\{ \frac{1}{3}, \frac{0}{-150} \right\} = \{0.3, 0\}$ 0 is min ratio corresponding to s_2.

$\max = \frac{1}{20}$
$x_1 = \frac{1}{25}$
$x_2 = 0$
$x_3 = 1$
$x_4 = 0$

Duality:-

Associated with every LPP there always exist another LPP which is based upon same data and having the same solution. This property of LPP is termed as duality in LPP.

The original problem is called as primal problem and associated problem is called as dual problem.

Any of this LPP can be taken as primal and other as dual therefore this problems are simultaneously called as primal dual pair.

Formulation of dual problem:-

For the formulation of dual problem from the primal problem following steps are used.

step 1] Convert the constraint of given LPP in the standard form.

2] Identify the decision variables for the dual variables. The no. of dual variable will be equal to the no. of constraints in primal problem.

3] Write the objective funⁿ for the dual problem by taking the constants on the right hand side of primal constraint as the cost coefficient for the dual problem.

If primal problem is maximization type then dual will be minimization type and vice versa.

Step 4] Define the constraint for the dual problem the column constraint coefficient of primal problem will become the row constraint coefficient of the dual problem. The cost coefficient of primal problem will be taken as the constant on the right hand side of dual constraint. If primal is of maximization type and then dual constraint must be of '$\geq$' type and if primal is of minimization type then dual constraint must be of '$\leq$' type.

Step 5] Dual variables will be unrestricted sign.

Let us consider the LPP as a maximization problem.

Primal

$\max z = c_1x_1 + c_2x_2 + \cdots + c_nx_n$

subject to $a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \leq b_1$

$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \leq b_m$

$x_1, x_2, \ldots x_n \geq 0.$

It's dual is,

Dual

$\min z^* = b_1 w_1 + b_2 w_2 + \cdots + b_m w_m$

subject to $a_{11} w_1 + a_{21} w_2 + \cdots + a_{m1} w_m \geq 0$

$a_{12} w_1 + a_{22} w_2 + \cdots + a_{m2} w_m \geq 0$

$\vdots$

$a_{1n} w_1 + a_{2n} w_2 + \cdots + a_{mn} w_m \geq 0.$

$w_1, w_2, \ldots w_m$ unrestricted in sign.

Let us consider the LPP as minimization problem.

Primal

$\min z = c_1 x_1 + c_2 x_2 + \cdots + c_n x_n$

subject to, $a_{11} x_1 + a_{12} x_2 + \cdots + a_{1n} x_n \geq b_1$

$a_{m1} x_1 + a_{m2} x_2 + \cdots + a_{mn} x_n \geq b_m$

$x_1, x_2, \ldots x_m \geq 0$

Dual

$\max z^* = b_1 w_1 + b_2 w_2 + \cdots + b_m w_m$

subject to, $a_{11} w_1 + a_{21} w_2 + \cdots + a_{m1} w_m \leq c_1$

$a_{12} w_1 + a_{22} w_2 + \cdots + a_{m2} w_m \leq c_2$

$\vdots$

$a_{1n} w_1 + a_{2n} w_2 + \cdots + a_{mn} w_m \leq c_n$

$w_1, w_2 \ldots w_m$ unrestricted in sign

Primal Problem	Dual Problem
$\max z = c^T x$	$\min z^* = b^T W$
subject to $AX = b$	subject to $A^T W \geq c$
$x \geq 0$	W is unrestricted in sign
$\min z = c^T x$	$\max z^* = b^T W$
subject to $AX = b$	subject to $A^T W \leq c$
$x \geq 0$	W is unrestricted in sign.

Q Write the dual of the given primal LPP.

$\max z = 5x_1 + 12x_2 + 4x_3$

subject to $x_1 + 2x_2 + x_3 \leq 10$

$2x_1 - x_2 + 3x_3 = 8$

$x_1, x_2, x_3 \geq 0$

→ Let,

$\max z = 5x_1 + 12x_2 + 4x_3 + 0s_1$

$x_1 + 2x_2 + x_3 + s_1 = 10$

$2x_1 - x_2 + 3x_3 = 8$

$x_1, x_2, x_3, s_1 \geq 0$

The no. of

$\min z^* = 10w_1 + 8w_2$

subject to $w_1 + 2w_2 \geq 5$

$2w_1 - w_2 \geq 12$

$w_1 + 3w_2 \geq 4$

$w_1 \geq 0$

w_1, w_2 unrestricted in sign.

$$\begin{bmatrix} 1 & 2 & 1 & 1 \\ 2 & -1 & 3 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 \\ 2 & -1 \\ 1 & 3 \\ 1 & 0 \end{bmatrix}$$

2] m~~ax~~in $z = 15x_1 + 12x_2$

subject to $x_1 + 2x_2 \geq 3$

Simplex $\quad 2x_1 - 4x_2 \leq 5$

$x_1, x_2 \geq 0$.

→ Let,

min ~~max~~ $z = 15x_1 + 12x_2$

subject to $x_1 + 2x_2 - s_1 = 3$

$2x_1 - 4x_2 + s_2 = 5$

$x_1, x_2 \geq 0$.

$$\begin{bmatrix} 1 & 2 & -1 & 0 \\ 2 & -4 & 0 & 1 \end{bmatrix}$$

$\max z^* = 3w_1 + 5w_2$

Simplex $\quad$ subject to,

$$\begin{bmatrix} 1 & 2 \\ 2 & -4 \\ -1 & 0 \\ 0 & 1 \end{bmatrix}$$

$w_1 + 2w_2 \leq 15$

$2w_1 - 4w_2 \leq 12$

$-w_1 \leq 0$

$w_2 \leq 0$

w_1, w_2, unrestricted in sign. —②.

Th^m 1:-

The dual of dual of given primal is the primal.

→ Proof:-

Consider the primal.

$\max z_x = c^T x$

subject to $Ax \leq b$

$x \geq 0$. —①

Dual of given primal is,

$\min z_w = b^T w$
subject to $A^T w \geq c$ — ②
$w \geq 0$

The corresponding primal is,
$\max (-z_w) = -b^T w$
subject to $-A^T w \leq -c$ — ③
$w \geq 0$.

Consider the dual of ③,
$\min (+z_u) = -c^T u$
subject to $-Au \geq -b$ — ④
$u \geq 0$.

The standard form of ④ is,
$\max (-z_u) = c^T u$ $\quad -z_u = z$
subject to $Au \leq b$ — ⑤
$u \geq 0$.

We observe that eqⁿ ① & ⑤ are equal
∴ Dual of dual is primal.

1] Take dual of example ② eqⁿ ②
→ Let,
$\min z = 15y_1 + 12y_2$
subject to, $y_1 + 2y_2 - s_1$

Procedure:-
General rules for converting any primal into its dual.

Step I] First convert the objective funⁿ to maximiza form.

Step II] If constraint has inequality sign '≥' then multiply both side by (-1), and make the inequality sign '≤'.

Step III] If constraint has an equality sign then it is replaced by two constraint involving the inequalities going in opposite direction simultaneously.

Step IV] Every unrestricted variable is replaced by the diffⁿ of two non-negative variable.

Step V] We get the standard primal form of given LPP in which,
i) All the constraints have '≤' sign where the objective funⁿ is of maximization form
ii) All the constraint have '≥' sign where the objective funⁿ is of minimization form.

Step VI] Finally the dual of any problem is obtained by.
i) Transposing the rows & colums of the constraint coefficient

ii) Transposing the coefficient of objective fun^n and right side constant.

iii) Changing the inequalities from '$\le$' to '$\ge$' ~~to~~ sign.

Minimizing the objective fun^n instead of maximizing it.

1) $\min z = 15x_1 + 12x_2$

subject to $x_1 + 2x_2 \ge 3$

$2x_1 \cancel{+} -4x_2 \le 5$

$x_1, x_2 \ge 0.$

→ Let,

$\max(-z) = -15x_1 - 12x_2$

subject to $-x_1 - 2x_2 \le -3$

$2x_1 - 4x_2 \le 5$

$x_1, x_2 \ge 0.$ —— (1)

Dual

$\min z^* = -3w_1 + 5w_2$

subject to $-w_1 + 2w_2 \ge -15$

$-2w_1 - 4w_2 \ge -12$

$w_1, w_2 \ge 0.$ —— (2)

The standard form of (2) is,

$\max(-z^*) = 3w_1 - 5w_2$

subject to $w_1 - 2w_2 \le 15$

$2w_1 + 4w_2 \le 12$

$w_1, w_2 \ge 0$

Dual

$\min(z^{**}) = 15y_1 + 12y_2$

subject to $y_1 + 2y_2 \geq 3$

$-2y_1 + 6y_2 \geq -5 \Rightarrow 2y_1 - 6y_2 \leq 5$

$\therefore$ Dual of dual is a primal.

2] $\min z^* = 2x_2 + 5x_3$

$x_1 + x_2 \geq 2$

$2x_1 + x_2 + 6x_3 \leq 6$

$x_1 - x_2 + 3x_3 = 4$

→ Let,

$\max(z) = -2x_2 - 5x_3$

$-x_1 - x_2 \leq -2$

$2x_1 + x_2 + 6x_3 \leq 6$

multi (-1) $-x_1 + x_2 - 3x_3 \leq -4$

$x_1 - x_2 + 3x_3 \leq 4$

Dual

$\min z^* = -2w_1 + 6w_2 - 4w_3 + 4w_4$

subject to $-w_1 + 2w_2 - w_3 + w_4 \geq 0$

$-w_1 + w_2 + w_3 - w_4 \geq -2$

$6w_2 - 3w_3 + 3w_4 \geq -5$

$w_1, w_2, w_3, w_4 \geq 0$

3] $\min z = 3x_1 - 2x_2 + 4x_3$

subject to $3x_1 + 5x_2 + 4x_3 \geq 7$

$6x_1 + x_2 + 3x_3 \geq 4$

$7x_1 - 2x_2 - 3x_3 \leq 10$

$x_1 - 2x_2 + 5x_3 \geq 3$

$4x_1 + 7x_2 - 2x_3 \geq 2$

$x_1, x_2, x_3 \geq 0.$

→ Let,

$\max(-z) = -3x_1 + 2x_2 - 4x_3$

subject to, $-3x_1 - 5x_2 - 4x_3 \cancel{\geq 7} \leq -7$

$-6x_1 - x_2 - 3x_3 \cancel{\geq 4} \leq -4$

$+7x_1 - 2x_2 - x_3 \cancel{\geq 10} \leq 10$

$-x_1 + 2x_2 - 5x_3 \cancel{\geq 3} \leq -3$

$-4x_1 - 7x_2 + 2x_3 \cancel{\geq 2} \leq -2$

$x_1, x_2, x_3 \geq 0.$

Dual

$\min(z^*) = -7w_1 - 4w_2 + 10w_3 - 3w_4 - 2w_5$

subject to $-3w_1 - 6w_2 + 7w_3 - w_4 - 4w_5 \geq -3$

$-5w_1 - w_2 - 2w_3 + 2w_4 - 7w_5 \geq 2$

$-4w_1 - 3w_2 - w_3 - 5w_4 + 2w_5 \geq -4$

$w_1, w_2, w_3, w_4 \geq 0,$

4] $\min z = 15x_1 + 12x_2$

subject to $x_1 + 2x_2 \geq 3$

$2x_1 - 4x_2 \leq 5$

$x_1, x_2 \geq 0.$

→ Let,

The standard form of LPP is, (primal)

$\max(-z) = -15x_1 - 12x_2 + 0s_1 + 0s_2$

subject to $\cancel{+x_1 + 12x_2}$ $x_1 + 2x_2 - s_1 = 3$

$2x_1 - 4x_2 + s_2 = 5$

$x_1, x_2 \geq 0$

Dual,

$\min z^* = 3w_1 + 5w_2$

subject to, $w_1 + 2w_2 \geq -15$

$2w_1 - 4w_2 \geq -12$ $\quad w_1, w_2 \geq 0,$

$-w_1 \geq 0$

$w_2 \geq 0$

The standard form of dual is,

$$\max(-z^*) = -3w_1 - 5w_2 + 0s_1 + 0s_2 + 0s_3 + 0s_4$$

subject to

$$w_1 + 2w_2 - s_1 = -15$$
$$2w_1 - 4w_2 - s_2 = -12$$
$$-w_1 - s_3 = 0$$
$$w_2 + s_4 = 0$$

The first simplex table for primal is,

BV	C_B	X_B	x_1	x_2	s_1	s_2	Min ratio
s_1	0	3	1	2	-1	0	
s_2	0	5	2	-4	0	1	
		$z = 0$	15	12	0	0	

The simplex table for dual is,

BV	C_B	X_B	w_1	w_2	s_1	s_2	s_3	s_4	Min ratio
s_1	0	-15	1	2	-1	0	0	0	
s_2	0	-12	2	-4	0	-1	0	0	
s_3	0	0	-1	0	0	0	-1	0	
s_4	0	0	0	1	0	0	0	-1	
		$z = 0$	3	5	0	0	0	0	

Th^m 2:-

If x is any feasible sol^n to the primal problem and W is any feasible sol^n to the dual problem then $C^T x \le b^T W$, $z_x \le z_w$

or $\sum_{i=1}^{n} c_i x_i \le \sum_{i=1}^{m} b_i w_i$

→ Proof:-

Consider the primal LPP,

$$\max z_x = c^T x$$

$$\text{s.t.} \quad Ax = b$$

$$x \geq 0$$

The dual of LPP is,

$$\min z_W = b^T W$$

$$\text{subject to} \quad A^T W = c$$

$$W \geq 0$$

The constraint of primal is matrix form is,

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

$$\sum_{j=1}^{n} a_{ij} x_j \le b_i \qquad i = 1, 2 \ldots m$$

The constaint of dual in matrix form is,

$$\begin{bmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & & a_{m2} \\ \vdots & \vdots & & \vdots \\ a_{1n} & a_{2n} & & a_{mn} \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_m \end{bmatrix} = \begin{bmatrix} \cancel{b_1} c_1 \\ \cancel{b_2} c_2 \\ \vdots \\ \cancel{b_n} c_n \end{bmatrix}$$

$$\sum_{p=1}^{m} a_{pk} w_p \geq c_k \qquad k = 1, 2 \ldots n$$

consider,

$$\sum_{i=1}^{n} c_i x_i = \sum_{i=1}^{n}\left(\sum_{p=1}^{m} a_{pi} w_p\right) x_i$$

$$= \sum_{p=1}^{m} w_p \left(\sum_{i=1}^{n} a_{pi} x_i\right)$$

$$\therefore \sum_{i=1}^{n} c_i x_i \leq \sum_{p=1}^{m} w_p b_p$$

$$\Rightarrow \sum_{i=1}^{n} c_i x_i \leq \sum_{i=1}^{m} w_i b_i$$

Th^m 3:-

If $\hat{\bar{x}}$ is a feasible solution to the primal and $\hat{\bar{w}}$ is a feasible solution to it's dual such that $\bar{c}\hat{\bar{x}} = \bar{b}\hat{\bar{W}}$ then $\hat{\bar{x}}$ is an optimal solution to the primal and $\hat{\bar{W}}$ is an optimal solution to the dual.

→ <u>Proof:-</u>

We know that if $\bar{x}$ is a feasible sol^n to the primal and $\hat{\bar{w}}$ is a feasible sol^n to it's dual then,

$$\bar{c}\bar{x} \leq \bar{b}\cdot\hat{\bar{W}}$$

But, $\bar{b}\hat{\bar{W}} = \bar{c}\hat{\bar{x}}$ (given)

$$\therefore \bar{c}\bar{x} \leq \bar{b}\hat{\bar{W}} = \bar{c}\hat{\bar{x}}$$

$$\bar{c}\bar{x} \leq \bar{c}\hat{\bar{x}}$$

$\therefore \bar{c}\cdot\hat{\bar{x}}$ is maximum value.

$\therefore \hat{\bar{x}}$ is a optimal sol^n to the problem

If $\bar{w}$ is any feasible sol^n to the sol^n then,

$$\bar{c}\cdot\hat{x} \le \bar{b}\cdot\bar{w}$$

But $\bar{c}\cdot\hat{x} = \bar{b}\,\hat{w}$

$\therefore\ \bar{c}\cdot\hat{x} = \bar{b}\,\hat{w} \le \bar{b}\cdot\bar{w}$

$\therefore\ \bar{b}\,\hat{w} \le \bar{b}\cdot\bar{w}$

$\therefore\ \bar{b}\,\hat{w}$ is minimum value.

$\therefore\ \hat{w}$ is optimal sol^n to the dual.

Th^m 4:-

If k^{th} constraint in the primal is an equality then the dual variable w_k is unrestricted in sign.

→ **Proof:-**

Consider the primal.

max $z_x = c^T x$

subject to $a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \le b_1$

$a_{k1}x_1 + a_{k2}x_2 + \cdots + a_{kn}x_n = b_k$

$\vdots$

$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \le b_m$

$x_i \ge 0 \qquad i = 1, 2, \ldots n$

max $z_x = c^T x$

subject to $a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \le b_1$

$a_{k1}x_1 + a_{k2}x_2 + \cdots + a_{kn}x_n \le b_k$

$a_{k1}x_1 \qquad \ge b_k$

$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \le b_m$

$\max z_x = c^T x$

subject to $a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \le b_1$

$a_{k1}x_1 + a_{k2}x_2 + \cdots + a_{kn}x_n \le b_k$

$-a_{k1}x_1 - a_{k2}x_2 - \cdots - a_{kn}x_n \le b_k$

$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \le b_m$

$x_i \geq 0, \quad i = 1, 2 \ldots n$

Dual

$\min z_w = b_1 w_1 + \cdots + b_k w_k' - b_k w_k'' + \cdots b_m w_m$

subject to $a_{11}w_1 + \cdots + a_{k1}w_k' - a_{k1}w_k'' + \cdots a_{m1}w_m \geq c_1$

$a_{12}w_1 + \cdots + a_{k2}w_k' - a_{k2}w_k'' + \cdots a_{m2}w_m \geq c_2$

$\vdots$

$a_{1n}w_1 + \cdots + a_{kn}w_k' - a_{kn}w_k'' + \cdots a_{mn}w_m \geq c_n$

$w_i \geq 0, \quad w_k', w_k'' \geq 0$

Put $w_k = w_k' - w_k''$ then w_k is unrestricted in sign.

$\therefore$ Dual of given LPP becomes,

$\min z_w = b_1 w_1 + \cdots + b_k w_k + \cdots + b_m w_m$

subject to $a_{11}w_1 + \cdots + a_{k1}w_k + \cdots + a_{m1}w_m \geq c_1$

$a_{21}w_1 + \cdots + a_{k2}w_k + \cdots + a_{m2}w_m \geq c_2$

$\vdots$

$a_{1n}w_1 + \cdots + a_{kn}w_k + \cdots + a_{mn}w_m \geq c_n$

$w_1, w_2 \ldots, w_{k-1}, w_{k+1}, \ldots w_n \geq 0$

w_k is unrestricted in sign.

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Basic Duality Theorem 5:-

If $x_0 (w_0)$ is an optimum solution to the primal (dual) then there exists a feasible sol^n $w_0 (x_0)$ to the dual (primal) such that,

$c^T x_0 =$

→ Proof:-

consider the primal max z

$$\max z_x = \bar{c} \cdot \bar{x}$$

subject to $A\bar{x} \leq b$

$\bar{x} \geq 0$

We can written as,

$$\max z_x = \bar{c} \cdot \bar{x}$$

subject to $A\bar{x} + I x_s = b$

$\bar{x} \geq 0$

$x_s \in R^m$ is the slack vector.

Let $x_0 = [x_B, 0]$ is an optimum sol^n to the primal where $x_B \in R^m$ is the optimum basic feasible sol^n given by, $x_B = B^{-1} b$.

Then the optimal primal sol^n is,

$$z = \bar{c} \cdot \bar{x} = \bar{c}_B \cdot \bar{x}_B$$

where $\bar{c}_B$ is a cost vector associated with $\bar{x}_B$.

Now the net evaluation in the optimal simplex table is given by,

$$z_j - c_j = c_B \bar{x}_j - c_j$$

$$= \begin{cases} \bar{c}_B B^{-1} e_j - 0 & \forall e_j \in A \\ \bar{c}_B B^{-1} a_j - c_j & \forall a_j \in A \end{cases}$$

Since x_B is a optimum we must have $z_j - c_j \geq 0 \quad \forall j$.

e_j = identity matrix

$A \cdot \bar{B} = A^T . B$

$\bar{c}_B B^{-1} e_j - 0 \geq 0$ and $\bar{c}_B B^{-1} a_j - c_j \geq 0$

$\bar{c}_B B^{-1} e_j \geq 0$ and $\bar{c}_B B^{-1} a_j > c_j \quad \forall j$

$c_B^T B^{-1} I \geq 0$ and $c_B^T B^{-1} A \geq c^T$

$c_B^T B^{-1} \geq 0$ and $c_B^T B^{-1} A \geq c^T$

We assume, $c_B^T B^{-1} = w_0^T$

$\therefore w_0^T \geq 0$ and $w_0^T A \geq c^T$

$\therefore w_0 \geq 0$ and $A^T w_0 \geq c$.

$\Rightarrow w_0$ is a feasible solⁿ to the dual problem. Moreover the corresponding dual ~~obj~~ objective funⁿ,

$= [x_B \; 0]$

$$b^T w_0 = w_0^T b = c_B^T B^{-1} b$$
$$= c_B^T x_B$$
$$= c_B^T x_0$$
$$= \bar{c}_B x_0$$

Thus given an optimum solⁿ x_0 to the primal there exist a feasible solⁿ w_0 to the dual such that,

$$c^T x_0 = b^T w_0$$

similarly starting with w_0 and the existence of x_0 can be proved.

Thᵐ 6 :-

If the p^{th} variable in primal is unrestricted in sign then p^{th} constraint of the dual is an equation.

$\Rightarrow$ **Proof:-**

Consider the primal LPP,

$\max z_x = c_1x_1 + c_2x_2 + \cdots + c_px_p + \cdots c_nx_n$

subject to,

$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1p}x_p + \cdots + a_{1n}x_n \leq b_1$

$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2p}x_p + \cdots + a_{2n}x_n \leq b_2$

$\vdots$

$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mp}x_p + \cdots + a_{mn}x_n \leq b_m$

$x_1, x_2, \ldots x_{p-1}, x_{p+1}, \ldots x_n \geq 0$

x_p unrestricted in sign.

$\therefore \quad x_p = x_p' - x_p'' \qquad x_p', x_p'' \geq 0$

$\therefore$ The primal becomes,

$\max z_x = c_1x_1 + c_2x_2 + \cdots + c_px_p' - c_px_p'' + \cdots + c_nx_n$

subject to,

$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1p}x_p' - a_{1p}x_p'' + \cdots + a_{1n}x_n \leq b_1$

$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2p}x_p' - a_{2p}x_p'' + \cdots + a_{2n}x_n \leq b_2$

$\vdots$

$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mp}x_p' - a_{mp}x_p'' + \cdots + a_{mn}x_n \leq b_m$

$x_1, x_2 \ldots x_{p-1}, x_p', x_p'', x_{p+1} \ldots x_n \geq 0.$

It's dual is,

$\min z_w = b_1w_1 + b_2w_2 + \cdots + b_pw_p + \cdots + b_mw_m$

subject to,

$a_{11}w_1 + a_{21}w_2 + \cdots + a_{p1}w_1 - a_{p1}w_1 + \cdots + a_{m1}w_1 \geq c_1$

p $a_{1p}w_1 + a_{2p}w_2 + \cdots + a_{pp}w_p + \cdots + \cdots + a_{mp}w_p \geq c_p$

p+1 $-a_{1p}w_1 - a_{2p}w_2 - \cdots - a_{pp}w_p - \cdots - a_{mp}w_p \geq -c_p$

$a_{1n}w_1 + a_{2n}w_2 + \cdots + a_{pn}w_p + \cdots + a_{mn}w_m \geq c_n$

$w_1, w_2, \ldots w_{p-1}, w_p, \ldots w_n \geq 0.$

Here P 4 P+1 constraints implies.

$a_{1p}\omega_1 + a_{2p}\omega_2 + \dots a_{pp}\omega_p \dots a_{mp}\omega_m = c_p$

Thus p^{th} constaints of the dual is an equation.

Revised Simplex ~~Table~~method:-

The usual simplex method used so far is lengthy algebraic procedure and the calculations in the usual simplex method are tedious and we have to following disadvantages.

i) It is very time consuming even when considered on the time scale of the electronic digital computers hence it is not an efficient computational procedure.

ii) In the usual simplex method many numbers are computate and stored which are either never used at the current iteration or are needed only in indirected way.

iii) Keeping this in mind a revised simplex method has been developed to overcome this disadvantages due to which speed of the calculation is increased by reducing the required amount of computational effort.

Ingeneral approach of the revised simplex method is identical to that of ordinary simplex method.

Standard forms for revised simplex method:-

There are two standard for the revised simplex method.

Standard form I]

In this form it assume that an identity basis matrix is obtained after introducing slack variables only.

Standard form II]

If artificial variables are needed for an initial identity basis matrix then two phase method of ordinary simplex method is used in a slightly diffⁿ way to handle artificial variables.

Formulation of LPP in standard form I]

A LPP in standard form.

$$\max z = c_1x_1 + c_2x_2 + \cdots + c_nx_n + 0x_{n+1} + 0x_{n+2} + \cdots + 0x_{n+m}$$

subject to,

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n + x_{n+1} = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n + x_{n+2} = b_2$$
$$\vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \quad x_{n+m} = b_m \qquad \text{--- (2)}$$

where the starting basis matrix b is an $m \times m$ identity matrix.

In the revised simplex form the objective funⁿ is also considered as another constraint in which z is as large as possible and unrestricted in sign. Thus eq^n ① & ② may be written in a compact form as,

$$z - c_1x_1 - c_2x_2 - \cdots - c_nx_n - 0x_{n+1} - 0x_{n+2} - \cdots - 0x_{n+m} = 0$$

subject to

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n + x_{n+1} = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n + x_{n+2} = b_2$$
$$\vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \quad x_{n+m} = b_m \qquad \text{③}$$

which can be considered as a system of $m+1$ simultaneous eq^{ns} in $n+m+1$ variables.
Here our aim is to find the sol^n of the system ③ such that z is as large as possible and unrestricted in sign. Now the system may be rewritten as,

$$1\cdot x_0 + a_{01}x_1 + a_{02}x_2 + \cdots + a_{0n}x_n + a_{0n+1}x_{n+1} + \cdots + a_{0n+m}x_{n+m} = 0$$

subject to,

$$0x_0 + a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n + x_{n+1} = b_1$$
$$0x_0 + a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \quad x_{n+2} = b_2$$
$$0x_0 + a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \quad x_{n+m} = b_m$$

where,

$$z_0 = x_0, \quad -c_j = a_{0j}, \quad j = 1, 2, \ldots, n+m$$

So write in this system ~~in~~ ④ in matrix form,

$$\begin{bmatrix} 1 & a_{01} & a_{02} & \cdots & a_{n+m} \\ 0 & a_{11} & a_{12} & & 0 \\ 0 & a_{21} & a_{22} & & \vdots \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & a_{m1} & a_{m2} & & 1 \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ \vdots \\ x_n \\ \vdots \\ x_{n+m} \end{bmatrix} = \begin{bmatrix} 0 \\ b_1 \\ b_2 \\ \vdots \\ \vdots \\ b_m \end{bmatrix}$$

Using the partitoning of matrix we get,

$$\begin{bmatrix} 1 & a_0 \\ 0 & A \end{bmatrix} \begin{bmatrix} x_0 \\ X \end{bmatrix} = \begin{bmatrix} 0 \\ b \end{bmatrix} \quad — ⑥$$

This matrix can be expressed in the original notation form.

$$\begin{bmatrix} 1 & -c \\ 0 & A \end{bmatrix} \begin{bmatrix} z \\ x \end{bmatrix} = \begin{bmatrix} 0 \\ b \end{bmatrix} \quad — ⑦$$

So eq^n ⑥ 4 ⑦ is referred as standard form (I) for revised simplex ~~tab~~ method.

Notations of standard form - I :-

It is observed that all the vectors have $m+1$ components hence subscript (1) is used for all vectors to show that they have m+1 components in standard form (I).

(I) Corresponding to a_j in A a new m+1 component vectors is represented by $a_j^{(1)}$ as

$$a_j^{(1)} = [-c_j, a_{1j}, a_{2j}, \cdots a_{mj}]$$
$$= [a_{0j}, a_{1j}, a_{2j}, \cdots a_{mj}].$$

$a_j^{(1)} = [a_{0j}, a_j]$, $j = 1, 2, \dots n$.

(ii) similarly corresponding to m component vector b in $Ax = b$ we shall represent the m+1 component vector by $\underline{\underline{b^{(1)}}}$

$b^{(1)} = [0, b_1, b_2, \dots, b_m]$
$= [0 \quad b]$.

(iii) The column vector corresponding to z is the m+1 component unit vector which is usually denoted by e and will always be in the first column of the basis matrix B_1.

$\therefore B_1 = [e_1, B_1^{(1)}, B_2^{(1)}, \dots B_m^{(1)}]$

If the basis matrix B for $Ax = b$ be represented by,

$$\begin{bmatrix} B_{11} & B_{12} & \dots & B_{1m} \\ B_{21} & B_{22} & & B_{2m} \\ & & & \\ B_{m1} & B_{m2} & & B_{mn} \end{bmatrix}$$

Then B_1 represented by,

$$B_1 = \left[\begin{array}{c|cccc} 1 & -c_{B_1} & -c_{B_2} & \dots & -c_{B_n} \\ 0 & B_{11} & B_{12} & \dots & B_{1n} \\ \vdots & & & & \vdots \\ 0 & B_{m1} & B_{m2} & \dots & B_{mn} \end{array}\right]$$

where $-c_{Bi}$ are the coefficient of x_{Bi} in the eq^n.

$z - c_1x_1 - c_2x_2 - \dots - c_nx_n = 0$

and $c_B = [c_{B_1}, c_{B_2}, \dots, c_{B_n}]$.

$\therefore B_1 = \begin{bmatrix} 1 & -c_B \\ 0 & B \end{bmatrix}$

$B_1^{-1} = \frac{1}{|B|} \begin{bmatrix} B & c_B \\ 0 & 1 \end{bmatrix}$

(19) And $B_1^{-1} = \begin{bmatrix} I & B^{-1} c_B \\ 0 & B^{-1} \end{bmatrix}$

(3) Any $a_j^{(1)}$ can be expressed as the linear combination of the column vectors, $(B_0^{(1)}, B_1^{(1)}, B_2^{(1)} \dots B_m^{(1)})$.

$a_j^{(1)} = x_{0j} B_0^{(1)} + x_{1j} B_1^{(1)} + \dots + x_{mj} B_m^{(1)}$
$= (x_{0j}, x_{1j}, \dots x_{mj})(B_0^{(1)}, B_1^{(1)}, \dots B_m^{(1)})$.

$a_j^{(1)} = x_j^{(1)} B_1 \qquad j = 1, 2 \dots n$

$x_j^{(1)} = B_1^{-1} a_j^{(1)}$.

$x_j^{(1)} = \begin{bmatrix} 1 & c_B B^{-1} \\ 0 & B^{-1} \end{bmatrix} \begin{bmatrix} -c_j \\ a_j \end{bmatrix}$

$= \begin{bmatrix} -c_j + c_B B^{-1} \\ B^{-1} a_j \end{bmatrix}$

$x_j^{(1)} = \begin{bmatrix} \Delta_j \\ x_j \end{bmatrix}$

(21) The $m+1$ component solution vector $x_B^{(1)}$ is given by,
$x_B^{(1)} = B_1^{-1} b^{(1)}$

$$x_B^{(1)} = \begin{bmatrix} 1 & c_B B^{-1} \\ 0 & B^{-1} \end{bmatrix} \begin{bmatrix} 0 \\ b \end{bmatrix}$$

$$= \begin{bmatrix} c_B B^{-1} b \\ B^{-1} b \end{bmatrix}$$

$$= \begin{bmatrix} c_B x_B \\ x_B \end{bmatrix}$$

$$x_B^{(1)} = \begin{bmatrix} z \\ x_B \end{bmatrix}$$

(vii) $x_B^{(1)}$ is a basic solution for the matrix eq^n corresponding to the basis matrix B_1. Also the first component of $x_B^{(1)}$ gives the value of the objective fun^n while the second component gives exactly the basic feasible sol^n to the original constraint, $Ax = b$ corresponding to it's basis matrix B.

(viii) The inverse of initial basis matrix is given by,

$$B_1^{-1} = \begin{bmatrix} 1 & c_B B^{-1} \\ 0 & B^{-1} \end{bmatrix}$$

But the initial basis matrix b for the original matrix is the identity matrix $B = I_m$.

$$\therefore B_1^{-1} = \begin{bmatrix} 1 & c_B \\ 0 & I_m \end{bmatrix}_{(m+1)\times(m+1)}.$$

$$\therefore c_{B_1} = c_{B_2} = \dots = c_{Bm} = 0.$$

$$\therefore B_1^{-1} = \left[\begin{array}{c|c} 1 & 0 \\ 0 & I_m \end{array}\right]$$

Thus the inverse of the initial basis matrix will be

$$B_1^{-1} = I_{m+1}$$

with which we start the revised simplex procedure then the initial basic sol^n is,

$$x_B^{(1)} = B_1^{-1} b^{(1)}$$
$$= I_{m+1} b^{(1)}$$
$$\therefore x_B^{(1)} = b^{(1)}$$

(viii) since x_0 should always be in the basis the first column $B_0^{(1)}$ of initial basis matrix inverse ie. B_1^{-1} will not be remove at any subsequent iteration. The remaining column vectors B_1^{-1} will be $B_1^{(1)}, B_2^{(1)} \ldots B_m^{(1)}$. The last column in the revised simplex table will be,

$$x_k^{(1)} = \begin{bmatrix} z_k - c_k \\ x_k \end{bmatrix} = \begin{bmatrix} \Delta_k \\ x_k \end{bmatrix}$$

variables in the Basis	e	B_1^{-1}: $B_1^{(1)}$	$B_2^{(1)}$	$\ldots B_m^{(1)}$	$x_B^{(1)}$	$x_k^{(1)}$
z	1	0	0	0	0	$z_k - c_k$
x_{B_1}	0	1	0	0	b_1	x_{1k}
⋮					⋮	
x_{B_m}	0	0		1	b_m	x_{mk}

The additional for those $a_j^{(1)}$ which are not included in the B_i^{-1} of starting table.

7] Solve the following simple LPP by revised simplex table method.

$\max z = 2x_1 + x_2$

subject to $3x_1 + 4x_2 \leq 6$

$6x_1 + x_2 \leq 3$

$x_1, x_2 \geq 0$.

→ Let,

Step I] Express the given problem in standard form after insuring that all b_i's ≥ 0 transferring the objective funⁿ of original problem for maximization of z and introduce non-negative slack variable to convert inequalities to eqⁿ and treat the objective funⁿ as a first constraint eqⁿ.

∴ Given problem becomes,

$z - 2x_1 - x_2 = 0$

$3x_1 + 4x_2 + s_1 = 6$

$6x_1 + x_2 + s_2 = 3$

$z, x_1, x_2, s_1, s_2 \geq 0$.

Step II] Construct the starting table in revised simplex form. The constraint eqⁿ can be expressed in the matrix form.

$$\begin{array}{ccccc} B_0^{(1)} & & & B_1^{(1)} & B_2^{(1)} \\ e & a_1^{(1)} & a_2^{(1)} & a_3^{(1)} & a_4^{(1)} \end{array}$$

$$\begin{bmatrix} 1 & -2 & -1 & 0 & 0 \\ 0 & 3 & 4 & 1 & 0 \\ 0 & 6 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} Z \\ x_1 \\ x_2 \\ s_1 \\ s_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 6 \\ 3 \end{bmatrix}$$

Here the columns $B_0^{(1)}$ $B_1^{(1)}$ $B_2^{(1)}$ will form the basis matrix B_1.

$\therefore B_1^{-1} = I_3$

The first revised simplex table is,

Variables in basis matrix	B_1^{-1}			$X_B^{(1)}$	$X_k^{(1)}$
	e	$B_1^{(1)}$	$B_2^{(1)}$		
Z	1	0	0	0	
s_1	0	1	0	6	
s_2	0	0	1	3	

$a_1^{(1)}$	$a_2^{(1)}$
-2	-1
3	4
6	1

<u>Step III]</u> First iteration. computation of Δj for $a_1^{(1)}$ and $a_2^{(1)}$.

Δj = (first row of B_1^{-1}) $\cdot$ a_j (not in the basis matrix)

$$\Delta_1 = (1 \quad 0 \quad 0)\begin{pmatrix} -2 \\ 3 \\ 6 \end{pmatrix} = -2$$

$$\Delta_2 = (+1 \quad 0 \quad 0)\begin{pmatrix} -1 \\ 4 \\ 1 \end{pmatrix} = -1$$

$$(1 \quad 0 \quad 0)\begin{pmatrix} -2 & -1 \\ 3 & 4 \\ 6 & 1 \end{pmatrix} = (-2, -1).$$

step IV] Apply test of optimality. Apply usual simplex rule to test the starting solⁿ for optimality. Since Δ_1, Δ_2 are both negative so the starting basic feasible solⁿ is not optimal.

step V] Determination of entering vector $a_k^{(1)}$:
Find such value of k for which
$\Delta_k = \min\{\Delta_j\}$,
For those j for which $a_j^{(1)}$ are not in the basis.

$$\Delta_k = \min\{\Delta_1, \Delta_2\}$$
$$= \min\{-2, -1\}$$
$$= -2$$
$$\therefore \quad \underline{\underline{k = 1}}$$

$\therefore$ $a_1^{(1)}$ enters the basis.
This indicates that the corresponding variable x_1 will enter the solⁿ.

step VI] compute the column vector x_k for $\underline{k=1}$.

$$\therefore \quad X_k^{(1)} = B_1^{-1} a_k^{(1)}$$
$$X_1^{(1)} = I \cdot a_1^{(1)}$$
$$X_1^{(1)} = \begin{pmatrix} -2 \\ 3 \\ 6 \end{pmatrix}$$

Variables in basis matrix	B_1^{-1}: e	$B_1^{(1)}$	$B_2^{(1)}$	$X_B^{(1)}$	$X_k^{(1)}$
z	1	0	0	0	-2
s_1	0	1	0	6 x_{B_1}	x_{11} 3
s_2	0	0	1	3 x_{B_2}	x_{21} 6

Step VII] Determination of the leaving vector $B_r^{(1)}$ given the entering vector $a_1^{(1)}$.

The vector $B_i^{(1)}$ to be removed from the basis is determined by using min ratio rule.

$$\frac{x_{Br}}{x_{rk}} = \min_i \left\{ \frac{x_{Bi}}{x_{ik}}, x_{ik} > 0 \text{ for } k=1 \right\}$$

$$\frac{x_{Br}}{x_{r1}} = \min_i \left\{ \frac{x_{Bi}}{x_{i1}}, x_{i1} > 0 \right\}$$

$$= \min_i \left\{ \frac{x_{B_1}}{x_{11}}, \frac{x_{B_2}}{x_{21}} \right\}$$

$$= \min \left\{ \frac{6}{3}, \frac{3}{6} \right\}$$

$$\frac{x_{B_2}}{x_{21}} = \frac{1}{2}$$

$\therefore r = 2$

The value of r is to this shows that the vector $B_2^{(1)}$ must leaves the basis.

Step VIII] Determination of the improved sol^n by transforming table :-

In order to bring uniformity with ordinary simplex method we we adopt simplex matrix. transformation rules which are easier for computation. The intermediate coefficient matrix can be written as,

$B_1^{(1)}$	$B_2^{(1)}$	$x_B^{(1)}$	$x_1^{(1)}$
0	0	0	-2
1	0	6	3
0	1	3	$\boxed{6}$

$R_3 \to R_3 \times \frac{1}{6}$

B_1	B_2	X_B	X_1
0	0	0	-2
1	0	6	3
0	1/6	1/2	1

$R_1 \to R_1 + 2R_3$ $\quad$ $R_2 \to R_2 - 3R_3$

B_1	B_2	X_B	X_1
0	1/3	1	0
1	-1/2	9/2	0
0	1/6	1/2	1

Now the vector B_2 has been removed from the basis matrix and it has entered in the place of x_1. In this way the process of entering a_1 & removing B_2 i.e. a_4 from the basis matrix has done.

Second Revised simplex table is,

Variables in	B_1^{-1}			$X_B^{(1)}$	$X_k^{(1)}$
Basis matrix	e	$B_1^{(1)}$	$B_2^{(1)}$		
z	1	0	1/3	1	
s_1	0	1	-1/2	9/2	
$\leftarrow x_1$	0	0	1/6	1/2	

removed x_1 added $a_4 = s_2$

add a_1 in the place of a_4 i.e. x

$a_4^{(1)} = s_2$	$a_2^{(1)}$
0	-1
0	4
1	1

The improved sol^n is,

$z = 1, \quad x_1 = \frac{1}{2}, \quad s_1 = \frac{9}{2}, \quad s_2 = 0, \quad x_2 = 0$

step IX] :- second iteration

$$\{\Delta_4 \quad \Delta_2\} = (1, 0, 1/3)\begin{pmatrix} 0 & -1 \\ 0 & 4 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1/3 \\ -2/3 \end{pmatrix} \begin{matrix} \to \Delta_4 \\ \to \Delta_2 \end{matrix}$$

min :. k=2

$\therefore k = 2$

Since Δ_2 is negative.

$\therefore$ Sol^n is not optimal.

step X] Determination of entering vector $a_k^{(1)}$.

Here $\Delta_k = \Delta_2$

$\therefore k = 2$

So $a_2^{(1)}$ should enter the solution, i.e. the variable x_2 will enter the basic sol^n

step XI] Determination of leaving vector

$$x_2^{(1)} = B_1^{-1} a_2^{(1)}$$

$$= \begin{bmatrix} 1 & 0 & 1/3 \\ 0 & 1 & -1/2 \\ 0 & 0 & 1/6 \end{bmatrix} \begin{bmatrix} -1 \\ 4 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} -2/3 \\ 7/2 \\ 1/6 \end{bmatrix}_{3\times 1}$$

Now we find the minimum ratio

$$\frac{x_{Br}}{x_{rk}} = \min\left\{ \frac{x_{Bi}}{x_{ik}} \quad x_{ik} > 0 \quad \text{for } k=2 \right\}$$

$$= \min\left\{ \frac{x_{Bi}}{x_{i2}} \quad x_{i2} > 0 \right\}$$

$$= \min \left\{ \frac{x_{B1}}{x_{12}}, \frac{x_{B2}}{x_{22}} \right\}$$

$$= \min \left\{ \frac{9/2}{7/2}, \frac{1/2}{1/6} \right\}$$

$$\frac{x_{B1}}{x_{11}} = \min \left\{ \frac{9}{7}, 3 \right\} = \frac{9}{7}$$

$\mu = 1$

Remove the vector $\overset{S_1}{B_1}$ from the basis

Variables in	B_1^{-1}			$X_B^{(1)}$	$X_1^{(1)}$
Basis matrix	e	B_1	B_2		
z	1	0	$1/3$	1	$-2/3$
s_1	0	1	$-1/2$	$9/2$	$\boxed{7/2}$
x_1	0	0	$1/6$	$1/2$	$1/6$

<u>Step XII]</u> Determination of the improved solⁿ

$B_1^{(1)}$	$B_2^{(1)}$	$X_B^{(1)}$	$X_1^{(1)}$
0	$\cancel{0}$ $1/3$	$\cancel{1/3}$ 1	$-2/3$
1	$-1/2$	$9/2$	$7/2$
0	$1/6$	$1/2$	$1/6$

$R_2 \times 2/7$

$B_1^{(1)}$	$B_2^{(1)}$	$X_B^{(1)}$	$X_1^{(1)}$
0	$1/3$	1	$-2/3$
$2/7$	$-1/7$	$9/7$	1
0	$1/6$	$1/2$	$1/6$

$R_1 + \frac{2}{3}R_2, \quad R_3 - \frac{1}{6}R_2$

$B_1^{(1)}$	$B_2^{(1)}$	$X_B^{(1)}$	$X_2^{(1)}$
$4/21$	$5/21$	$13/7$	0
$2/7$	$-1/7$	$9/7$	1
$-1/21$	$8/42$	$2/7$	0

Variables in Basis matrix	B_1^{-1}			$X_B^{(1)}$	$X_2^{(1)}$
	e	B_1			
z	1	$4/21$	$5/21$	$13/7$	0
x_2	0	$2/7$	$-1/7$	$9/7$	1
x_1	0	$-1/21$	$8/42$	$2/7$	0

$a_4^{(1)}$	$a_3^{(1)}$
0	0
0	1
1	0

step XIV]

$$[\Delta_4 \quad \Delta_3] = \begin{pmatrix} 1 & 4/21 & 5/21 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 5/21 \\ 4/21 \end{pmatrix} \begin{array}{l} \Delta_4 \\ \Delta_3 \end{array}$$

All the Δ_j are positive here

$\therefore$ Given LPP has optimal sol^n.

$\therefore$ The 2nd simplex table is,

BV	C_B	X_B	x_1	x_2	s_1	s_2	s_3	Min ratio
(s_1	0	6	1	0	1	-1	0)	$6/1 = 6$
x_2	4	14	$1/3$	1	0	$1/3$	0	$14/\frac{1}{3} = 42$
s_3	0	7	$2/3$	0	0	$-1/3$	1	$7/\frac{2}{3} = 21/2$
		$z=56$	$-2/3$	0	0	$4/3$	0	

$$R_2 - \frac{1}{3}R_1, \quad R_3 - \frac{2}{3}R_1, \quad R_4 + \frac{2}{3}R_1$$

BV	C_B	X_B	x_1	x_2	s_1	s_2	s_3	min ratio
x_1	2	6	1	0	1	-1	0	
x_2	4	12	0	1	$-1/3$	$2/3$	0	
s_3	0	5	0	0	$-1/3$	$1/3$	1	
		$z=54$	0	0	$2/3$	$2/3$	0	

$\therefore$ All Δ_j are positive.

$\therefore$ Given LPP has optimal solution

$z = \underline{54}$.

1) $\max z = x_1 + 2x_2$

subject to $x_1 + x_2 \le 3$

$x_1 + 2x_2 \le 5$

$3x_1 + x_2 \le 6$

$x_1, x_2 \geq 0.$

→ Let,

$$\max z - x_1 - 2x_2 = 0$$
$$x_1 + x_2 + s_1 = 3$$
$$x_1 + 2x_2 + s_2 = 5$$
$$3x_1 + x_2 + s_3 = 6$$
$$x_1 \; x_2 \; s_1 \; s_2 \; s_3 \geq 0$$

$$z - x_1 - 2x_2 = 0$$
$$x_1 + x_2 + s_1 = 3$$
$$x_1 + 2x_2 + s_2 = 5$$
$$3x_1 + x_2 + s_3 = 6 \qquad x_1, x_2, s_1, s_2, s_3 \geq 0.$$

$$\begin{array}{cccccc} \beta_0^{(1)} & & & \beta_1^{(1)} & \beta_2^{(1)} & \beta_3^{(1)} \\ e & a_1^{(1)} & a_2^{(1)} & a_3^{(1)} & a_4^{(1)} & a_5^{(1)} \end{array}$$
$$\begin{bmatrix} 1 & -1 & -2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 3 & 1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} z \\ x_1 \\ x_2 \\ s_1 \\ s_2 \\ s_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ 5 \\ 6 \end{bmatrix}$$

Variables in Basis matrix	B_1^{-1}				$X_B^{(1)}$	$X_K^{(1)}$
	e	$B_1^{(1)}$	$B_2^{(1)}$	$B_3^{(1)}$		
z	1	0	0	0	0	
s_1	0	1	0	0	3	
s_2	0	0	1	0	5	
s_3	0	0	0	1	6	

$a_1^{(1)}$	$a_2^{(1)}$
-1	-2
1	1
1	2
3	1

I] First iteration.

i) Δ_j = (first row of B_1^{-1}) $\cdot$ a_j

$$\{\Delta_1, \Delta_2\} = (1\ 0\ 0\ 0)\begin{pmatrix} -1 & -2 \\ 1 & 1 \\ 1 & 2 \\ 3 & 1 \end{pmatrix} = \{-1\ \ -2\}$$

Most negative $= \Delta_2$

$\therefore\ k = 2$

Δ_1 & Δ_2 are both negative.

$\therefore$ sol^n is not optimal

$k = 2$

$\therefore$ c

II] Determination of entering vector:-

To find the entering vector $a_i^{(1)}$ we apply the rule,

$$\Delta_k = \min\{\Delta_1, \Delta_2\}$$
$$= \min\{-1, -2\}$$
$$= -2$$
$$= \Delta_2$$

$\therefore\ k = 2$

$\therefore$ The vector a_2 must enter the basis

This shows that x_2 will enter the basic feasible sol^n

Determination of Leaving vector.

$$x_k^{(1)} = B_1^{-1} a_k^{(1)}$$
$$x_2^{(1)} = B_1^{-1} a_2^{(1)}$$
$$= I_m a_2^{(1)}$$
$$= a_2^{(1)}$$
$$x_2 = \begin{pmatrix} -2 \\ 1 \\ 2 \\ 1 \end{pmatrix}$$

Variables in basis matrix	B_1^{-1}				$x_B^{(1)}$	$x_2^{(1)}$	Min ratio
	e	B_1	B_2	B_3			
z	1	0	0	0	0	-2	
s_1	0	1	0	0	3	1	3
s_2	0	0	1	0	5	2	5/2
s_3	0	0	0	1	6	1	6

key elem.

By minimum ratio rule B_2 must leave the basis matrix Hence s_2 is outgoing variable.

Determination of improved sol^n.

$R_{3/2}$

B_1	B_2	B_3	x_B	x_2
0	0	0	0	-2
1	0	0	3	1
0	1/2	0	5/2	1
0	0	1	6	1

$R_1 + 2R_3$, $R_3 - R_3$ $R_4 - R_3$

B_1	B_2	B_3	x_B	x_2
0	1	0	5	0
1	$-1/2$	0	$1/2$	0
0	$1/2$	0	$5/2$	1
0	$-1/2$	1	$7/2$	0

Variables in	B_1^{-1}				x_B	$X_K^{(1)}$
Basis matrix	e	B_1	B_2	B_3		
z	1	0	1	0	5	
s_1	0	1	$-1/2$	0	$1/2$	
x_2	0	0	$1/2$	0	$5/2$	
s_3	0	0	$-1/2$	1	$7/2$	

$a_1^{(1)}$	$a_4^{(1)}$
-1	0
1	0
1	1
3	0

The improved sol^n is,

$z = 5, \quad s_1 = \frac{1}{2}, \quad x_2 = \frac{5}{2}, \quad s_3 = \frac{7}{2}, s_1 = 0, x_1 = 0$

second

$$\{\Delta_1 \quad \Delta_4\} = (1 \; 0 \; 1 \; 0) \begin{pmatrix} -1 & 0 \\ 1 & 0 \\ 1 & 1 \\ 3 & 0 \end{pmatrix} = (\underset{\Delta_1}{0}, \underset{\Delta_4}{1})$$

$\Delta_1 = 0, \quad \Delta_4 = 1$

$\because$ Δ_1 & Δ_4 are non-negative.

$\therefore$ sol^n is optimal. But $\Delta_1 = 0$ shows that

$x_1 = 0$
$x_2 = \frac{5}{2}$
$z = 5$

the LPP has alternate optimum solⁿ.

2) $\max z = 6x_1 - 2x_2 + 3x_3$

$$2x_1 - x_2 + 3x_3 \le 2$$
$$x_1 - 4x_3 \le 4$$
$$x_1, x_2, x_3 \geq 0.$$

→ Let

The standard form of LPP is,

$$z - 6x_1 + 2x_2 - 3x_3 = 0$$
$$2x_1 - x_2 + 3x_3 + s_1 = 2$$
$$x_1 - 4x_3 + s_2 = 4$$
$$x_1 \quad x_2 \quad x_3 \quad s_1 \quad s_2 \geq 0.$$

$$\begin{array}{cccccc} B_0^{(1)} & & & & B_1^{(1)} & B_2^{(1)} \\ e & a_1^{(1)} & a_2^{(1)} & a_3^{(1)} & a_4^{(1)} & a_5^{(1)} \end{array}$$

$$\begin{bmatrix} 1 & -6 & 2 & -3 & 0 & 0 \\ 0 & 2 & -1 & 3 & 1 & 0 \\ 0 & 1 & +0 & -4 & 0 & 1 \end{bmatrix} \begin{bmatrix} z \\ x_1 \\ x_2 \\ x_3 \\ s_1 \\ s_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 4 \end{bmatrix}$$

Variables in Basis matrix	B_1^{-1}			$X_B^{(1)}$	$X_K^{(1)}$
	e	$B_1^{(1)}$	$B_2^{(1)}$		
z	1	0	0	0	
s_1	0	1	0	2	
s_2	0	0	1	4	

a_1	a_2	a_3
-6	2	-3
2	-1	3
1	0	-4

I] First Iteration :-

$$\{\Delta_1, \Delta_2, \Delta_3\} = (1 \quad 0 \quad 0)\begin{pmatrix} -6 & 2 & -3 \\ 2 & -1 & 3 \\ 1 & 0 & 4 \end{pmatrix} = (\underset{a_1}{\underset{\downarrow}{-6}}, 2, -3)$$

Δ_1 and Δ_3 are both negative.

$\therefore$ solution is not optimal.

The most negative Δj.

$$\{\Delta_1\ \Delta_2\ \Delta_3\} = \langle -6, 2, -3 \rangle$$
$$= -6$$
$$= \Delta_1$$
$$\therefore k = 1$$

II] Determination of entering vector :-

To find the entering vector $a_k^{(1)}$ we apply the rule

$$\Delta_k = \min\{\Delta_1\ \Delta_2\ \Delta_3\}$$
$$= \min\{-6, 2, -3\}$$
$$= -6$$
$$= \Delta_1$$

$$\therefore \quad k = 1$$

$\therefore$ The vector $\overset{x_1}{a_1}$ must enter the basis.

This shows that x_1 will enter the basic feasible sol^n.

III] Determination of leaving vector.

$x_k^{(1)} = B_1^{-1} a_k^{(1)}$

$x_1^{(1)} = B_1^{-1} a_1^{(1)}$

$= I_m \cdot a_1^{(1)}$

$= a_1^{(1)}$

$= \begin{pmatrix} -6 \\ 2 \\ 1 \end{pmatrix}$

Variables in basis matrix	B_1^{-1}					
	e	$B_1^{(1)}$	$B_2^{(1)}$	$X_B^{(1)}$	$X_1^{(1)}$	Min
z	1	0	0	0	-6	
s_1	0	1	0	2	$\underline{\underline{2}}$	$2/2 = \underline{1}$
s_2	0	0	1	4	1	$4/1 = 4$

$B_1^{(1)} = s_1$ is outgoing vector. (leaving vector).

IV] Determination of improved solⁿ.

B_1	B_2	x_B	x_1
0	0	0	-6
1	0	2	$\underline{\underline{2}}$
0	1	4	1

$R_2/2$

B_1	B_2	x_B	x_1
0	0	0	-6
$1/2$	0	1	1
0	1	4	1

$R_1 + 6R_2$, $R_3 - R_2$

B_1	B_2	X_B	X_1
		6	0
3	0	1	1
1/2	0	3	0
-1/2	1		

variables in basis matrix	e	B_1	B_2	X_B	X_1
z	1	3	0	6	0
x_1	0	1/2	0	1	1
S_2	0	-1/2	1	3	0

s_1	a_2	a_3
0	2	-3
1	-1	3
0	0	4

II] Second Iteration :-

$$(\Delta_2 \quad \Delta_3) = (1 \quad 3 \quad 0)\begin{pmatrix} 0 & 2 & -3 \\ 1 & -1 & 3 \\ 0 & 0 & 4 \end{pmatrix} = \begin{pmatrix} 3 & \underset{\Delta_2}{-1} & 6 \end{pmatrix}$$

-3.4

$\therefore k = 2$

since Δ_2 is negative.

III] Determination of entering vector :- $a_k^{(1)}$

$\Delta_k = \Delta_2$

$k = 2$.

$\therefore a_2(x_2)$ is entering vector.

VII] Determination of Leaving Vector :-

$$X_2^{(1)} = B_1^{-1} a_2^{(1)}$$

$$= \begin{pmatrix} 1 & 3 & 0 \\ 0 & 1/2 & 0 \\ 0 & -1/2 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}$$

$$= \begin{bmatrix} -1 \\ -1/2 \\ 1/2 \end{bmatrix}$$

variables in basis matrix	B_1^{-1}			$X_B^{(1)}$	$X_2^{(1)}$	Min ratio
	e	B_1	B_2			
z	1	3	0	6	-1	
x_1	0	1/2	0	1	-1/2	
s_2	0	-1/2	1	3	1/2	6

$\therefore B_2 = S_2$ is outgoing vector.

VIII] Determination of improved solⁿ :-

B_1	B_2	X_B	X_2
3	0	6	-1
1/2	0	1	-1/2
-1/2	1	3	1/2

$R_3/\frac{1}{2}$

B_1	B_2	X_B	X_2
3	0	6	-1
1/2	0	1	-1/2
-1	2	6	1

$R_1 + R_3$, $R_2 + \frac{1}{2}R_3$

B_1	B_2	X_B	Y_2
2	2	812	0
0	1	64	0
-1	2	86	1

variables in Basis matrix	B_1^{-1}			X_B
	e	B_1	B_2	
z	1	2	8 2	012
x_1	0	0	4 1	04
x_2	0	-1	6 2	1 6

s_1	$s_2 = a_4$	a_3
1	0	-3
0	1	3
0	0	4

Step IX]

$$(\Delta_4 \quad \Delta_3) = (1 \quad 2 \quad 2)\begin{pmatrix} 1 & 0 & -3 \\ 0 & 1 & 3 \\ 0 & 0 & 4 \end{pmatrix} = (1 \quad 2 \quad 11).$$

All the Δj are positive here.

$\therefore$ Given LPP has optimal solution.

Procedure:-

Step I] If the problem is of minimization, convert it into the maximization problem.

Step II] Express the given problem in standard form - I.
After ensuring that all $b_i \geq 0$, express the given problem in in revised simplex form I.

Step III] Find the initial basic feasible sol^n and the basis matrix B_1.
In this step we proceed to obtain the initial basis matrix B_1 as an identity matrix. Thus the initial solution given by,

$$x_B^{(1)} = \{0, b_1, b_2, \ldots b_m\}$$

Step IV] Construct the starting table for revised simplex table as explained in.

Step V] Test the optimality of current BFS.
This is done by computing,

$\Delta j = z_j - c_j$ for all $a_j^{(1)}$ not in the basis B_1 by the formula.

$$\Delta j = (\text{first row of } B_1^{-1}) \times a_j^{(1)} \text{ (not in the basis)}$$

The BFS is optimal only when all $\Delta j \geq 0$.

If current BFS is neither optimal nor unbounded, proceed to improve it in the next step.

Step VI] Improve the BFS.

In this step we first find the incoming (entering) vector and the leaving (outgoing) vector to obtain key element. Then we determine the improved solution like regular simplex method as follows:

i] To find incoming vector :-

The incoming vector will be taken as $a_k^{(1)}$ if $\Delta_k = \min_j (\Delta_j)$ for those which $a_j^{(1)}$ are not in the Basis B_1.

ii] To find out-going vector :-

For this first we compute $x_k^{(1)}$ by the formula :

$x_k^{(1)} = B_1^{-1} a_k^{(1)} = [\Delta_k, x_{1k}, x_{2k}, \dots x_{mk}]$

The vector $\beta_r^{(1)}$ to be removed from the the Basis is determined by using the minimum ratio rule. Then selected corresponding to such value of r for which,

$$\frac{x_{Br}}{x_{rk}} = \min_i \left\{ \frac{x_{Bi}}{x_{ik}}, \; x_{ik} > 0 \right\}$$

Step VII] Now again test the optimality of above improved BFS as in step V. If this solution is not optimal, then repeat step VI until an optimal solⁿ is finally obtained.

3] max $z = x_1 + x_2$

subject to $3x_1 + 3x_2 \le 6$

$x_1 + 4x_2 \le 4$

$x_1, x_2 \geq 0$

→ Let,

The standard form of LPP is,

$z - x_1 - x_2 = 0$

$3x_1 + 3x_2 + s_1 = 6$

$x_1 + 4x_2 + s_2 = 4$

$x_1, x_2, s_1, s_2 \geq 0$

$$\begin{array}{ccccc} B_0^{(1)} & & & B_1^{(1)} & B_2^{(1)} \\ e & a_1^{(1)} & a_2^{(1)} & a_3^{(1)} & a_4^{(1)} \end{array}$$

$$\begin{bmatrix} 1 & -1 & -1 & 0 & 0 \\ 0 & 3 & 3 & 1 & 0 \\ 0 & 1 & 4 & 0 & 1 \end{bmatrix} \begin{bmatrix} z \\ x_1 \\ x_2 \\ s_1 \\ s_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 6 \\ 4 \end{bmatrix}$$

Variables in	B_1^{-1}			$X_B^{(1)}$	$X_k^{(1)}$
Basis matrix	e	$B_1^{(1)}$	$B_2^{(1)}$		
z	1	0	0	0	
s_1	0	1	0	6	
s_2	0	0	1	4	

$a_1^{(1)}$	$a_2^{(1)}$
-1	-1
3	3
1	4

I] First Iteration:-

$$\{\Delta_1 \ \Delta_2\} = (1 \ 0 \ 0)\begin{pmatrix} -1 & -1 \\ 3 & 3 \\ 1 & 4 \end{pmatrix} = (-1 \quad -1)$$

Most negative $= \Delta_1, \Delta_2$ are both negative

$\therefore$ sol^n is not optimal.

II] Determination of entering vector:-

To find the entering vector $a_k^{(1)}$ we apply the rule.

$$\Delta_k = \min\{\Delta_1, \Delta_2\}$$
$$= \min\{-1, -1\}$$
$$= -1$$
$$= \Delta_1$$

$\therefore k = 1$

$\therefore$ The vector a_1 must enter the basis. This shows that x_1 will enter the basic feasible sol^n.

III] Determination of Leaving vector

$$x_k^{(1)} = B_1^{-1} a_k^{(1)}$$
$$x_1^{(1)} = B_1^{-1} a_1^{(1)}$$
$$= I_3 \ a_1^{(1)}$$
$$= \begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 \\ 3 \\ 1 \end{bmatrix}$$

Variables in Basis matrix	e	B_i^{-1} B_1	B_2	$x_B^{(1)}$	$x_1^{(1)}$	Min ratio
z	1	0	0	0	-1	
(s_1	0	1	0	6	3	) $6/3 = 2$
s_2	0	0	1	4	1	$4/1 = 4$

By minimum ratio B_1 must leave the basis matrix. ∴ s_1 is leaving vector.

IV] Determination of improved solⁿ.

B_1	B_2	$x_B^{(1)}$	$x_1^{(1)}$
0	0	0	-1
1	0	6	3
0	1	4	1

$R_2/3$

B_1	B_2	$x_B^{(1)}$	$x_1^{(1)}$
0	0	0	-1
$1/3$	0	2	1
0	1	4	1

$R_1 + R_2$, $R_3 - R_2$

B_1	B_2	$x_B^{(1)}$	$x_1^{(1)}$
$1/3$	0	2	0
$1/3$	0	2	1
$-1/3$	1	2	0

Variables in basis matrix	e	B_1^{-1} B_1	B_2	$X_B^{(1)}$	$X_1^{(1)}$
z	1	$1/3$	0	2	0
x_1	0	$1/3$	0	2	1
s_2	0	$-1/3$	1	2	0

$a_3^{(1)}$	$a_2^{(1)}$
0	-1
0	3
1	4

II] Second iteration

$$\{\Delta_3 \quad \Delta_2\} = [1 \quad 1/3 \quad 0]\begin{bmatrix} 0 & -1 \\ 0 & 3 \\ 1 & 4 \end{bmatrix} = (0 \quad 0).$$

$\Delta_3 = 0, \quad \Delta_2 = 0$

Δ_3 are Δ_2 are non-negative.

b) $\max z = x_1 + 2x_2$

subject to $x_1 + 2x_2 \le 3$

$x_1 + 3x_2 \le 1$

$x_1, x_2 \geq 0$

$\rightarrow$ Let,

The standard form of LPP is,

or $z - x_1 - 2x_2$

$x_1 + 2x_2 + s_1 = 3$

$x_1 + 3x_2 + s_2 = 1$

$x_1, x_2\ s_1\ s_2 \geq 0$

$$\begin{array}{ccccc} B_0^{(1)} & & & B_1^{(1)} & B_2^{(1)} \\ e & a_1^{(1)} & a_2^{(1)} & a_3^{(1)} & a_4^{(1)} \end{array}$$

$$\begin{bmatrix} 1 & -1 & -2 & 0 & 0 \\ 0 & 1 & 3 & 1 & 0 \\ 0 & 1 & 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} z \\ x_1 \\ x_2 \\ s_1 \\ s_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix}$$

variables in Basis matrix	B_1^{-1}			$X_B^{(1)}$	$X_k^{(1)}$
	e	B_1	B_2		
z	1	0	0	0	
s_1	0	1	0	3	
s_2	0	0	1	1	

$a_1^{(1)}$	$a_2^{(1)}$
-1	-2
1	2
1	3

I] First iteration:-

$$\{\Delta_1, \Delta_2\} = (1\,0\,0)\begin{pmatrix} -1 & -2 \\ 1 & 2 \\ 1 & 3 \end{pmatrix} = (-1, -2)$$

Δ_1 and Δ_2 are both negative.

$\therefore$ sol^n is not optimal.

II] Determination of entering vector:-

To find the entering vector $a_k^{(1)}$ we apply the rule.

$$\Delta_k = \min\{\Delta_1, \Delta_2\}$$
$$= \min\{-1, -2\}$$
$$= -2$$
$$= \Delta_2$$

$\therefore k = 2$

$\therefore$ The vector a_2 must enter the vector basis. This shows that x_2 will enter the basic feasible sol^n.

III] Determination of Leaving vector:-

$$x_k^{(1)} = B_1^{-1}\, a_k^{(1)}$$

$$x_2^{(1)} = B_1^{-1}\, a_2^{(1)} \qquad = I_{3\times3} \cdot a_2^{(1)}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -2 \\ 2 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} -2 \\ 2 \\ 3 \end{bmatrix}$$

Variables in Basis matrix	B_1^{-1}			X_B	$X_B^{(1)}$	Min ratio
	e	B_1	B_2			
z	1	0	0	0	-2	
s_1	0	1	0	3	2	$3/3 = 1$
s_2	0	0	1	1	3	$1/3 = 0.3$

By minimum ratio B_2 must leave the basis matrix. Hence s_2 is leaving variable.

IV] Determination of improved solⁿ.

$R_{3/3}$

e	B_1	B_2	X_B	X_2
	0	0	0	-2
	1	0	3	2
	0	$1/3$	$1/3$	1

$R_1 + 2R_3 \quad R_2 - 3R_3$

B_1	B_2	X_B	X_2
0	$2/3$	$2/3$	0
1	-1	2	0
0	$1/3$	$1/3$	1

Variables in Basis matrix	B_1^{-1}			X_B	X_2
	e	B_1	B_2		
z	1	0	$2/3$	$2/3$	0
s_1	0	1	-1	2	0
x_2	0	0	$1/3$	$1/3$	1

$a_1^{(1)}$	$a_4^{(1)}$
-1	0
1	0
1	1

V] Second iteration:-

$$(\Delta_1 \quad \Delta_4) = [1 \quad 0 \quad 2/3]\begin{bmatrix} -1 & 0 \\ 1 & 0 \\ 1 & 1 \end{bmatrix} = [-1/3 \quad 2/3]$$

$\therefore k = 1$

since Δ_1 is negative.

sol^n is not optimal.

VI] Determination of entering vector, $a_k^{(1)}$

Here $\Delta_k = \Delta_1$

$k = 1$

So $a_1^{(1)}$ should enter the sol^n i.e. variable x_1 will enter the basic sol^n.

VII] Determination of leaving vector.

$$X_k^{(1)} = B_1^{-1} \, a_k^{(1)}$$

$$X_1^{(1)} = B_1^{-1} \cdot a_1^{(1)}$$

$$= I_{3m} \, a_1^{(1)}$$

$$= \begin{bmatrix} 1 & 0 & 2/3 \\ 0 & 1 & -1 \\ 0 & 0 & 1/3 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1/3 \\ 0 \\ 1/3 \end{bmatrix}$$

Variables in Basis matrix	B_1^{-1}			X_B	$X_1^{(1)}$	Min ratio
	e	B_1	B_2			
Z	1	0	2/3	2/3	-1/3	
S_1	0	1	-1	2	0	
x_2	0	0	1/3	1/3	1/3	$\frac{1/3}{1/3} = 1$

VIII] Determination of improved sol^n

B_1	B_2	x_B	x_1

Integer Linear Programming:-

Integer programming is the special class of linear programming problem where all or some of the variables in the optimal solⁿ are restricted to non-negative integer value.

Types of integer linear Programming problem:-

There are basically three types of integer problems.

i) Pure IPP:-

An IPP is said to be a pure IPP if all the decision variables are restricted to be integer.

ii) Mixed IPP:-

An IPP is said to be a mixed IPP if some but not all of its decision variables are restricted to be integer.

iii) Zero-one IPP:-

In which all the decision variables are restricted to integer zero or one.

Method to solve the IPP:-

There are two methods

i) Gomory's cutting plane method.

ii) Branch and Bound method.

Gomory's Cutting Plane Method:-

Gomory's cutting plane method was developed by R.E Gomory in 1956. This method is used for solving an pure integer LPP. This method starts without taking into consideration the integer requirements. If the solⁿ is integral then the current solⁿ is optimum. However if some of the basic variables are not integers value then additional linear constraints called as Gomory's constraint (fractional cut) is generated.

After having generated linear constraint, is added as the last row of the optimum simplex table indicating that the solⁿ is no longer feasible. The modified problem is then solved by using dual simplex method. An optimum integer solⁿ is obtained if all the variables in the solⁿ are integer values. otherwise another Gomory constraint is added and we repeat the procedure.

Construction of Gomory's constraint:-

BV	C_B	X_B	Basic BV				Non-Basic BV.	
			x_1	$x_2 \dots$	$x_i \dots$	x_m	$x_{m+1} \dots$	x_n
x_1	C_{B_1}	X_{B_1}	1	0	0	0	$x_{1,m+1}$	$x_{1,n}$
x_2	C_{B_2}	X_{B_2}	0	1	0	0	$x_{2,m+1}$	x_{2n}
$\vdots$	$\vdots$	$\vdots$						
x_i	C_{B_i}	X_{B_i}	0	0	1	0	$x_{i,m+1}$	x_{in}
$\vdots$	$\vdots$	$\vdots$						
x_m	C_{Bm}	X_{Bm}	0	0	0	1	$x_{m,m+1}$	x_{mn}
	$z = C_B X_B$		0	0	0	0	Δ_{m+1}	Δ_n

Let ith i^{th} basic variable x_{Bi} posses a non-integer value which is given by, the constraint eqn.

$$x_{Bi} = 0x_1 + 0x_2 + \dots + 0x_i + \dots 0x_m + x_{i\,m+1}x_{m+1} \dots x_{in}x_n$$

$$\left.\begin{aligned} x_{Bi} &= x_i + \sum_{j=m+1}^{n} x_{ij}x_j \\ x_i &= x_{Bi} - \sum_{j=m+1}^{n} x_{ij}x_j \end{aligned}\right\} \quad \text{—— (1)}$$

Let,

$$x_{Bi} = I_{Bi} + f_{Bi}$$
$$x_{ij} = I_{ij} + f_{ij}$$

where I_{Bi} and I_{ij} are the largest integral parts of x_{Bi} and x_{ij} resp such that,

$$I_{Bi} \leq x_{Bi}$$
$$I_{ij} \leq x_{ij}$$

and $0 < f_{Bi} < 1$ and $0 \leq f_{ij} < 1$.

where f_{Bi} is a strictly positive fraction and f_{ij} is a non-negative fraction.

$$x_i = I_{Bi} + f_{Bi} - \sum_{j=m+1}^{n} (I_{ij} + f_{ij})x_j$$

$$f_{Bi} - \sum_{j=m+1}^{n} f_{ij}x_j = x_i - I_{Bi} + \sum_{j=m+1}^{n} I_{ij}x_j \quad \text{—— (2)}$$

For all the variables x_i & x_j to be integer value the right hand side of the above eqn must be integer.

ie.

$f_{Bi} - \sum_{j=m+1}^{n} f_{ij} x_j$ must be integer.

Since, $0 < f_{Bi} < 1$ and $\sum f_{ij} x_j \geq 0$

$\therefore$ Therefore

$$f_{Bi} - \sum_{j=m+1}^{n} f_{ij} x_j \leq 0 \quad \text{--- (3)}$$

This is true bcoz

$$f_{Bi} - \sum_{j=m+1}^{n} f_{ij} x_j < f_{Bi} < 1.$$

This quantity can be eithere zero or negative integer.

$\therefore$ Eq^n (3) becomes,

$$f_{Bi} - \sum_{j=m+1}^{n} f_{ij} x_j + g_i = 0 \quad \text{--- (4)}$$

where g_i is a non-negative Gomorian slack variable which by def^n must be an integer. The constraint (4) is called the Gomory's cutting plane.

Gomory's cutting plane algorithm:-

Step (I) If the IPP is in minimization form, convert it into maximization form.

Step (II) Then convert the inequalities into equations by introducing slack/surplus variable and obtain the optimum sol^n of the LPP.

Step (III) 1) If the optimum sol^n contains all integer values then an optimum integer basic feasible sol^n has been achieved.

2) If not go to next step.

Step (IV) Examine the constraint eq^n corresponding to the current optimal sol^n.
Let these constraints be expressed by

$$x_{B_i} = x_i + \sum_{j=m+1}^{n} x_{ij} x_j$$

Select the largest fun^n of x_{Bi} i.e. find max $[f_{Bi}]$.

Step (V) Construct the Gomorian constraint.

$$f_{Bi} - \sum_{j=m+1}^{n} f_{ij} x_j \le 0.$$

i.e. $$f_{Bi} - \sum_{j=m+1}^{n} f_{ij} x_j + g_i = 0.$$

Step (VI) Starting with this set of constraint eq^n, obtain the new optimum sol^n by using dual simplex method. (Choose a variable to enter into the new sol^n having the smallest ratio $\{c_j - z_j / y_{ij} : y_{ij} < 0\}$ and return to step (IV)

step (V)

Step (VII) If this new optimum sol^n for the modified LPP is an all integer sol^n, it is also feasible & optimum for the given LPP otherwise we return to step (IV) and repeat the entire process until an optimum feasible integer sol^n is obtained.

17) $\max z = 7x_1 + 9x_2$

subject to $-x_1 + 3x_2 \le 6$

$7x_1 + x_2 \le 35$

$x_1, x_2 \geq 0.$

→ Let,

The standard form of LPP is,

$\max z = 7x_1 + 9x_2$

subject to $-x_1 + 3x_2 + s_1 = 6$

$7x_1 + x_2 + s_2 = 35$

$x_1, x_2, s_1, s_2 \geq 0.$

BV	C_B	X_B	x_1	x_2	s_1	s_2	Min ratio
s_1	0	6	-1	3	1	0	$6/3 = 2$
s_2	0	35	7	1	0	1	$35/1 = 35$
		$z = 0$	-7	-9	0	0.	

$R_1/3 \quad R_2 - R_1, \quad R_3 + 9R_1,$

X_B	x_1	x_2	s_1	s_2
2	$-1/3$	1	$1/3$	0
33	$22/3$	0	$-1/3$	1
$z = 18$	-10	0	3	0

BV	C_B	X_B	x_1	x_2	s_1	s_2	Min ratio
x_2	9	2	$-1/3$	1	$1/3$	0	
s_2	0	33	$22/3$	0	$-1/3$	1	$33/\frac{22}{3} = 99/2 = 19.$
		$z = 18$	-10	0	3	0	

$R_2/\frac{22}{3}$

X_B	x_1	x_2	s_1	s_2
2	$-1/3$	1	$1/3$	0
$9/2$	1	0	$-1/22$	$3/22$
$z = 18$	-10	0	3	0

$+R_1 + \frac{1}{3}R_2$, $R_3 + 10R_2$

	x_B	x_1	x_2	s_1	s_2
x_2	$7/2$	0	1	$7/22$	$1/22$
x_1	$9/2$	1	0	$-1/22$	$3/22$
	$z=63$	0	0	$28/11$	$15/11$

BV	C_B	x_B	x_1	x_2	s_1	s_2
x_2	9	$7/2$	0	1	$7/22$	$1/22$
x_1	7	$9/2$	1	0	$-1/22$	$3/22$
		$z=63$	0	0	$28/11$	$15/11$

Here x_1 and x_2 is not an integer.

We select the constraint corresponding to.

$\max(f_{B_i}) = \max\{f_{B_1}, f_{B_2}\}$

$x_{B_1} = \frac{9}{2} = I_{B_1} + f_{B_1}$

$= 4 + \frac{1}{2}$

$f_{B_1} = \frac{1}{2}$

$x_{B_2} = \frac{7}{2} = I_{B_2} + f_{B_2} = 3 + \frac{1}{2}$

$f_{B_2} = \frac{1}{2}$

$\therefore \max\{\frac{1}{2}, \frac{1}{2}\} = \frac{1}{2}$.

Here both eq^n of same value of f_{B_i}

Hence one of the two eq^n can be used.

Consider first row of optimum table to construct Gomorian constraint.

So Gomorian constraint is given by,

$f_{B_i} - \sum_{j=m+1}^{n} f_{ij} x_j + g_i = 0$

$m = 2, \quad i = 1, \quad f_{B_1} = \frac{1}{2}$

$$\frac{1}{2} - \sum_{j=3}^{4} f_{ij} x_j + g_i = 0.$$

$$\frac{1}{2} - f_{13} x_3 - f_{14} x_4 + g_1 = 0.$$

$$\frac{1}{2} - \frac{7}{22} x_3 - \frac{1}{22} x_4 + g_1 = 0$$

$$\frac{-7}{22} x_3 - \frac{1}{22} x_4 + g_1 = \frac{-1}{2}.$$

Adding this new constraint to optimum table.

BV	C_B	X_B	x_1	x_2	x_3	x_4	G_1
x_2	9	$7/2$	0	1	$7/22$	$1/22$	0
x_1	7	$9/2$	1	0	$-1/22$	$3/22$	0
g_1	0	$-1/2$	0	0	$-7/22$	$-1/22$	1
		$z = 63$	0	0	$28/11$	$15/11$	0

We apply the dual simplex method, leaving vector is G_1. So entering vector is

$$\max\left\{\frac{\Delta_3}{x_{33}}, \frac{\Delta_4}{x_{34}}\right\}$$

$$= \max\left\{\frac{28/11}{-7/22}, \frac{15/11}{-1/22}\right\}$$

$$= \max\{-8, -30\}$$

$$= -8$$

$\therefore$ The entering variable is x_3.

x_B	x_1	x_2	x_3	x_4	g_1
$7/2$	0	1	$7/22$	$1/22$	0
$9/2$	1	0	$-1/22$	$3/22$	0
$-1/2$	0	0	$-7/22$	$-1/22$	1
$z = 63$	0	0	$28/11$	$15/11$	0

$R_3 / -7/22$

x_B	x_1	x_2	x_3	x_4	g_1
$7/2$	0	1	$7/22$	$1/22$	0
$9/2$	1	0	$-1/22$	$3/22$	0
$11/7$	0	0	1	$1/7$	$-22/7$
63	0	0	$28/11$	$15/11$	0

$g_1 - x_3$

$R_1 - \frac{7}{22} R_3$, $R_2 + \frac{1}{22} R_3$, $R_4 - \frac{28}{11} R_3$

	x_B	x_1	x_2	x_3	x_4	g_1
x_2	3	0	1	0	0	1
x_1	$32/7$	1	0	0	$1/7$	$-1/7$
x_3	$11/7$	0	0	1	$1/7$	$-22/7$
	$z = 59$	0	0	0	1	8

$\therefore\ z = 59,\quad x_1 = \frac{32}{7},\quad x_2 = 3,\quad x_3 = \frac{11}{7}$

The optimal solⁿ as obtained by dual simplex method is still non-integer thus a new Gomory's constraint is to be constructed again

$x_{B_2} = \frac{32}{7} = I_{B_2} + f_{B_2} = 4 + \frac{4}{7} \Rightarrow 4, f_{B_2} = \frac{4}{7}$

$x_{B_3} = \frac{11}{7} = I_{B_3} + f_{B_3} = 1 + \frac{4}{7} \Rightarrow f_{B_3} = \frac{4}{7}$

$\max f_{Bi} = \max \{ f_{B_2}, f_{B_3} \} = \min\{\frac{4}{7}, \frac{4}{7}\} = \frac{4}{7}$.

$i=2$ corresponding row

consider first row of optimum table to construct Gomorian constraint.

So Gomorian constraint is given by.

$$f_{Bi} - \sum_{j=m+1}^{n} f_{ij} x_j + g_i = 0$$

$-\frac{1}{7} = -1 + \frac{6}{7}$ corresponding to f_{25}

$m = 3, \quad i = 2 \quad f_{B_2} = \frac{4}{7}$

$$\therefore \frac{4}{7} - \sum_{j=4}^{5} f_{2j} x_j + g_2 = 0$$

$$\frac{4}{7} - f_{24} x_4 - f_{25} x_5 + g_2 = 0$$

$$\frac{4}{7} - \frac{1}{7} x_4 - \frac{6}{7} g_1 + g_2 = 0$$

$$-\frac{1}{7} x_4 - \frac{6}{7} g_1 + g_2 = -\frac{4}{7}$$

Adding this new constraint to optimum table.

BV	C_B	X_B	x_1	x_2	x_3	x_4	G_1	g_2
x_2	9	3	0	1	0	0	1	0
x_1	7	32/7	1	0	0	1/7	-1/7	0
x_3	0	11/7	0	0	1	1/7	-22/7	0
g_2	0	-4/7	0	0	0	-1/7	-6/7	1
		$z=59$	0	0	0	1	8	0.

$\Delta_4 \quad \Delta_5$

$$\max\left\{\frac{\Delta_4}{x_{44}}, \frac{\Delta_5}{x_{45}}\right\}$$

$$= \max\left\{\frac{1}{-1/7}, \frac{8}{-6/7}\right\}$$

$$= \max\left\{\underset{\text{corresponding to } x_4}{-7}, \frac{-56}{6}\right\}$$

$$= -7$$

$\therefore$ The entering variable is x_4.

	x_B	x_1	x_2	x_3	x_4	G_1	G_2
1	3	0	1	0	0	1	0
2	32/7	1	0	0	1/7	-1/7	0
3	11/7	0	0	1	1/7	-22/7	0
4	-4/7	0	0	0	-1/7	-6/7	1
	59	0	0	0	1	8	0

$R_4 / -1/7$

	x_B	x_1	x_2	x_3	x_4	G_1	G_2
1	3	0	1	0	0	1	0
2	32/7	1	0	0	1/7	-1/7	0
3	11/7	0	0	1	1/7	-22/7	0
4	4	0	0	0	1	6	-7
5	z=59	0	0	0	1	8	0

$R_2 - \frac{1}{7}R_4$, $R_3 - \frac{1}{7}R_4$, $R_5 - R_4$

	x_B	x_1	x_2	x_3	x_4	G_1	G_2
x_2	3	0	1	0	0	1	0
x_1	4	1	0	0	0	-1	1
x_3	1	0	0	1	0	-1	1
x_4	4	0	0	0	1	6	-7
	z=55	0	0	0	0	2	7

$\therefore z = 55, \quad x_1 = 1 \quad x_2 = 3 \quad x_3 = 1 \quad x_4 = 4$

$\therefore$ The optimal sol^n obtained by dual simplex are all integers.

2] $\max z = x_1 + 2x_2$

subject to $2x_2 \le 7$

$x_1 + x_2 \le 7$

$2x_1 \le 11$

$x_1, x_2 \geq 0$.

$\rightarrow$ Let,

The standard form of LPP is,

$\max z = x_1 + 2x_2$

subject to $2x_2 + s_1 = 7$

$x_1 + x_2 + s_2 = 7$

$2x_1 + s_3 \le 11$

$s_1, s_2, s_3, x_1, x_2 \geq 0$.

BV	C_B	x_B	x_1	x_2	s_1	s_2	s_3	Min Ratio
s_1	0	7	0	2	1	0	0	$7/2 = 3.5$
s_2	0	7	1	1	0	1	0	$7/1 = 7$
s_3	0	11	2	0	0	0	1	
		$z = 0$	-1	-2	0	0	0	

$R_1 - 2R_2 \quad R_4 + 2R_2$

BV	C_B	x_B	x_1	x_2	s_1	s_2	s_3
s_1	0	-7	-2	0	1	-2	0
x_2	2	7	1	1	0	1	0
s_3	0	11	2	0	0	0	1
		$z = 14$	1	0	0	2	0

$R_1/2$

X_B	x_1	x_2	s_1	s_2	s_3
$7/2$	0	1	$1/2$	0	0
7	1	1	0	1	0
11	2	0	0	0	1
$z = 0$	-1	-2	0	0	0

$R_2 - R_1 \quad R_4 + 2R_1$

BV	C_B	X_B	x_1	x_2	s_1	s_2	s_3	Min ratio
x_2	2	$7/2$	0	1	$1/2$	0	0	
s_2	0	$7/2$	1	0	$-1/2$	1	0	$\frac{7/2}{1} = \frac{7}{2} = 3.5$
s_3	0	11	2	0	0	0	1	$11/2 = 5.5$
		$z = 7$	-1	0	1	0	0	

X_B	x_1	x_2	s_1	s_2	s_3
$7/2$	0	1	$1/2$	0	0
$7/2$	1	0	$-1/2$	1	0
11	2	0	0	0	1
$z = 7$	-1	0	1	0	0

$R_3 - 2R_2 \quad R_4 + R_2$

	BV	C_B	X_B	x_1	x_2	s_1	s_2	s_3
① f_{B_1}	x_2	2	$7/2$	0	1	$1/2$	0	0
② f_{B_2}	x_1	1	$7/2$	1	0	$-1/2$	1	0
	s_3	0	4	0	0	1	-2	1
			$z = 21/2$	0	0	$1/2$	1	0

Here x_1 and x_2 are not integers
We select the constraint corresponding to

$$\max(f_{B_i}) = \max\{f_{B_1}, f_{B_2}\}$$

$$x_{B_1} = \frac{7}{2} = I_{B_1} + f_{B_1}$$
$$= 3 + \frac{1}{2} \Rightarrow f_{B_1} = \frac{1}{2}$$

$$x_{B_2} = \frac{7}{2} = I_{B_2} + f_{B_2} = 3 + \frac{1}{2} \Rightarrow f_{B_2} = \frac{1}{2}$$

$\therefore \max\left\{\frac{1}{2}, \frac{1}{2}\right\} = \frac{1}{2}$

Hence both eq^n have same value of f_{Bi}. Hence one of the two eq^n can be used.

Consider first row of optimum table to construct Gomorian constraint.

So Gomorian constraint is given by,

$$f_{Bi} - \sum_{j=m+1}^{n} f_{ij} x_j + g_i = 0$$

$m = 2 \quad i = 1 \quad j \,\&\, n = 5.$

$$f_{B_1} - \sum_{j=3}^{5} f_{1j} x_j + g_1 = 0$$

$$f_{B_1} - f_{13} x_3 - f_{14} x_4 - f_{15} x_5 + g_1 = 0$$

$$\frac{1}{2} - \frac{1}{2} x_3 - 0 x_4 - 0 x_5 + g_1 = 0$$

$$\frac{-1}{2} x_3 + g_1 = \frac{-1}{2}$$

Adding this new constraint to optimum table.

BV	C_B	X_B	x_1	x_2	x_3	x_4	x_5	G_1
x_2	2	7/2	0	1	1/2	0	0	0
x_1	1	7/2	1	0	-1/2	1	0	0
s_3	0	4	0	0	1	-2	1	0
g_1	0	-1/2	0	0	-1/2	0	0	1
		$z = 21/2$	0	0	1/2	1	0	0

$\Delta_3 \quad \Delta_4 \quad \Delta_5$

We add the dual simplex method, leaving vector is G_1, so entering vector is,

$$\max\left\{\frac{\Delta_3}{x_{43}}, \frac{\Delta_4}{x_{44}}\right\}$$

$$= \max\left\{\frac{1/2}{-1/2}, \frac{1}{0}\right\}$$

$$= \max\{-1, \infty\}$$

$$= -1$$

$\therefore$ The entering variable is x_3.

$\cancel{x_B}$ $R_4/-1/2$

x_B	x_1	x_2	x_3	x_4	x_5	G_1
7/2	0	1	1/2	0	0	0
7/2	1	0	-1/2	1	0	0
4	0	0	1	-2	1	0
1	0	0	1	0	0	-2
$z = 21/2$	0	0	1/2	1	0	0

$R_1 - \frac{1}{2}R_4$, $R_2 + \frac{1}{2}R_4$ $R_3 - R_4$ $R_5 - \frac{1}{2}R_4$

BV	C_B	x_B	x_1	x_2	x_3	x_4	x_5	G_1
x_2	2	3	0	1	0	0	0	1
x_1	1	4	1	0	0	1	0	-1
s_3	0	3	0	0	0	-2	1	2
x_3	0	1	0	0	1	0	0	-2
		$z = 10$	0	0	0	1	0	1

$\therefore z = 10, \; x_1 = 4, \; x_2 = 3, \; x_3 = 1$

$\therefore$ The optimal sol^n obtained by dual simplex are all integers.

Branch and Bound Method :-

The Branch and Bound method was first developed by A.H. Land and A.G. Doig and it was further studied by J.O.C. Little. This method can be used to solved all integer mixed integer and zero one linear problems. This is the most general technique for the sol^n of IPP in which a few or all the variable are constrainted by their upper & lower bound.

Step I] Consider the following all integer programming problem.

$$\max z = c_1x_1 + c_2x_2 + \dots + c_nx_n$$

$$\text{subject to } \quad a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n \le b_1$$

$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n \le b_2$$

$$\vdots$$

$$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n \le b_m$$

Obtain the optimal sol^n of the given problem. If the sol^n to this LP problem is infeasible or unbounded, the sol^n to the given all IPP is also infeasible or unbounded otherwise examine optimal feasible sol^n if the answer satisfies the integer restrictions. The optimal integer sol^n has been obtained. If one or more basic variable do not satisfy integer requirement then go to step II.

Step II] a) Let the optimal value of objective funⁿ of LP-A be z_1. This value provides an initial upper bound on objective funⁿ for integer LP problem. It is denoted by z_u.

The lower bound on integer LP can be obtain by trucating to all integer values of the variables and the lower bound is denoted by z_l.

b) Let x_k be the basic variable having largest fractional value.

c) Branch the LP-A into two new LP subproblems (also called nodes), best on integer value of x_k.

$x_k \le [x_k]$

$x_k \geq [x_k] + 1$

To the original LP-problem.

Here $[x_k]$ is the integer partition of the current non-integer value of the variable x_k. This is done to exclude the non-integer value of the variable x_k. Then two new LP subproblem are as follows.

LP - sub problem B.

$$\max z = \sum_{j=1}^{n} c_j x_j$$

subject to $\sum a_{ij} x_j = b_i$

$x_k \le [x_k]$

$x_j \geq 0$.

LP - subproblem C

$$\max z = \sum_{j=1}^{n} c_j x_j$$

subject to $\sum a_{ij} x_j = b_i$

$x_k \geq [x_k]+1$

$x_j \geq 0.$

Step III] Bound step :-

obtain optimal sol^n of subproblem B & C. Let the optimal value of the objective fun^n of LP-B and LP-C be z_2 and z_3 resp.

Step IV] Examine sol^n of both LP-B and LP-C

i) Exclude a subproblem from further consideration if it has an infeasible sol^n

ii) If a subproblem needs a sol^n i.e. feasible but not an integer then for this subproblem written to step II.

iii) If subproblem needs a feasible integer sol^n examine the value of objective fun^n if this value is equal to upper bound z_u and optimal so^n has reached. But if it is not equal to upper bound z_u but exceed the lower bound z_l. This value is considered as a new upper bound and written to step II].

Finally if it is less than the lower bound terminate this branch.

Step 4] The procedure of branching and bounding continue until no further subproblems remains to be examined at this stage the integer solⁿ corresponding to the current lower bound is the optimal all integer programming problem solⁿ.

1] Solve the following all IPP using the Branch and Bound method.

$\max z = 3x_1 + 5x_2$

subject to $2x_1 + 4x_2 \le 25$

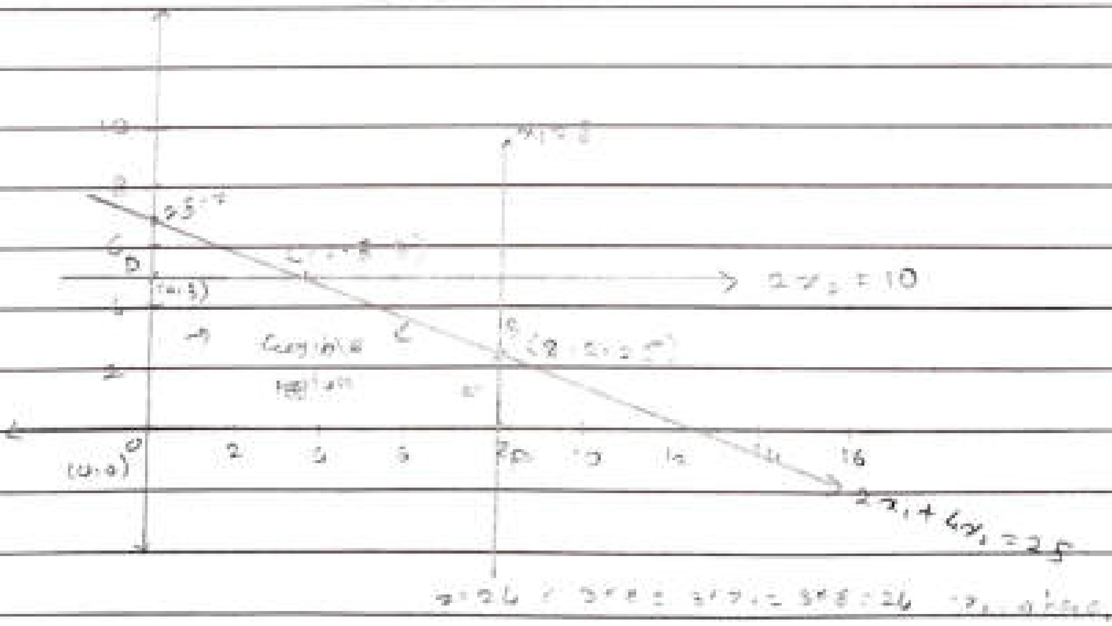
$x_1 \le 8$

$2x_2 \le 10$

$x_1, x_2 \ge 0$ & integer.

→ Let.

Relaxing the integer requiredments the optimal non-integer solⁿ of the given integer problem obtain by graphical problem.



$8 \times 3 + 2.25 \times 5 = 35.25$

The optimal sol^n is,

$z_1 = 35.25$, $x_1 = 8$, $x_2 = 2.25$

The value of z_1 represent the initial upper bound z_u. The lower bound z_L is obtain by truncating the sol^n values to $x_1 = 8$, $x_2 = 2$

$\therefore z_n = 34$.

The value x_2 is the only non-integer sol^n value. Therefore it is selected for divided the given problem into two subproblem LP-B & LP-C. Two new constraint,

$x_2 \le 2$, $x_2 \ge 3$ are created.

$\therefore$ subproblems are,

$\max z = 3x_1 + 5x_2$

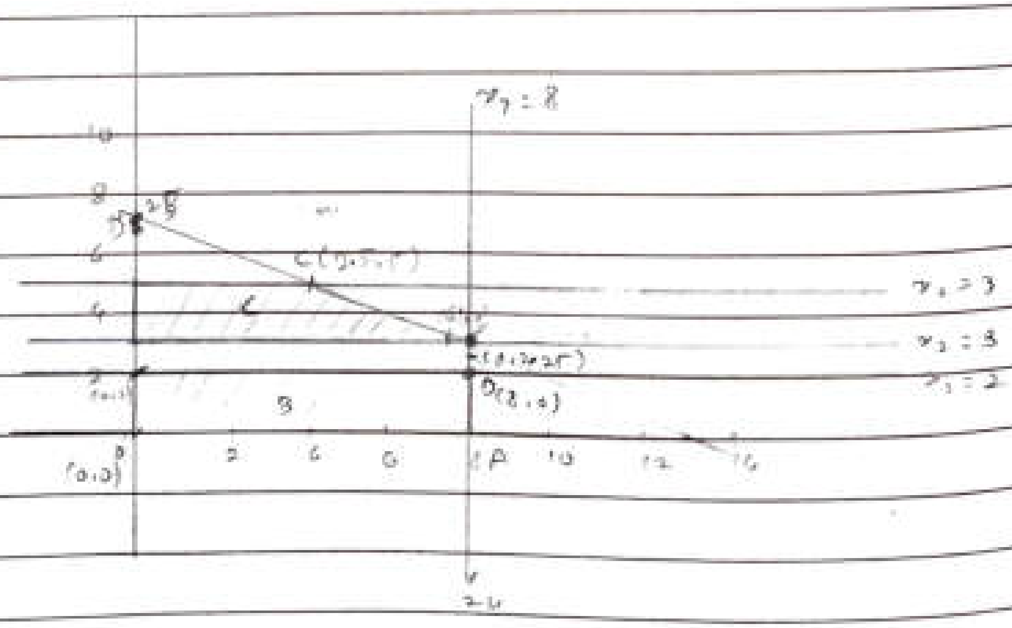
subject to $2x_1 + 4x_2 \le 25$

$x_1 \le 8$

$2x_2 \le 10$

$x_2 \le 2$, $x_2 \ge 3$

$x_1, x_2 \ge 0$ and integer.



The sol^n to the subproblem B is,
$x_1 = 8$, $x_2 = 2$, $z_2 = 34$.

The sol^n to the subproblem C is,
$x_1 = 6.5$, $x_2 = 3$, $z_3 = 34.5$.

Both the values of z lower than that of original LP problem. Since the sol^n of subpro B is an all integer so we stop the search of this subproblem. The value of $z_1 = \underline{\underline{34}}$ becomes the new lower bound on the IP problem.

A non-integer sol^n of subproblem 'c' indicate that the further branching is necessary. The upper bound now text the value $\underline{\underline{34.5}}$. Now the subproblem C is branched into two new subproblems D & E, and are obtained by adding.

$$\max z = 3x_1 + 5x_2$$

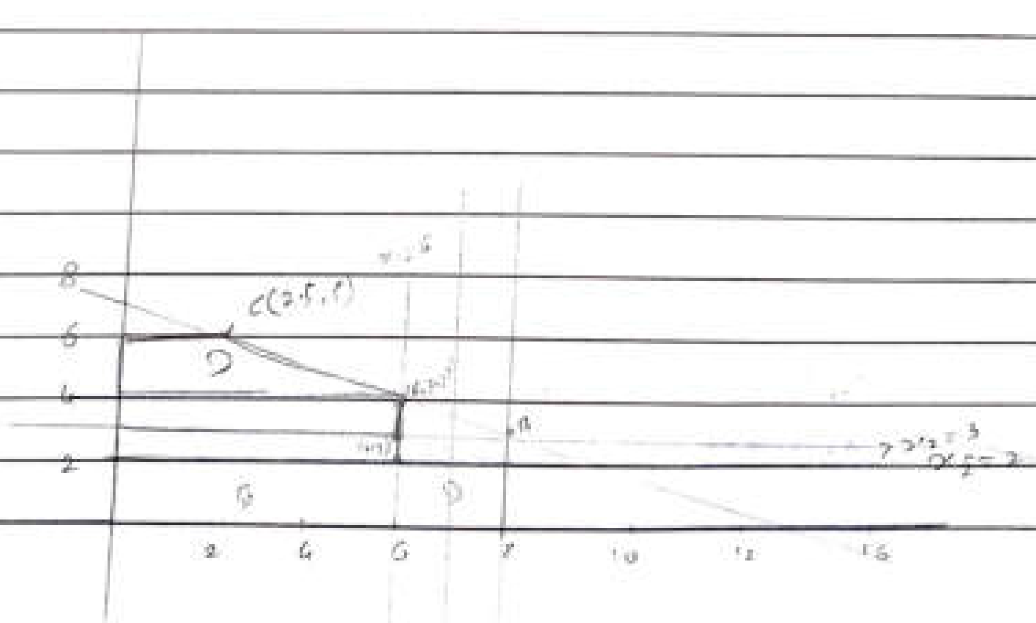
$$2x_1 + 4x_2 \le 25$$

$$2x_2 \le 10$$

$$x_1 \le 8$$

$$x_2 \ge 3$$

$$x_1 \le 6 \quad , \quad x_1 \ge 7$$

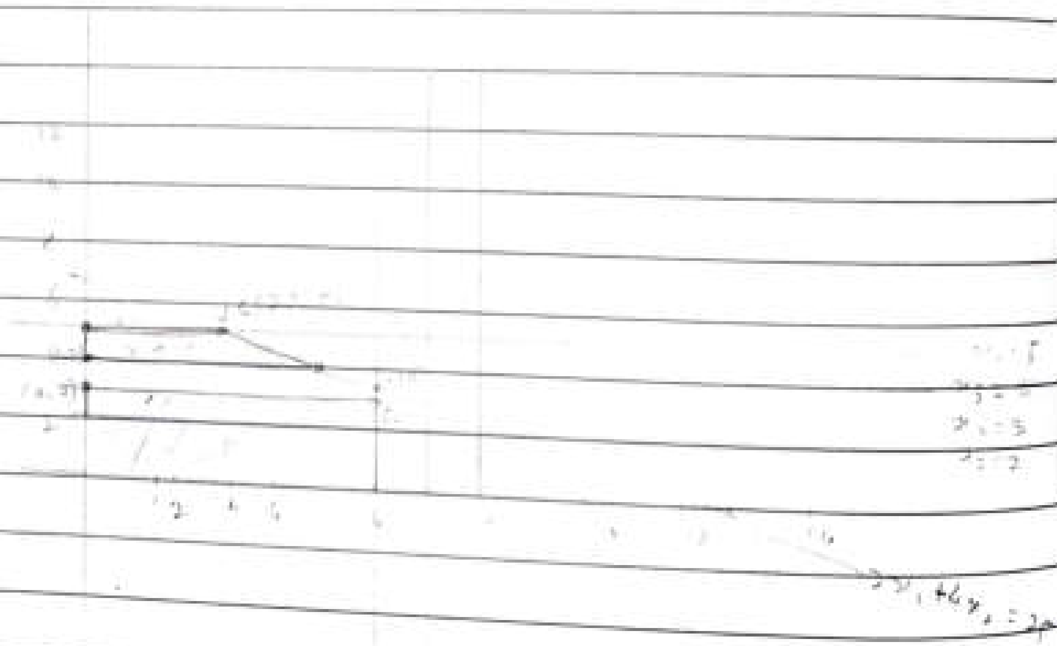


The sol^n of LP-B is,
$x_1 = 6$, $x_2 = 3.25$, $Z_4 = 34.25$

No feasible sol^n exist to LP-E bcoz the constraint $x_1 > 7$ and $x_2 > 3$ do not satisfy $2x_1 + 4x_2 \leq 25$. So this branch is terminated.

In problem D, x_2 is not an integer sol^n so we create new subproblem F & G from problem D with two new constraint.

F	G
$\max z = 3x_1 + 5x_2$	$\max z = 3x_1 + 5x_2$
$2x_1 + 4x_2 \leq 25$	$2x_1 + 4x_2 \leq 25$
$2x_2 \leq 10$	$2x_2 \leq 10$
$x_1 \leq 8$	$x_1 \leq 8$
$x_1 \leq 6$	$x_1 \leq 6$
$x_2 \leq 3$	$x_2 \geq 4$.



$x_1 = 6$ $\quad x_2 = 3$ $\quad Z_5 = 33$ $\quad$ - F

$x_1 = 4.25$ $\quad x_2 = 4$ $\quad Z_6 =$ $\quad$ - G

The branching process is terminated when new upper bound is less than or equal to the lower bound of previous soln. Although the soln at node G is non-integer, no additional branching is required from this node bcoz $z_6 < z_4$.

The Branch & Bound Algorithm is terminated.

The optimal integer soln is $x_1 = 8$, $x_2 = 2$ and $z_7 = 34$.