WELCOME
Special Theory of Relativity

By

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Special Theory of Relativity

The theory of relativity put forward by Einstein in the year 1905 revolutionized physical concepts during twentieth century.

*It is well known that physics is a '**science of measurements**' of various physical quantities.*

The theory of relativity reveals that the measurements depend upon the state of motion of the observer as well as upon the quantities that are being measured. The idea of relativity incorporated into mechanics gives rise to relativistic mechanics in which we come across some peculiar phenomenon, particularly when the particles forming a system are moving with high velocity comparable to that of light.

Newtonian Relativity

According to Newton,

'*Absolute motion is the translation of the body from one absolute place to another absolute place'*.

But what is meant by 'absolute place'? Special Theory of Relativity

Newton's view: A translatory motion can be detected only in the form of a motion relative to the other material bodies.

Motion involves the passage of time. According to Newton, 'absolute, true and mathematical time, of itself and by its own nature, flows uniformly on, without regard to anything external'. Thus single time scale would be valid everywhere. This is Newtonian relativity.

Let us represent the position of a point in terms of position vector r = r (x, y, z) and let t denote the time. Then r and t specify the position and the instant at which some kind of event occurs.

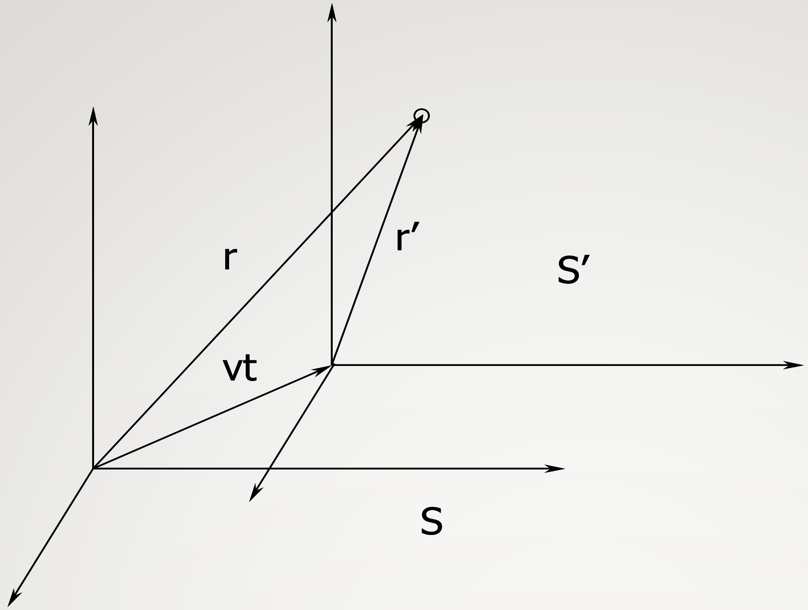
If S and S' are two frames of reference in uniform translatory motion with respect to each other, we get transformation equations.

$\mathbf{r}' = \mathbf{r} - \mathbf{v}t$ and $t' = t$

The velocities and accelerations are given by and respectively.

From above equation it is clear that the velocity of a particle is different in the two different systems.

The transformation given by $\mathbf{r}' = \mathbf{r} - \mathbf{v}t$ is called **Galilean transformation**.



Suppose that a source of light is placed at the origin of system S. Let r represent the position vector of a point on the wave surface which is spherical. The velocity of light is then expressed as

Where is the unit vector along r. The velocity of the light with respect to S′ would be

This equation shows that the magnitude of the velocity of light is not equal to c. Moreover it depends upon the direction of propagation of light. Hence, the wave surface will not be spherical. A series of experiments were carried out to verify this conclusion. The most crucial of these is the experiment performed by

Out come of the Michelson and Morley Experiment:

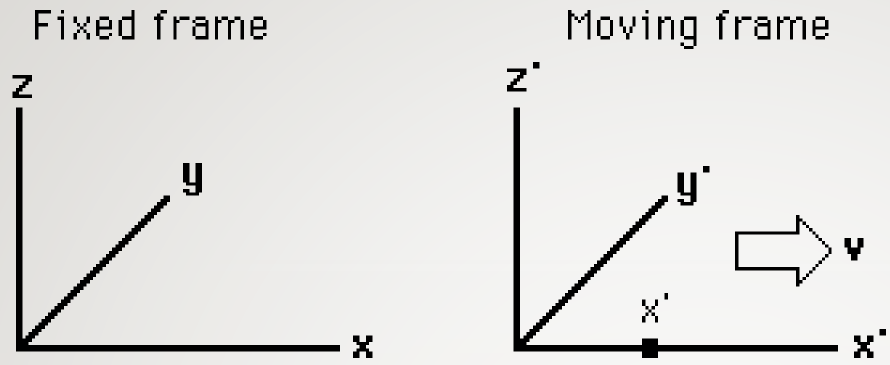
1. The hypothesis of existence of ether was rendered untenable by demonstrating that ether has no measurable properties.
2. The speed of light in free space is constant irrespective of the motion of the source or observer.

Special Theory of Relativity

The theory of relativity is based on the following postulates:

1. The laws of Physics should be expressed in equations having the same form in all frames of reference moving with a uniform velocity with respect to one another.
2. The speed of light in free space has the same value for all the observers irrespective of their state of motion.

Lorentz Transformations



The primed frame moves with velocity v in the x direction with respect to the fixed reference frame. The reference frames coincide at t = t' =0. The point x' is moving with the primed frame.

$$x' = \frac{x - vt}{\sqrt{1 - \frac{v^2}{c^2}}}, \qquad y' = y, \qquad z' = z \qquad \text{and} \qquad t' = \frac{t - \frac{vx}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}}$$

The reverse transformation is:

$$x = \frac{x' + vt'}{\sqrt{1 - \frac{v^2}{c^2}}} \qquad \text{and} \qquad t = \frac{t' + \frac{vx'}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}}$$

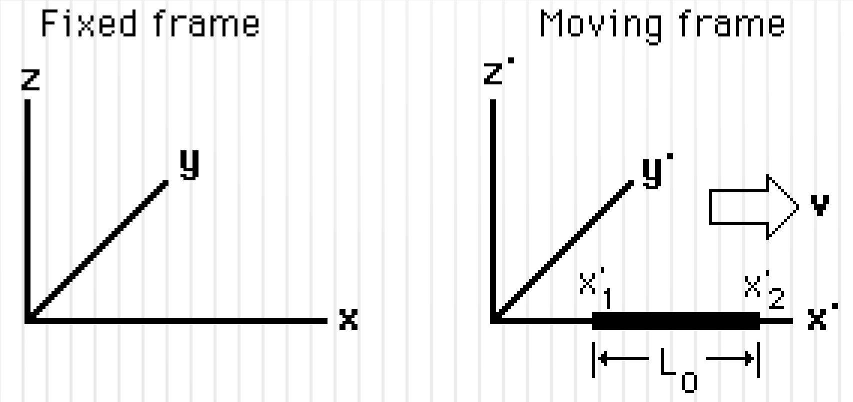
Much of the literature of relativity uses the symbols β and γ as defined here to simplify the writing of relativistic relationships.

$$\beta = \frac{v}{c} \qquad \text{and} \qquad \gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$$

Consequences of the Lorentz transformations:

[1] Length Contraction:

The length of any object in a moving frame will appear foreshortened in the direction of motion, or contracted. The amount of contraction can be calculated from the Lorentz transformation. The length is maximum in the frame in which the object is at rest.

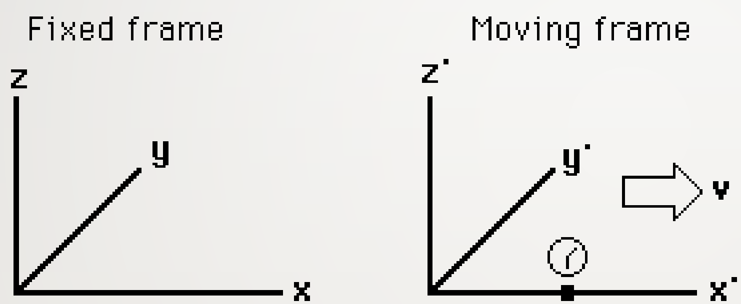


If the length   is measured in the moving reference frame, then   can be calculated using the Lorentz transformation.

$$L_0 = x_2' - x_1' = \frac{x_2 - vt_2 - x_1 + vt_1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

But since the two measurements made in the fixed frame are made simultaneously in that frame, $t_2 = t_1$, and $L = L_0\sqrt{1 - \frac{v^2}{c^2}} = \frac{L_0}{\gamma}$

[2] Time Dilation:



A clock in a moving frame will be seen to be running slow, or "dilated" according to the Lorentz transformation. The time will always be shortest as measured in its rest frame. The time measured in the frame in which the clock is at rest is called the "proper time”.

$$T_0 = t_2' - t_1'$$

$T = t_2 - t_1$ If the time interval is measured in the moving reference frame, then can be calculated using the Lorentz transformation.

$$T = t_2 - t_1 = \frac{t_2' + \frac{vx_2'}{c^2} - t_1' + \frac{vx_1'}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}}$$

The time measurements made in the $T = t_2 - t_1 = \frac{t_2' + \frac{vx_2'}{c^2} - t_1' + \frac{vx_1'}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}}$ moving frame

are made at the same location, so the expression reduces to

$$T = \frac{T_0}{\sqrt{1 - \frac{v^2}{c^2}}} = T_0\gamma$$

For small velocities at which the relativity factor is very close to 1, then the time dilation can be expanded in a binomial expansion to get the approximate expression:

$$T \approx T_0\left[1 + \frac{v^2}{2c^2}\right]$$

[3] Relativistic Mass:

The increase in effective mass with speed is given by the expression

$$m = \frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}} = \gamma m_0 \qquad m_0 = \text{"}\mathbf{Rest\ mass}\text{"}$$

It follows from the Lorentz transformation when collisions are described from a fixed and moving reference frame, where it arises as a result of conservation of momentum.

The increase in relativistic effective mass makes the speed of light c the speed limit of the universe. This increased effective mass is evident in cyclotrons and other accelerators where the speed approaches c. Exploring the calculation above will show that you have to reach 14% of the speed of light, or about 42 million m/s before you change the mass by 1%.

[4] Twin Paradox :

The story is that one of a pair of twins leaves on a high speed space journey during which he travels at a large fraction of the speed of light while the other remains on the Earth. Because of time dilation, time is running more slowly in the spacecraft as seen by the earthbound twin and the traveling twin will find that the earthbound twin will be older upon return from the journey. The common question: Is this real? Would one twin really be younger?

The basic question about whether time dilation is real is settled by the muon experiment. The clear implication is that the traveling twin would indeed be younger, but the scenario is complicated by the fact that the traveling twin must be accelerated up to traveling speed, turned around, and decelerated again upon return to Earth. Accelerations are outside the realm of special relativity and require general relativity.

Despite the experimental difficulties, an experiment on a commercial airline confirms the existence of a time difference between ground observers and a reference frame moving with respect to them.

Reference: http://hyperphysics.phy-astr.gsu.edu/hbase/relativ/ltrans.html#c2

Lagrangian and Hamiltonian of relativistic particle:

The total energy E associated with the mass m of a relativistic particle is given by famous Einstein relation E = mc2. Here E and m are measured in a given inertial frame iof reference. The relativistic momentum of a free particle is p = mv. The mass is frame dependent.

For free particle (potential energy = 0),

$$E = \vec{p}\cdot\vec{v} - L(\vec{r},\vec{v}) \qquad (1)$$

OR

$$mc^2 = m\vec{v}\cdot\vec{v} - L(r,v)$$

$$mc^2 = mv^2 - L(r,v)$$

Therefore,

$$L(r,v) = mv^2 - mc^2 = m\left(v^2 - c^2\right) \qquad (2)$$

Which gives

$$L(r,v) = \frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}}\left(v^2 - c^2\right) \qquad \text{since} \qquad m = \frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}} \qquad (3)$$

OR

$$L(r,v) = -m_0 c\sqrt{c^2 - v^2} = -m_0 \frac{c^2}{c}\sqrt{c^2 - v^2} = -m_0 c^2 \sqrt{\frac{c^2 - v^2}{c^2}} = -m_0 c^2 \sqrt{1 - \frac{v^2}{c^2}}$$

$$L(r,v) = -m_0 c^2 \sqrt{1-\beta^2} \qquad (4)$$

This is Lagrangian of a relativistic free particle.

If the particle is moving in a conservative potential field given by $V(r)$, the relativistic Lagrangian of the particle may be written as

$$L(r,v) = -m_0 c^2 \sqrt{1-\beta^2} - V(r) \qquad (5)$$

That this is the correct Lagrangian can be shown by demonstrating that resultant Lagrange equations

$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial \dot{q}_j} = 0$ i.e. $\frac{d}{dt}\left(\frac{\partial L}{\partial v}\right) - \frac{\partial L}{\partial x} = 0$ in the present case, agree with $\frac{dp_j}{dt} = F_j$

$q_j = q_1 = x$ **and** $\dot{q}_j = \dot{x} = v$

Now, $L(r,v) = -m_0 c^2 \sqrt{1-\frac{v^2}{c^2}} - V(r)$

$$\frac{\partial L}{\partial v} = -m_0 c^2 \frac{1}{2\sqrt{1-\frac{v^2}{c^2}}}\left(-\frac{1}{c^2}\right)2v = \frac{m_0 v}{\sqrt{1-\frac{v^2}{c^2}}} = mv = p \quad \text{and} \quad \frac{\partial L}{\partial x} = -\frac{\partial V}{\partial x}$$

Thus Lagrange equations results

$$\frac{d}{dt}\left(\frac{mv}{\sqrt{1-\beta^2}}\right) - \left(-\frac{\partial V}{\partial x}\right) = 0$$

i.e $\quad \frac{d}{dt}(p) = \left(-\frac{\partial V}{\partial x}\right)$

OR $\quad \frac{d}{dt}(p) = F$

It can be noted that the Lagrangian is no longer $L = T - V$ but that the partial derivative of L is still the momentum. Thus our assumption that equation (5) defines the Lagrangian for the relativistic free particle in potential field $V(r)$ is correct.

In non-relativistic limit, ($v \ll c$), equation (5) reduces to

$$L(r,v) = -m_0c^2\left(1-\beta^2\right)^{1/2} - V(r)$$

$$L(r,v) = -m_0c^2\left[1 - \frac{1}{2}\frac{v^2}{c^2} + \frac{1}{8}\frac{v^4}{c^4} + .. + ...\right] - V(r)$$

According Binomial expansion of the term $(1+x)^n = 1 + nx + \frac{n(n-1)}{2}x^2 + ...$

$$L(r,v) = -m_0c^2\left[1 - \frac{1}{2}\frac{v^2}{c^2}\right] - V(r)$$ Neglecting higher order terms.

$$L(r,v) = -m_0c^2\left[1 - \frac{1}{2}\frac{v^2}{c^2}\right] - V(r)$$

$$L(r,v) = \left[-m_0c^2 + \frac{1}{2}m_0c^2\frac{v^2}{c^2}\right] - V(r)$$

$$L(r,v) = \frac{1}{2}m_0v^2 - m_0c^2 - V(r)$$

OR

$$L(r,v) = T - m_0c^2 - V(r) \qquad (6)$$

Except for the constant term, $-m_0c^2$, this is expected classical Lagrangian for a particle moving in $V(r)$.

If a relativistic charges particle is moving in an electromagnetic field, the Lagrangian for this particle can be subtracting the generalized potential. Thus we get Lagrangian for **relativistic charged particle** as

$$L(r,v) = -m_0 c\sqrt{c^2 - v^2} - e\phi + e\vec{A}\cdot\vec{v} \tag{7}$$

where e is the electric charge on the particle, $A = A\,(r, t)$ is the electromagnetic vector potential and $\phi = \phi\,(r,\ t)$ is the scalar electric potential at the location (r) of the particle at time t.

We can extend the Lagrangian defined by equation (5) to systems of many particles and change from Cartesian to any desired set of generalized coordinates q_j. The canonical momenta will still be defined by

$$p_j = \frac{\partial L}{\partial \dot{q}_j} \tag{8}$$

so that the connection between cyclic coordinates and conservation of the corresponding momenta remains just as in case the nonrelativistic theory.

Further, if L does not contain time explicitly there exists a constant of motion h, defined by

$$h = \sum_j p_j \dot{q}_j - L \tag{9}$$

Let's show this function defines the total energy (*E, i.e.* Hamiltonian *H*) of the system.

$$h = \frac{m_0 v}{\sqrt{1-\frac{v^2}{c^2}}} v - \left(-m_0 c^2 \sqrt{1-\frac{v^2}{c^2}} - V \right) \quad (10)$$

$$h = \frac{m_0 v^2}{\sqrt{1-\frac{v^2}{c^2}}} + m_0 c^2 \sqrt{1-\frac{v^2}{c^2}} + V = \frac{m_0 v^2 + m_0 c^2 \left(1-\frac{v^2}{c^2}\right)}{\sqrt{1-\frac{v^2}{c^2}}} + V$$

$$h = \frac{m_0 v^2 + m_0 c^2 - m_0 v^2}{\sqrt{1-\frac{v^2}{c^2}}} + V = \frac{m_0 c^2}{\sqrt{1-\frac{v^2}{c^2}}} + V \quad (11)$$

$$h = mc^2 + V \quad \text{OR} \quad h = T + V = E = H \quad (12)$$

The quantity *h* is again seen to be the total energy *E*, which is therefore constant of motion under these conditions. Equation (11) defines the Hamiltonian of the relativistic free particle.

It can be shown that Hamiltonian of the relativistic free particle given by equation (11) can be written as

$$H = \sqrt{p^2c^2 + m_0^2c^4} + V(r) \qquad (13)$$

We have $H = \dfrac{m_0c^2}{\sqrt{1-\dfrac{v^2}{c^2}}} + V = \sqrt{\dfrac{m_0^2c^4}{1-\dfrac{v^2}{c^2}}} + V$

$$H = \sqrt{\frac{m_0^2c^4 + m_0^2c^2v^2 - m_0^2c^2v^2}{1-\dfrac{v^2}{c^2}}} + V$$

OR

$$H = \sqrt{\frac{m_0^2c^2v^2 + \left(m_0^2c^4 - m_0^2c^2v^2\right)}{1-\dfrac{v^2}{c^2}}} + V$$

$$H = \sqrt{\frac{m_0^2c^2v^2}{1-\dfrac{v^2}{c^2}} + \frac{\left(m_0^2c^4 - m_0^2c^2v^2\right)}{1-\dfrac{v^2}{c^2}}} + V$$

$$H = \sqrt{\frac{m_0^2 c^2 v^2}{1-\frac{v^2}{c^2}} + \frac{m_0^2 c^4 \left(1-\frac{v^2}{c^2}\right)}{1-\frac{v^2}{c^2}}} + V$$

OR

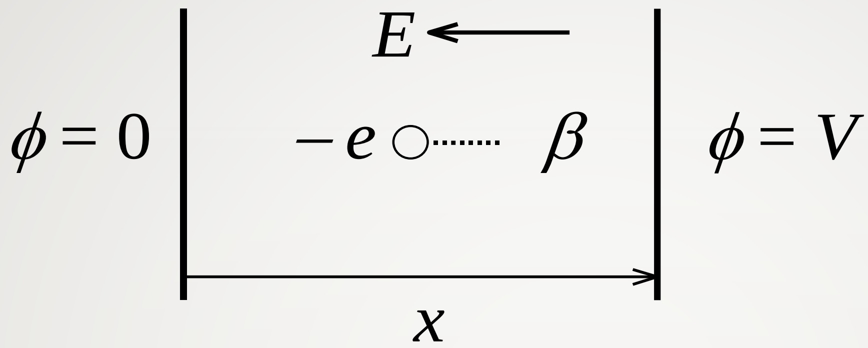
$$H = \sqrt{p^2 c^2 + m_0^2 c^4} + V \qquad (14)$$

where $p = \dfrac{m_0 v}{\sqrt{1-\frac{v^2}{c^2}}}$

Problem :

Particle accelerating under constant force:

Suppose that electron is being accelerated in an electric field E as shown



The potential energy in this case is $V = -eEx$. The relativistic Lagrangian is given by

$$L(r,v) = -mc^2\sqrt{1-\beta^2} + eEx \qquad (1)$$

and Lagranges equations in this case would be $\frac{d}{dt}\left(\frac{\partial L}{\partial v}\right) - \frac{\partial L}{\partial x} = 0 \qquad (2)$

$$\frac{\partial L}{\partial v} = -mc^2 \frac{1}{2\sqrt{1-\frac{v^2}{c^2}}}\left(-\frac{1}{c^2}\right)2v = \frac{mv}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{mc\left(\frac{v}{c}\right)}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{mc\beta}{\sqrt{1-\beta^2}} \qquad (3)$$

$$\frac{\partial L}{\partial x} = eE \qquad (4)$$

Putting equations (3) and (4) in equation (2), we get

$$\frac{d}{dt}\left(\frac{mc\beta}{\sqrt{1-\beta^2}}\right) - eE = 0 \qquad (5)$$

$$\frac{d}{dt}\left(\frac{mc\beta}{\sqrt{1-\beta^2}}\right) = eE \qquad \text{OR}$$

$$\frac{d}{dt}\left(\frac{\beta}{\sqrt{1-\beta^2}}\right) = \frac{eE}{mc} \qquad (6)$$

Integrating with respect to *t*, we get

$$\frac{\beta}{\sqrt{1-\beta^2}} = \frac{eE}{mc}t + C\,; \qquad \text{C being constant of integration} \qquad (7)$$

Assuming at $t = 0$, $x = 0$ and $v = 0$, This gives C = 0. Therefore,

$$\frac{\beta}{\sqrt{1-\beta^2}} = \frac{eE}{mc}t \qquad (8)$$

Squaring both sides of equation
(8)

$$\frac{\beta^2}{1-\beta^2} = \left(\frac{eE}{mc}t\right)^2 \Rightarrow 1+\frac{\beta^2}{1-\beta^2} = 1+\left(\frac{eE}{mc}t\right)^2 \Rightarrow \frac{1-\beta^2+\beta^2}{1-\beta^2} = 1+\left(\frac{eE}{mc}t\right)^2$$

$$\Rightarrow \frac{1}{1-\beta^2} = 1+\left(\frac{eE}{mc}t\right)^2 \Rightarrow 1-\beta^2 = \frac{1}{1+\left(\frac{eE}{mc}t\right)^2} \Rightarrow \beta^2 = 1-\frac{1}{1+\left(\frac{eE}{mc}t\right)^2}$$

$$\Rightarrow \beta^2 = \frac{1+\left(\frac{eE}{mc}t\right)^2 - 1}{1+\left(\frac{eE}{mc}t\right)^2} \Rightarrow \beta^2 = \frac{\left(\frac{eE}{mc}t\right)^2}{1+\left(\frac{eE}{mc}t\right)^2} \Rightarrow \beta = \frac{\left(\frac{eE}{mc}t\right)}{\sqrt{1+\left(\frac{eE}{mc}t\right)^2}} \qquad (9)$$

OR

$$\frac{v}{c} = \frac{\left(\frac{eE}{mc}t\right)}{\sqrt{1+\left(\frac{eE}{mc}t\right)^2}} \Rightarrow v = \frac{\left(\frac{eE}{m}t\right)}{\sqrt{1+\left(\frac{eE}{mc}t\right)^2}} = \dot{x} = \frac{dx}{dt}$$

Thus

$$x=\int \frac{\left(\frac{eE}{m}t\right)}{\sqrt{1+\left(\frac{eE}{mc}t\right)^2}}dt \quad \Rightarrow \quad x=\int \frac{cu}{\sqrt{1+u^2}}\frac{du}{\frac{eE}{mc}}$$

after putting $\frac{eE}{mc}t=u$

$$x=\frac{mc^2}{eE}\int \frac{udu}{\sqrt{1+u^2}}$$

$$x=\frac{mc^2}{eE}\sqrt{1+u^2}+C'$$

C' being constant of integration

$$x=\frac{mc^2}{eE}\sqrt{1+\left(\frac{eE}{mc}t\right)^2}+C' \qquad (10)$$

Assuming at $t=0$, $x=0$ and $v=0$, This gives $C=-\frac{mc^2}{eE}$ Therefore,

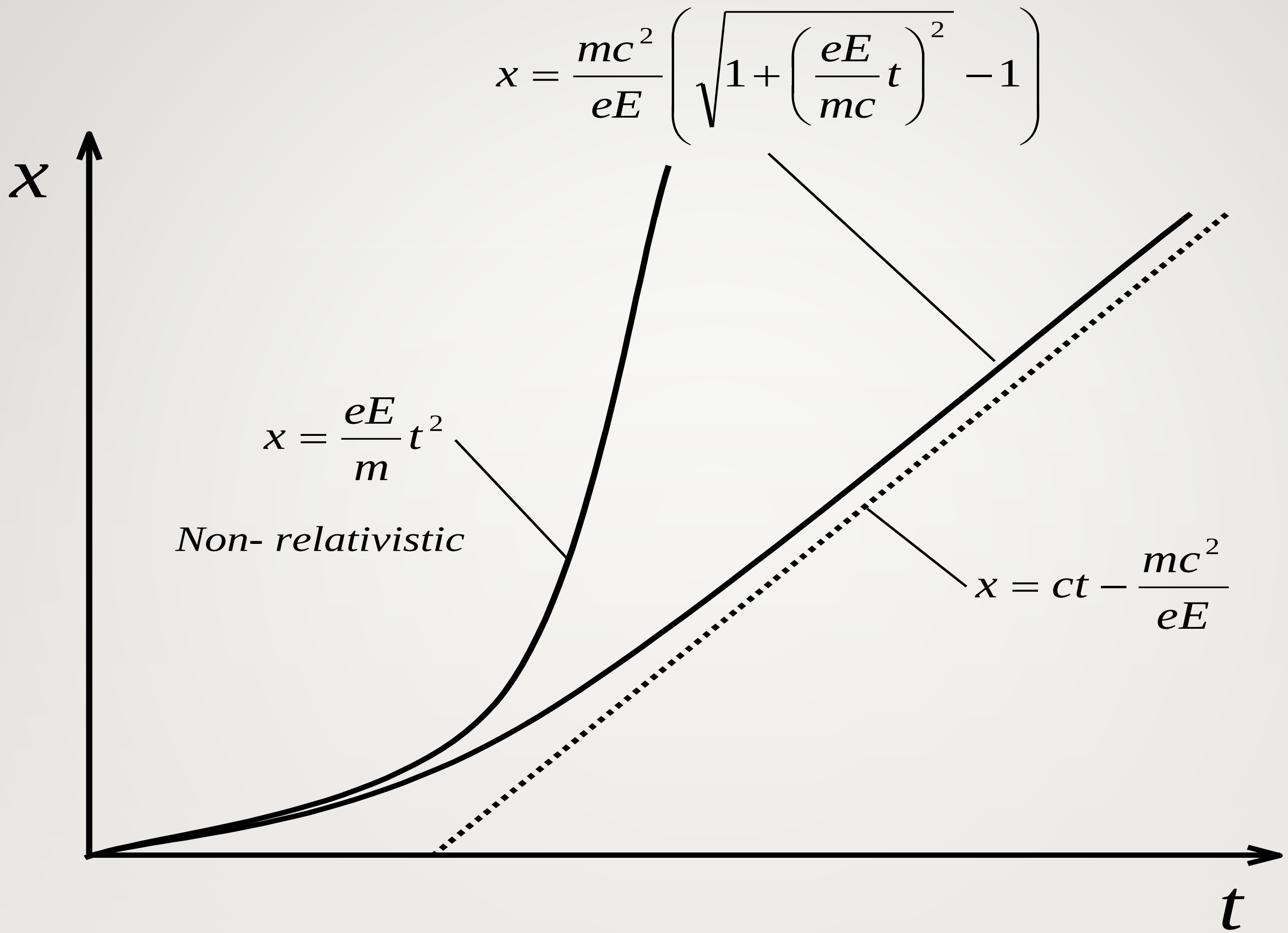
$$x=\frac{mc^2}{eE}\sqrt{1+\left(\frac{eE}{mc}t\right)^2}-\frac{mc^2}{eE}$$

$$x = \frac{mc^2}{eE}\left(\sqrt{1+\left(\frac{eE}{mc}t\right)^2} - 1\right) \tag{11}$$

Special cases

(i) Non-relativistic limit (v << c), $v = \frac{eE}{m}t$, therefore, $x = \frac{eE}{m}t^2$ (12)

(ii) As $t \to \infty$, $\to \beta = 1$, $\to x = ct$ (13)



$x = \frac{mc^2}{eE}\left(\sqrt{1+\left(\frac{eE}{mc}t\right)^2}-1\right)$
x
$x = \frac{eE}{m}t^2$
Non- relativistic
$x = ct - \frac{mc^2}{eE}$
t

THANK YOU