

Classical Mechanics

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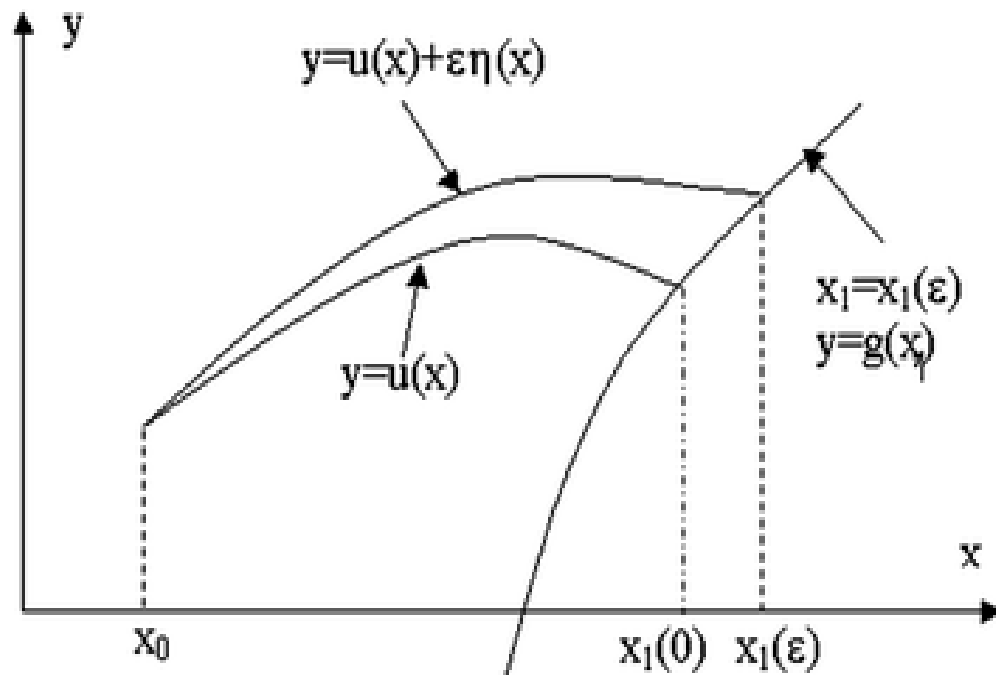
Variational principle

Introduction

- ▶ The Variational principle is a principle help in the calculus of variations, it is a general method to find maximum value of quantities which are depend on the functions.
- ▶ This principle is based on the “wave function” and finding the values of these parameters which is the expectation value.
- ▶ “A formulation of mechanics which is independent of the choice of coordinate system”
- ▶ Consider quantum system is described by Hamiltonian series with Eigen state with corresponds to Eigen values of energy.
- ▶ $H\varphi_n = E_n\varphi_n$ (which is Schrodinger equation)

Variational method

- ▶ $E_g \leq \langle \varphi | H | \varphi \rangle = \langle H \rangle$
- ▶ The above equation is the expected value of H in the state of ψ and it is over estimate for the ground state level energy.
- ▶ If ψ have the excited state , then $\langle H \rangle$ exceed the E_g .



Configuration space

- ▶ The parameters that define the configuration of a system is called generalized coordinates and the vector space defined by these coordinates is called the configuration space.
- ▶ the position of a single particle moving in ordinary 3-space.
- ▶ The vector $q = (x, y, z)$ and configuration space is $Q = \mathbb{R}^3$.
- ▶ The conventional in Hamilton formulation of classical mechanics and langrangian theorem. The p is indicated as momenta.

$$\dot{q} = \frac{dq}{dt}$$

If the particle is in rigid body and free to swing about its origin, it is lie in a space. The configuration space is the subset of coordinates in \mathbb{R}^3 on the sphere S^2 .

We can say that the manifold Q id in the space so $Q = S^2$.

Calculus of variation

- ▶ It is a field of mathematical analysis that uses variations which are small change in function to find minima and maxima.
- ▶ It is the set of function of the real numbers. This function minima or maxima function found by using the Euler-Lagrange equation.
- ▶ We can find the curve of shortest distance by using two points even though the curve is in a space.
- ▶ The maxima or minima are concerned for the calculus of variations and it is also called extrema.
- ▶ A function $J[y]$ is the extrema function with respect to the function country.
- ▶ $\Delta J = J[y] - J[f]$ has the sign for all y in an extreme at function j . And arbitrary function f is called external function.
- ▶ The $J[f]$ is called local maximum if $\Delta J \leq 0$

- ▶ If $\Delta J \geq 0$, extrema of corresponding functions are called weak extrema or strong extrema, depending on the first derivatives of the continuous functions.

Technique of the calculus of variation

- ▶ The maxima and minima function are located by finding two points and when we find derivative is zero.
- ▶ This is done by Euler-lagrange's equation,
- ▶ $J[y] = \int_{x_1}^{x_2} L(x, y(x), y'(x)) dx.$

where x_1, x_2 are constant.

$y(x)$ is double differentiable

$$Y'(x) = \frac{dy}{dx}$$

$L(x, y(x), y'(x))$ is double differentiate with the arguments x, y, y' .

If function $J[y]$ attain local minimum at f and $\eta(x)$ is an arbitrary function that has one derivative and become zero at x_1 and x_2 , then for any number ε close to 0.

- ▶ $J[f] \leq J [f + \varepsilon\eta]$.
- ▶ The term $\varepsilon\eta$ is variation of the function f and denoted by δf .
- ▶ $f + \varepsilon\eta$ for y in the functional $j[y]$ and the result is a function ε ,
 $\phi(\varepsilon) = J [f + \varepsilon\eta]$.

since the function $j[y]$ has minimum for $y=f$, for the function $\phi(\varepsilon)$ has minimum at $\varepsilon=0$ and therefore,

$\phi'(0) = \frac{d\phi}{d\varepsilon} \Big|_{\varepsilon=0} = \int_{x_1}^{x_2} \frac{dL}{d\varepsilon} dx =$ total derivative of $L [x,y,y']$, where $y = f + \varepsilon\eta$ and $y' = f' + \varepsilon\eta'$ are considered as function of ε rather than x ,

$$\frac{dL}{d\varepsilon} = \frac{\delta L}{\delta y} \frac{dy}{d\varepsilon} + \frac{\delta L}{\delta y'} \frac{dy'}{d\varepsilon}$$

$$\frac{dy}{d\varepsilon} = \eta \text{ and } \frac{dy'}{d\varepsilon} = \eta'$$

$$\blacktriangleright \frac{dL}{d\varepsilon} = \frac{\delta L}{\delta y} \eta + \frac{\delta L}{\delta y'} \eta'$$

$$\begin{aligned} \text{Therefore, } \int_{x_1}^{x_2} \frac{dL}{d\varepsilon} \bigg|_{\varepsilon=0} dx &= \int_{x_1}^{x_2} \left(\frac{\delta L}{\delta y} \eta + \frac{\delta L}{\delta y'} \eta' \right) dx \\ &= \int_{x_1}^{x_2} \left(\frac{\delta L}{\delta y} \eta dx + \frac{\delta L}{\delta f'} \eta \bigg|_{x_1}^{x_2} - \int_{x_1}^{x_2} \eta \frac{d}{dx} \frac{\delta L}{\delta f'} dx \right) \\ &= \int_{x_1}^{x_2} \left(\frac{\delta L}{\delta f} \eta - \eta \frac{d}{dx} \frac{\delta L}{\delta f'} \right) dx \end{aligned}$$

- ▶ where $L[x,y,y'] = L[x,f,f']$ where $\varepsilon=0$.
- ▶ The second derivative is zero, $\eta = 0$
- ▶ $\int_{x_1}^{x_2} \eta(x) \left(\frac{\delta L}{\delta f} - \frac{d}{dx} \frac{\delta L}{\delta f'} \right) dx = 0$ which is Euler-lagrange equation.

The delta notation

- ▶ In the calculus of variation may vary with the functions and it is denoted by δ -delta notation.

- ▶
$$\frac{\delta J}{\delta x} = \int_{x_1}^{x_2} \left[\frac{df}{dy} - \frac{d}{dx} \frac{df}{dy'} \right] \frac{\delta y}{\delta x} dx$$

- ▶ Multiplying both sides with dx,

- ▶
$$\frac{\delta J}{\delta x} dx = \int_{x_1}^{x_2} \left[\frac{df}{dy} - \frac{d}{dx} \frac{df}{dy'} \right] \frac{\delta y}{\delta x} dx \cdot dx$$

- ▶
$$\frac{\delta J}{\delta x} dx = \delta J$$

- ▶
$$\frac{\delta y}{\delta x} dx = \delta y$$

- ▶ So we have,
$$\delta J = \delta \int_{x_1}^{x_2} f(y, y', x) dx = 0$$

► Take δ in the integral sign,

►
$$\delta J = \int_{x_1}^{x_2} \delta f dx$$

$$= \int_{x_1}^{x_2} \left[\frac{\delta f}{\delta y} \delta y + \frac{\delta f}{\delta y'} \delta y' \right] dx$$

Now, $dy' = \delta \left(\frac{dy}{dx} \right) = \frac{d}{dx} (\delta y)$

Integrate the equation on right hand side we will get,

$$\delta J = \int_{x_1}^{x_2} \left[\frac{\delta f}{\delta y} - \frac{d}{dx} \frac{\delta f}{\delta y'} \right] \delta y dx$$

The variation of y at the end point is zero. If dy is arbitrary than $\delta J = 0$ and this is same as Euler-lagrange's equation.

Application of the variable principle

- ▶ 1. Consider extreme value of integer,

$$J = \int_{x_1}^{x_2} f(y, y', x) dx$$

where $f = \left(\left(\frac{dy}{dx} \right)^2 \right)$ and $y(x) = x$

- ▶ $y(\alpha, x) = x + \alpha\eta(x)$

Where $\eta(x) = (x - x_1)(x - x_2)$ and satisfied boundary condition,

$$\eta(x_1) = \eta(x_2) = 0$$

$$\frac{dy}{dx} = 1 - \alpha(x_1 + x_2) + 2\alpha x - 1 - \alpha a - 2\alpha x, \text{ where } a = x_1 + x_2$$

▶ Square and substitute the value, then we get $a = 0, J = (x_2 - x_1)$ and the value of $j(a) > j(0)$ for any value of α .

▶ $\left. \frac{dJ}{da} \right|_{a=0} = 0$ for $a = 0$.

▶ 2. Euler's Lagrange's equation becomes,

▶ $\frac{d}{dx} \frac{\partial f}{\partial y'} = 0, \frac{\partial f}{\partial y'} = \text{constant} = \frac{1}{\sqrt{2a}}$ by assumption.

▶ $\frac{y'^2}{x(1+y'^2)} = \frac{1}{2a}$

▶ $y' = \frac{\sqrt{x}}{\sqrt{2a-x}}$

▶ Now integrate, $y = \int \frac{\sqrt{x}}{\sqrt{2a-x}} dx$

▶ $x = a(1 - \cos \theta) d\theta$

▶ $y = \int a(1 - \cos \theta) d\theta = \int a(1 - \sin \theta) + \text{const.}$

- ▶ The constant term=0, when $x=y=0$ and $\theta = 0$
- ▶ So equation is $x = a(1 - \cos \theta)$

$$y = a(\theta - \sin \theta)$$

This equation is cycloid equation passing from the two points and found to be minimum for the Variational principle is extreme. These are the two possibilities which are applicable on Variational principle.

References:

1. R.G. Takwalw , classical mechanics book
2. Slideshare.net
3. Wikipedia.org

Thank you