Second Order Homogeneous Differential Equation with Constant Coefficients

Example

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2.4 Second Order Homogeneous Differential Equation with Constant Coefficients

The general form of the second order differential equation with constant coefficients is

$$A\frac{d^2y}{dx^2} + B\frac{dy}{dx} + Cy = G(x) \qquad \dots (2.3)$$

where A, B, C are constants with $A > 0$ and $G(x)$ is a function of x only.

If $G(x) = 0$, we get homogeneous second order linear differential equation with constant coefficients.

$$A\frac{d^2y}{dx^2} + B\frac{dy}{dx} + Cy = 0$$

OR
$$\frac{d^2y}{dx^2} + \frac{B}{A}\frac{dy}{dx} + \frac{C}{A}y = 0$$

OR
$$\frac{d^2y}{dx^2} + p\frac{dy}{dx} + qy = 0 \qquad \dots (2.4)$$

where p and q are constants.

Since the constant functions p and q are continuous on $(-\infty, +\infty)$, there exist linearly independent solution $y_1(x)$ and $y_2(x)$ on that interval. The general solution will then be given by $y(x) = C_1y_1(x) + C_2y_2(x)$ where C_1 and C_2 are arbitrary constants. [*Two functions f(x) and g(x)* independent.]

We will start by looking possible solution to equation (2.4) of the form $y = e^{mx}$ if the constant m is suitably chosen. This is motivated by the fact that the first and second derivatives of this function (e^{mx}) are multiples of y.

To find 'm', we substitute

$$y = e^{mx},\ \frac{dy}{dx} = me^{mx},\ \frac{d^2y}{dx^2} = m^2e^{mx} \qquad \dots (2.5)$$

into equation (2.4) to obtain

$$(m^2 + pm + q)\, e^{mx} = 0 \qquad \dots (2.6)$$

Since e^{mx} is

... (2.6)

Since, e^{mx} is never zero, equation (2.6) holds if and only if

$$m^2 + pm + q = 0 \qquad \text{... (2.7)}$$

Equation (2.7) is called the auxiliary equation for equation (2.4) which can be obtained from equation (2.4) by replacing $\frac{d^2y}{dx^2}$ by m^2, $\frac{dy}{dx}$ by $m(= m^1)$ and y by 1 $(= m^0)$. The roots m_1 and m_2 of the auxiliary equation can be obtained by factoring or by the quadratic formula. These roots are

$$m_1 = \frac{-p + \sqrt{p^2 - 4q}}{2}, \quad m_2 = \frac{-p - \sqrt{p^2 - 4q}}{2}$$

Depending on whether $p^2 - 4q$ is positive, zero or negative, these roots will be distinct and real, equal and real, or complex conjugates. We will consider each of these cases separately.

(I) Distinct Real Roots :

Roots m_1 and m_2 are distinct real number if and only if $p^2 - 4q > 0$, then equation (2.4) has two solutions

$$y_1 = e^{m_1 x}, \quad y_2 = e^{m_2 x}$$

Neither of the functions $e^{m_1 x}$ and $e^{m_2 x}$ is a constant multiple of the other, so these functions are linearly independent and

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x} \qquad \ldots (2.5)$$

is the general solution of equation (2.4).

(II) Equal Real Roots :

It is evident that the roots m_1 and m_2 are equal and real number if and only if $p^2 - 4q = 0$.

If m_1 and m_2 are equal real roots, say $m_1 = m_2$ (= m), then the auxiliary equation yields only one solution of equation (2.4).

$$y_1(x) = e^{mx}$$

Now, we will show that $y_2(x) = xe^{mx}$ is a second linearly independent solution, since $p^2 - 4q = 0$.

The roots are $m = m_1 = m_2 = -\frac{p}{2}$.

$$\therefore \qquad y_2(x) = xe^{(-p/2)x}$$

Differentiating yields,

$$y_2'(x) = \left(1 - \frac{p}{2}x\right)e^{(-p/2)x} \text{ and } y_2''(x) = \left(\frac{p^2}{4}x - p\right)e^{(-p/2)x}$$

So $$y_2''(x) + py_2'(x) + qy_2(x) = \left[\left(\frac{p^2}{4}x - p\right) + p\left(1 - \frac{p}{2}x\right) + qx\right]e^{(-p/2)\,x}$$

$$= \left[-\frac{p^2}{4} + q\right]xe^{(-p/2)\,x}$$

But $p^2 - 4q = 0$ implies that $\left(\frac{-p^2}{4} + q\right) = 0$, so

$$y_2''(x) + py_2'(x) + qy_2(x) = 0$$

which tell us that $y_2(x) = xe^{mx}$ is a solution of equation (2.4).

Thus, it can be shown that

$$y_1(x) = e^{mx} \text{ and } y_2(x) = xe^{mx}$$

are linearly independent. So the general solution of equation (2.4) in this case is

$$y = c_1 e^{mx} + c_2 xe^{mx} \qquad \ldots (2.6)$$

Example 2.8 : *Find the general solution of* $\frac{d^2y}{dx^2} - 8\frac{dy}{dx} + 16y = 0.$

Solution : The auxiliary equation is

$$m^2 - 8m + 16 = 0 \text{ or equivalently } (m - 4)^2 = 0$$

So $m = 4$ is the only root.

Thus, the general solution of differential equation is

$$y = c_1 e^{4x} + c_2 xe^{4x}$$

Example 2.9 : *Find the solution of* $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4x = 0.$

Solution : The auxiliary equation is

$$m^2 - 4m + 4 = 0 \text{ or } (m - 2)^2 = 0$$

$$(m - 2)(m - 2) = 0$$

In this case, we have repeated root, $m = 2$

So, $y = c_1 e^{2x} + c_2 xe^{2x}$ is the solution.

(III) Distinct Complex Roots :

The roots m_1 and m_2 are distinct complex numbers if and only if $p^2 - 4q < 0$.

In this case, auxiliary equation has complex roots $m_1 = a + ib$ and $m_2 = a - bi$. By Euler's formula,

$$e^{-i\theta} = \cos\theta + i\sin\theta$$

Our two solutions are

$$e^{m_1 x} = e^{(a+ib)x} = e^{ax}e^{ibx} = e^{ax}(\cos bx + i\sin bx) \qquad \dots (2.7)$$

and $$e^{m_2 x} = e^{(a-ib)x} = a^{ax}e^{-ibx} = e^{ax}(\cos bx - i\sin bx) \qquad \dots (2.8)$$

We are interested in solutions that are real valued functions. We can add equations (2.7) and (2.8) and divide by 2, and subtract and divide by 2i, to obtain

$$e^{ax}\cos bx \text{ and } e^{ax}\sin bx$$

These solutions are linearly independent, so the general solution of equation (2.4) in this case is

$$y = e^{ax}(c_1\cos bx + c_2\sin bx) \qquad \dots (2.9)$$

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But $p^2 - 4q = 0$ implies that $\left(\frac{-p^2}{4} + q\right) = 0$, so

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Thus, it can be shown that

$$y_1(x) = e^{mx} \text{ and } y_2(x) = xe^{mx}$$

are linearly independent. So the general solution of equation (2.4) in this case is

$$y = c_1 e^{mx} + c_2 xe^{mx} \qquad \ldots (2.6)$$