Lagrange's Equations with Constraints

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7.3 Constrained Systems in General

- Let's number a system of N particles as $\alpha = 1, \ldots, N$. The positions of these N particles are $\mathbf{r}_\alpha$. We say that the parameters $q_1, \ldots, q_n$ are a set of n **generalized coordinates** for the system if each position $\mathbf{r}_\alpha$ can be expressed as a function of $q_1, \ldots, q_n$ and possibly time t,

$$\mathbf{r}_\alpha = (q_1, \cdots, q_n, t) \qquad [\alpha = 1, \cdots, N],$$

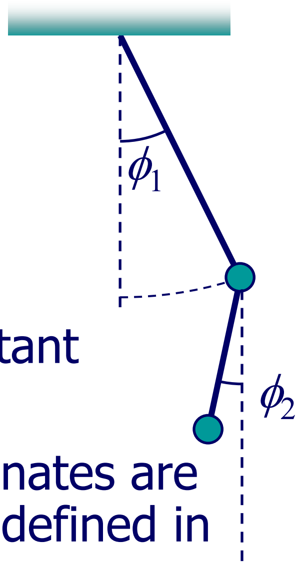
  and conversely each q_i can be expressed in terms of $\mathbf{r}_\alpha$ and possibly t,

$$q_i = q_i(\mathbf{r}_1, \cdots, \mathbf{r}_N, t) \qquad [i = 1, \cdots, n].$$

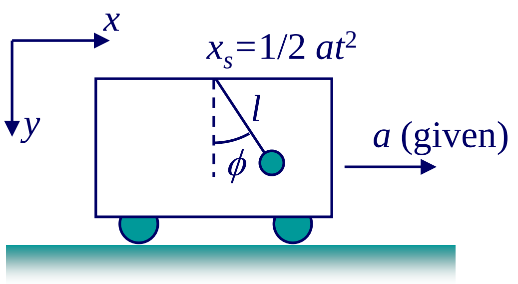
- In addition, we require that the number n of the generalized coordinates is the smallest number that allows the system to be parametrized in this way. In three dimensions, the number of generalized coordinates for N particles is certainly no more than $3N$, and for constrained systems is usually less. For a rigid body of, say 10^{23} particles, for example, the number of generalized coordinates is $n = 6$, three for the position of the center of mass, and three for the orientation.
- For the case of the pendulum we discussed last time, there is one body (the pendulum bob), and two coordinates (x and y), but there is only one generalized coordinate, ϕ, since $\mathbf{r} = (x, y) = (l \sin \phi, l \cos \phi)$.

Generalized Coordinates-2

- Consider the double pendulum, with two bobs confined to motion in a plane.
- Now we have two particles, four coordinates (x_1, y_1, x_2, y_2), but only two generalized coordinates ϕ_1 and ϕ_2.



- In these two examples, the transformation between Cartesian coordinates and generalized coordinates did not depend on time, but here is an example that does.
- Consider a pendulum hanging from a car that is undergoing a constant acceleration a to the right.
- Because Lagrange's equation was derived assuming that the coordinates are defined in an inertial frame, we are not allowed to use coordinates defined in the frame of the accelerating car!
- However, we can express them relative to the ground.
- In this case, the conversion from Cartesian to generalized coordinates is



$$\mathbf{r} = (x, y) = (l\sin\phi + \tfrac{1}{2}at^2, l\cos\phi) = \mathbf{r}(\phi, t).$$

- Generalized coordinates that do not depend on t are called *natural*.

Degrees of Freedom

- The number of degrees of freedom of a system is the number of coordinates that can be independently varied, i.e. the number of "directions" a system can move in small displacements from any initial configuration.
- A simple pendulum has one degree of freedom, while the double pendulum has two. A free particle has three, while a system of N free particles has $3N$ degrees of freedom, i.e. each particle has complete freedom.
- When the number of degrees of freedom of a system of N particles is less than $3N$, we say that the system is **constrained**. A system of free particles constrained to move in two dimensions has $2N$ degrees of freedom. Some further examples: a rigid body has 6 degrees of freedom, a bead on a wire has 1 degree of freedom, and a particle on a surface has 2 degrees of freedom.
- In each of these examples, the number of degrees of freedom equals the number of generalized coordinates (and so the number of Lagrange equations that apply).
- A system with this natural-seeming property is said to be **holonomic**. This course will only treat holonomic systems, which are easier to solve.

A Non-Holonomic System

- You might think that a system that does not have this natural property must be rare and bizarrely complicated. However, there are some simple examples of a nonholonomic system. Here is one.
  - Imaging a rubber ball free to roll (but not slide or spin) on a 2-d surface.
  - Starting at position (x, y) on the 2-d surface, it can only move in two independent directions and you might think that only two coordinates are necessary to completely describe its configuration, the coordinates x and y of its center.
  - But consider the following—place the ball at the origin O, and paint a dot on its top. Then roll it a distance equal to its circumference c along the x axis, so that the dot is again on top.
  - Now roll it a distance c in the y direction to a point P where its dot returns to the top.
  - Finally, roll it along the hypotenuse back to the origin. Now the dot is not on top, even though its position is again at O.
- Evidentally, the two coordinates x and y are not enough to uniquely specify the configuration. In fact, we need three more, the orientation of the ball.
- So 5 coordinates are needed, even though the ball has only two degrees of freedom. Such a system is nonholonomic.

7.4 Proof of Lagrange's Equations with Constraints

- We are now ready to prove Lagrange's equations for any holonomic system. We will prove it for one particle, but it can easily be extended to an arbitrary number (see Prob. 7.13).
- Let's take a particle constrained to move on a surface, so that it has two degrees of freedom and hence two independent generalized coordinates q_1 and q_2.
- There are two types of forces on the particle—forces of constraint (whatever forces are keeping the particle constrained), which we'll denote $\mathbf{F}_{\text{cstr}}$, and all other forces $\mathbf{F}$. The key is that the forces $\mathbf{F}_{\text{cstr}}$ can do no work on the particle. Note that the $\mathbf{F}_{\text{cstr}}$ forces may not be conservative, but this does not matter, since the Lagrange equations are not going to include them.
- We shall assume that the non-constraint forces do satisfy the second condition, at least, of conservative forces, i.e. that they can be derived from the gradient of a potential energy, $U(\mathbf{r}, t)$:

$$\mathbf{F} = -\nabla U(\mathbf{r}, t).$$

- If all forces $\mathbf{F}$ are really conservative, then they do not depend on t, but we do not need to assume this. The total force on the particle is $\mathbf{F}_{\text{tot}} = \mathbf{F}_{\text{cstr}} + \mathbf{F}$.

Action Integral Stationary on Right Path

- Consider a constrained particle that moves through two points $\mathbf{r}_1$ and $\mathbf{r}_2$ at times t_1 and t_2. We will denote $\mathbf{r}(t)$ as the position when the particle is on the "right" path and $\mathbf{R}(t)$ as the position along any neighboring "wrong" path.
- For a small displacement $\boldsymbol{\varepsilon}(t)$ between the right and wrong path, we have
$$\mathbf{R}(t) = \mathbf{r}(t) + \boldsymbol{\varepsilon}(t).$$
- Note that $\boldsymbol{\varepsilon}(t) = 0$ at the end points $\mathbf{r}_1$ and $\mathbf{r}_2$, since both paths go through these points. Note also that $\mathbf{r}(t)$ and $\mathbf{R}(t)$ are in the surface, so $\boldsymbol{\varepsilon}(t)$ is also. We denote the action integral by
$$S = \int_{t_1}^{t_2} \mathcal{L}(\mathbf{R}, \dot{\mathbf{R}}, t)\,dt,$$
taken along any path $\mathbf{R}(t)$ lying in the surface, and by S_o the corresponding integral taken along the right path $\mathbf{r}(t)$.
- We wish to prove that the integral S is stationary when $\mathbf{R}(t) = \mathbf{r}(t)$, i.e. when $\boldsymbol{\varepsilon}(t) = 0$. Another way to say this is that the difference in the integrals $\delta S = S - S_o$, is zero to first order in $\boldsymbol{\varepsilon}$.
- Now $\delta S = \int_{t_1}^{t_2} \delta\mathcal{L}\,dt$, where $\delta\mathcal{L} = \mathcal{L}(\mathbf{R}, \dot{\mathbf{R}}, t) - \mathcal{L}(\mathbf{r}, \dot{\mathbf{r}}, t)$.
- We can substitute $\mathbf{R}(t) = \mathbf{r}(t) + \boldsymbol{\varepsilon}(t)$, and $\mathcal{L}(\mathbf{r}, \dot{\mathbf{r}}, t) = T - U = \frac{1}{2}m\dot{\mathbf{r}}^2 - U(\mathbf{r}, t)$.

Action Integral Stationary on Right Path-2

- This gives

$$\delta \mathcal{L} = \tfrac{1}{2} m \left[\left(\dot{\mathbf{r}} + \dot{\boldsymbol{\varepsilon}} \right)^2 - \dot{\mathbf{r}}^2 \right] - \left[U(\mathbf{r} + \boldsymbol{\varepsilon}, t) - U(\mathbf{r}, t) \right]$$

$$= m\dot{\mathbf{r}} \cdot \dot{\boldsymbol{\varepsilon}} - \boldsymbol{\varepsilon} \cdot \nabla U + O(\varepsilon^2).$$

- Putting this back into the integral, and integrating the first term by parts, we get

$$\delta S = \int_{t_1}^{t_2} \delta \mathcal{L} dt = \int_{t_1}^{t_2} \left[m\dot{\mathbf{r}} \cdot \dot{\boldsymbol{\varepsilon}} - \boldsymbol{\varepsilon} \cdot \nabla U \right] dt = -\int_{t_1}^{t_2} \boldsymbol{\varepsilon} \cdot \left[m\ddot{\mathbf{r}} + \nabla U \right] dt,$$

  where the end-point term is zero, as usual.
- By Newton's second law $m\ddot{\mathbf{r}} = \mathbf{F}_{\text{tot}} = \mathbf{F}_{\text{cstr}} + \mathbf{F},$ and $\nabla U = -\mathbf{F}$, so

$$\delta S = -\int_{t_1}^{t_2} \boldsymbol{\varepsilon} \cdot \mathbf{F}_{\text{cstr}} dt,$$

- But recall that $\boldsymbol{\varepsilon}$ is in the surface, while $\mathbf{F}_{\text{cstr}}$ is perpendicular to the surface. Therefore $\boldsymbol{\varepsilon} \cdot \mathbf{F}_{\text{cstr}} = 0,$ and we have proved that $\delta S = 0$. We have thus proved Hamilton's Principle, that the action integral is stationary at the path that the particle actually follows.
- Notice that this is only true for paths in the surface, i.e. *consistent with the constraints*. Thus, it is not true for any coordinates x, y, z, say, but only for our generalized coordinates q_1 and q_2.

The General Result

- For any holonomic system, with n degrees of freedom and n generalized coordinates, and with the nonconstraint forces derivable from a potential energy $U(q_1, \ldots, q_n, t)$, the path followed by the system is determined by the n Lagrange equations

$$\frac{\partial \mathcal{L}}{\partial q_i} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} \qquad [i = 1, \cdots, n],$$

  where $\mathcal{L}$ is the Lagrangian $\mathcal{L} = T - U$ and $U(q_1, \ldots, q_n, t)$ is the total potential energy corresponding to all of the forces excluding the forces of constraint.
- You might ask what you should do in the case that some force on the particle is not conservative, i.e. friction. In that case you must modify the Lagrange equations (see Prob. 7.12), but the result is not elegant and we will not consider such cases.
- We are now going to take a look at a number of examples where the above Lagrange equations hold. You should try as many examples as you can, to get a feel for how to do these problems. You WILL see these sorts of problems on the exam.