Ordinary Differential Equations

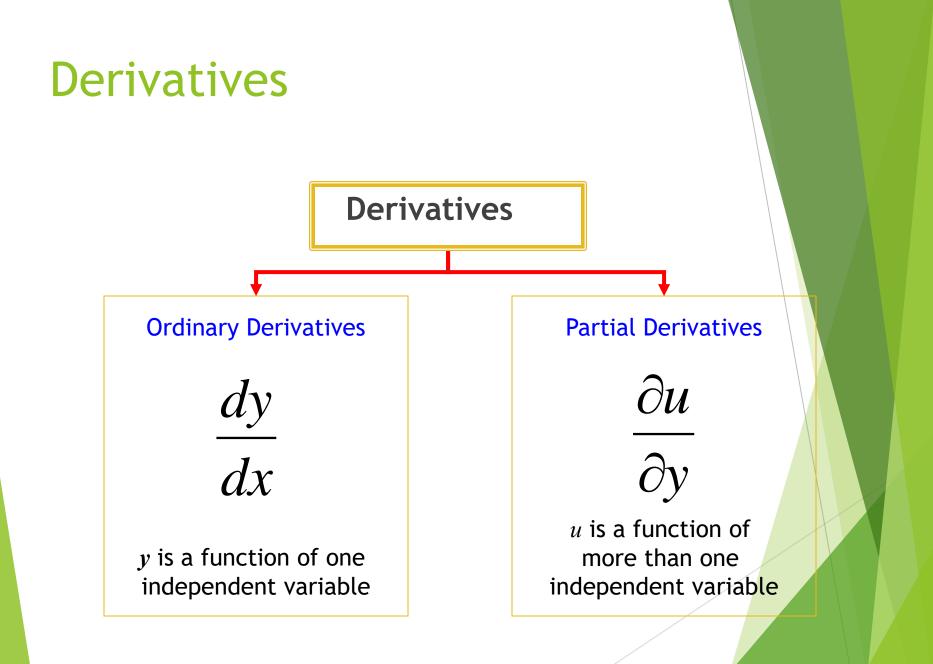
BY MR. S. V. MALGAONKAR M.Sc. (Assist. Prof.)

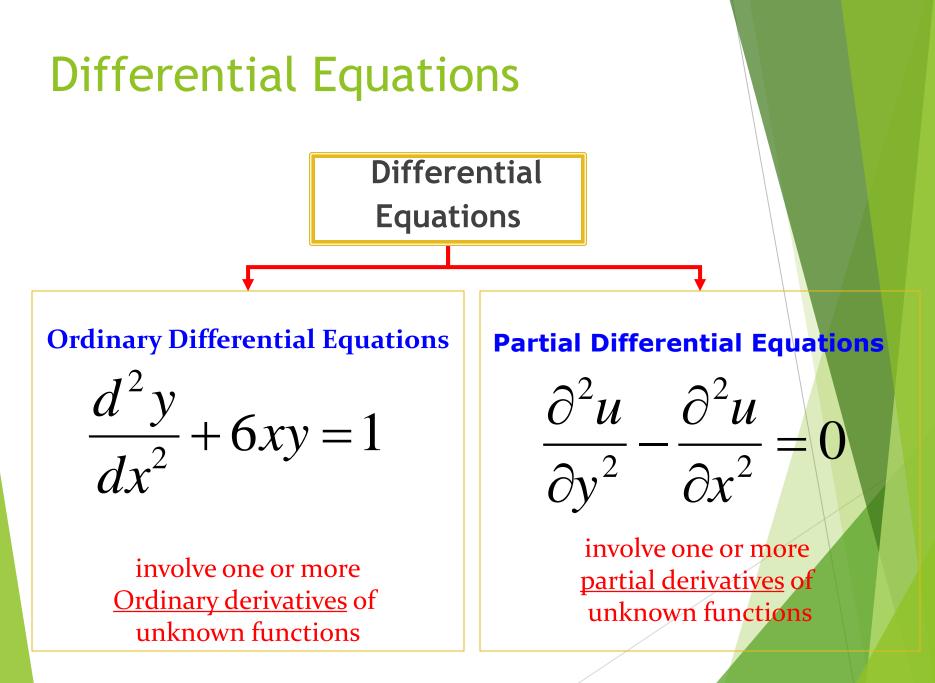
DEPARTMENT OF PHYSICS VIVEKANAND COLLEGE, KOLHAPUR (Autonomous)

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Introduction to Ordinary Differential Equations (ODE)

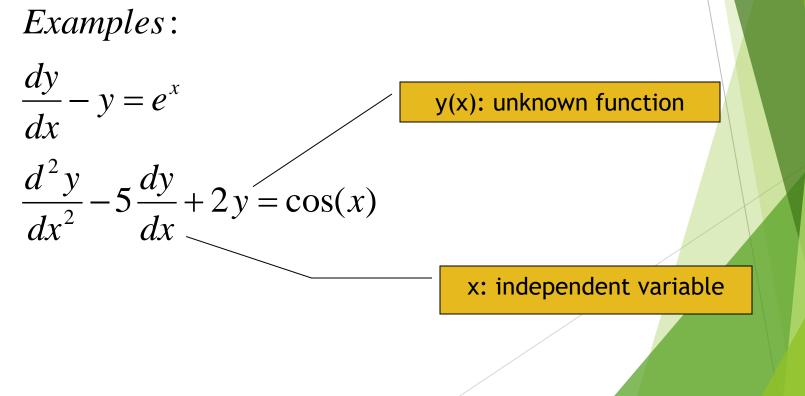
- Recall basic definitions of ODE,
 - ▶ order
 - linearity
 - initial conditions
 - solution
- Classify ODE based on(order, linearity, conditions)
- Classify the solution methods





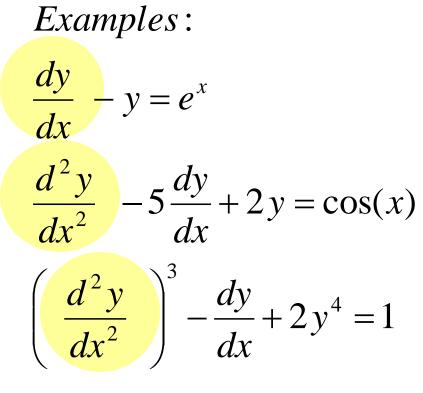
Ordinary Differential Equations

Ordinary Differential Equations (ODE) involve one or more ordinary derivatives of unknown functions with respect to one independent variable



Order of a differential equation

The **order** of an ordinary differential equations is the order of the highest order derivative



First order ODE

Second order ODE

Second order ODE

Solution of a differential equation

A **solution** to a differential equation is a function that satisfies the equation.

Example: $\frac{dx(t)}{dt} + x(t) = 0$ $\frac{dx(t)}{dt} = -e^{-t}$ $\frac{dx(t)}{dt} + x(t) = -e^{-t} + e^{-t} = 0$

Linear ODE

An ODE is linear if the unknown function and its derivatives appear to power one. No product of the unknown function and/or its derivatives

$$a_n(x)y^n(x) + a_{n-1}(x)y^{n-1}(x) + \dots + a_1(x)y'(x) + a_0(x)y(x) = g(x)$$

Examples:

 $\frac{dy}{dx} - y = e^x$ Linear ODE $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 2x^2y = \cos(x)$ $\left(\frac{d^2 y}{dx^2}\right)^3 - \frac{dy}{dx} + \sqrt{y} = 1$

Linear ODE

Non-linear ODE

Boundary-Value and Initial value Problems

Initial-Value Problems

The auxiliary conditions are at one point of the independent variable

Boundary-Value Problems

- The auxiliä ty to nd tip to not y(0) = 1, y(2) = 1.5 at one point of the independent variable
- More difficult to solve than initial value problem

different

$$y''+2y'+y = e^{-2x}$$

 $y(0) = 1, y'(0) = 2.5$
same

Classification of ODE

ODE can be classified in different ways

Order

- First order ODE
- Second order ODE
- Nth order ODE

Linearity

- Linear ODE
- Nonlinear ODE

Auxiliary conditions

- Initial value problems
- Boundary value problems



- Analytical Solutions to ODE are available for linear ODE and special classes of nonlinear differential equations.
- Numerical method are used to obtain a graph or a table of the unknown function
- We focus on solving first order linear ODE and second order linear ODE and Euler equation

Def: A first order differential equation is said to be *linear* if it can be written

$$y' + p(x)y = g(x)$$

How to solve first-order linear ODE ?

$$y' + p(x)y = g(x)$$
 (1)

Sol:

Multiplying both sides by $\mu(x)$, called an integrating factor, gived $\frac{dy}{dx} + \mu(x)p(x)y = \mu(x)g(x)$ (2) assuming $\mu(x)p(x) = \mu'(x)$, (3) we get $\mu(x)\frac{dy}{dx} + \mu'(x)y = \mu(x)g(x)$ (4)

By product rule, (4) becomes

$$(\mu(x)y(x))' = \mu(x)g(x) \quad (5)$$

$$\Rightarrow \mu(x)y(x) = \int \mu(x)g(x)dx + c_1$$

$$\Rightarrow y(x) = \frac{\int \mu(x)g(x)dx + c_1}{\mu(x)} \quad (6)$$

Now, we need to solve $\mu(x)$ from (3)

$$\mu(x)p(x) = \mu'(x) \Longrightarrow \frac{\mu'(x)}{\mu(x)} = p(x)$$

$$\frac{\mu'(x)}{\mu(x)} = p(x) \Longrightarrow (\ln \mu(x))' = p(x)$$

$$\Rightarrow \ln \mu(x) = \int p(x)dx + c_2$$
$$\Rightarrow \mu(x) = e^{\int p(x)dx + c_2} = c_3 e^{\int p(x)dx} \quad (7)$$

to get rid of one constant, we can use

$$\mu(x) = e^{\int p(x)dx} \quad (8)$$

The solution to a linear first order differential equation is then

$$y(x) = \frac{\int \mu(x)g(x)dx + c}{e^{\int p(x)dx}} \quad (9)$$

Summary of the Solution Process

- Put the differential equation in the form (1)
- Find the integrating factor,
- Multiply both sides of (1) by
- Integrate both sides $(\mu(x)y(x))'$ Solve for the solution

using &and write the left side of (1) as (X)

y(x)

Example 1
$$y' - y = e^{2x}$$

Sol:

$$y(x) = e^{-\int p(x)dx} \left[\int e^{\int p(x)dx} g(x)dx + c \right]$$
$$= e^{-\int (-1)dx} \left[\int e^{\int (-1)dx} e^{2x}dx + c \right]$$
$$= e^x \left[\int e^{-x} e^{2x}dx + c \right]$$
$$= e^x \left[e^x + c \right]$$
$$= ce^x + e^{2x}$$

Example 2

$$xy'+2y = x^2 - x, y(1) = \frac{1}{2}$$

Sol: $\Rightarrow y'+\frac{2}{x}y = x-1$
 $\Rightarrow y(x) = e^{-\int p(x)dx} \left[\int e^{\int p(x)dx} g(x)dx + c \right]$
 $= e^{-\int \frac{2}{x}dx} \left[\int e^{\int \frac{2}{x}dx} (x-1)dx + c \right] = x^{-2} \left[\int x^2 (x-1)dx + c \right]$
 $= x^{-2} \left[\frac{1}{4}x^4 - \frac{1}{3}x^3 + c \right] = \frac{1}{4}x^2 - \frac{1}{3}x + cx^{-2}$
Apply the initial condition to get c,
 $\Rightarrow \frac{1}{2} = \frac{1}{4} - \frac{1}{3} + c \Rightarrow c = \frac{7}{12}.$

Second Order Linear Differential Equation

Homogeneous Second Order Linear Differential Equations

- real roots, complex roots and repeated roots
- Non-homogeneous Second Order Linear Differential Equations
 - Undetermined Coefficients Method
- Euler Equations

Second Order Linear Differential Equations

The general equation can be expressed in the form

ay''+by'+cy = g(x)

where a, b and c are constant coefficients

Let the dependent variable y be replaced by the sum of the two new variables: y = u + vTherefore

$$[au''+bu'+cu]+[av''+bv'+cv]=g(x)$$

If v is a particular solution of the original differential equation [au''+bu'+cu]=0

purpose

The general solution of the linear differential equation will be the sum of a "<u>complementary function</u>" and a "<u>particular solution</u>".

The Complementary Function (solution of the homogeneous equation)

 $\frac{dy}{dx} = re^{rx}$

$$ay''+by'+cy=0$$

Let the solution assumed to be:

 $e^{rx}(ar^2+br+c)=0$

$$y = e^{rx}$$

$$\frac{d^2 y}{dx^2} = r^2 e^{rx}$$

Real, distinct roots Double roots Complex roots Real, Distinct Roots to Characteristic Equation

• Let the roots of the characteristic equation be real, distinct and of values r_1 and r_2 . Therefore, the solutions of the characteristic equation are:

$$y = e^{r_1 x} \qquad \qquad y = e^{r_2 x}$$

• The general solution will be

$$y = c_1 e^{r_1 x} + c_2 e^{r_2 x}$$

• Example

$$y''-5y'+6y = 0 \longrightarrow r^2 - 5r + 6 = 0$$

$$r_1 = 2$$

$$r_2 = 3 \longrightarrow y = c_1 e^{2x} + c_2 e^{3x}$$

Equal Roots to Characteristic Equation

• Let the roots of the characteristic equation equal and of value $r_1 = r_2 = r$. Therefore, the solution of the characteristic equation is: $y = e^{rx}$

Let
$$y = Ve^{rx} \Rightarrow y' = e^{rx}V' + rVe^{rx}$$
 and $y'' = e^{rx}V'' + 2re^{rx}V' + r^2Ve^{rx}$
where V is a
function of x
 $ay'' + by' + cy = 0$
 $ar^2 + br + c = 0$ $2ar + b = 0$
 $V''(x) = 0 \rightarrow V = cx + d$
 $y = be^{rx} + (cx + d)e^{rx} = c_1e^{rx} + c_2xe^{rx}$

Complex Roots to Characteristic Equation

Let the roots of the characteristic equation be complex in the form $r_{1,2} = \lambda \pm \mu i$. Therefore, the solution of the characteristic equation is: $y_1 = e^{(\lambda + \mu i)x} = e^{\lambda x} (\cos(\mu x) + i \sin(\mu x)),$

$$y_2 = e^{(\lambda - \mu i)x} = e^{\lambda x} (\cos(\mu x) - i\sin(\mu x)).$$

$$u(x) = \frac{1}{2}(y_1 + y_2) = e^{\lambda x} \cos(\mu x), \quad v(x) = \frac{1}{2i}(y_1 - y_2) = e^{\lambda x} \sin(\mu x)$$

It is easy to see that *u* and *v* are two solutions to the differential equation. Therefore, the geneal solution to the d.e. is :

$$y(x) = c_1 e^{\lambda x} \cos(\mu x) + c_2 e^{\lambda x} \sin(\mu x).$$

(I) Solve y''+6y'+9y=0(II) Solve y''-4y+5y=0characteristic equation characterist $r^2 - 4r + 5 = 0$ $r^2 + 6r + 9 = 0$ $r_1 = r_2 = -3$ $r_{1.2} = 2 \pm i$ $y = e^{2x}(c_1 \cos x + c_2 \sin x)$ $y = (c_1 + c_2 x)e^{-3x}$

Non-homogeneous Differential Equations (Method of Undetermined Coefficients)

$$ay''+by'+cy = g(x)$$

When g(x) is constant, say k, a particular solution of equation is

y = k / c

When g(x) is a polynomial of the form $a_0 + a_1x + a_2x^2 + ... + a_nx^n$ where all the coefficients are constants. The form of a particular solution is

$$y = \alpha_0 + \alpha_1 x + \alpha_2 x^2 + \ldots + \alpha_n x^n$$

Solve $y''-4y'+4y = 4x + 8x^3$ complementary function y''-4y'+4y = 0 $y = p + qx + rx^{2} + sx^{3}$ $y' = q + 2rx + 3sx^{2}$ y'' = 2r + 6sx $r^2 - 4r + 4 \neq 0$ characterist ation $(2r+6sx) - 4(q+2rx+3sx^{2}) + 4(p+qx+rx^{2}+sx^{3}) = 4x+8x^{3}$ equating coefficients of equal powers of x $y_c = (c_1 + c_2 x)e^{2x}$ 2r - 4q + 4p = 06s - 8r + 4q = 44r - 12s = 04s = 8 $y = y_c + y_p$ $= (c_1 + c_2 x)e^{2x} + 7 + 10x + 6x^2 + 2x^3$ $y_n = 7 + 10x + 6x^2 + 2x^3$

Non-homogeneous Differential Equations (Method of Undetermined Coefficients)

• When g(x) is of the form Te^{rx} , where T and r are constants. The form of a particular solution is $y = Ae^{rx}$

• When g(x) is of the form Csinnx + Dcosnx, where C and D are constants, the form of a particular solution is

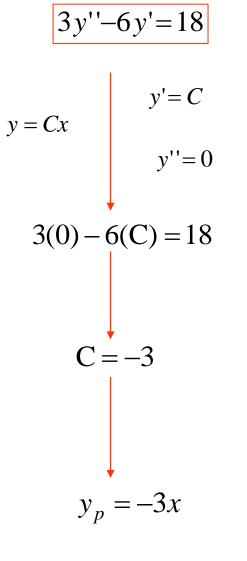
 $A = \frac{T}{ar^2 + br + c}$

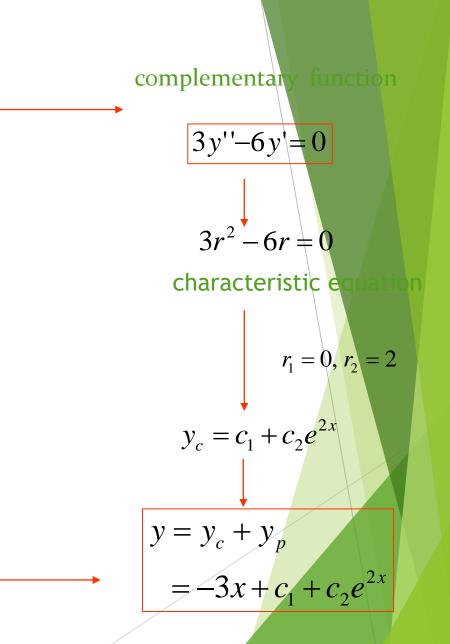
 $y = E\sin nx + F\cos nx$

$$E = \frac{(c - n^{2}a)C + nbD}{(c - n^{2}a)^{2} + n^{2}b^{2}}$$

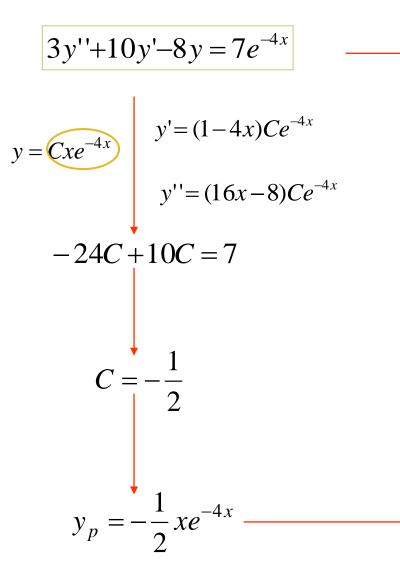
$$F = \frac{(c - n^{2}a)C - nbD}{(c - n^{2}a)^{2} + n^{2}b^{2}}$$

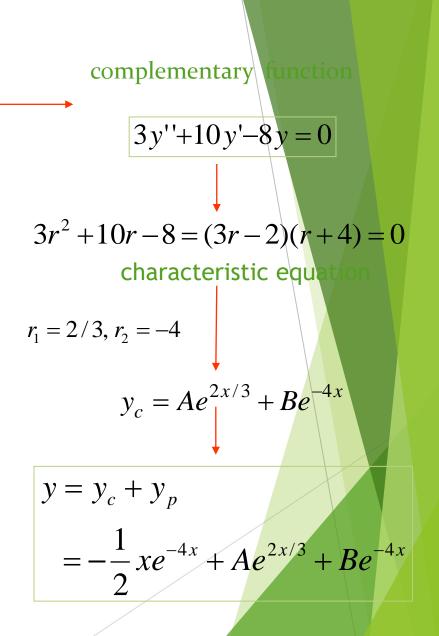
Solve





Solve





Solve
$$y''+y'-6y = 52 \cos 2x$$

 $y = C \cos 2x + D \sin 2x$
 $y'' = -2(C \sin 2x - D \cos 2x)$
 $y'' = -4(C \cos 2x + D \sin 2x)$
 $-10C + 2D = 52$
 $-2C - 10D = 0$
 $C = -5$
 $D = 1$
 $y_{p} = -5 \cos 2x + \sin 2x$
 $y_{p} = -5 \cos 2x + \sin 2x$
 $y_{p} = -5 \cos 2x + \sin 2x$
 $y_{p} = -6 \cos 2x + \sin 2x$

Euler Equations

• Def: *Euler* equations $ax^2y''+bxy'+cy=0$

Assuming x>0 and all solutions are of the form y(x) = x^r

Plug into the differential equation to get the characteristic equation ar(r-1) + b(r) + c = 0.

Solving Euler Equations: (Case

• The characteristic equation has two different real solutions r_1 and r_2 .

• In this case the functions $y = x^{r_1}$ and $y = x^{r_2}$ are both solutions to the original equation. The general solution is: $y(x) = c_1 x^{r_1} + c_2 x^{r_2}$

Example:

 $2x^2y''+3xy'-15y = 0$, the characteristic equation is :

$$2r(r-1) + 3r-15 = 0 \Longrightarrow r_1 = \frac{5}{2}, r_2 = -3.$$
$$\Rightarrow y(x) = c_1 x^{\frac{5}{2}} + c_2 x^{-3}.$$

Solving Euler Equations: (Case II

- The characteristic equation has two equal roots $r_1 = r_2 = r_2$.
- In this case the functions $y = x^r$ and $y = x^r \ln x$ are both solutions to the original equation. The general solution is: $y(x) = x^r (c_1 + c_2 \ln x)$

Example:

 $x^{2}y''-7xy'+16y = 0$, the characteristic equation is : $r(r-1)-7r+16 = 0 \Rightarrow r = 4.$ $\Rightarrow y(x) = c_{1}x^{4} + c_{2}x^{4} \ln x.$

Solving Euler Equations: (Case

- The characteristic equation has two complex roots $r_{1,2} = \lambda \pm \mu i$.
 - $x^{\lambda+\mu i} = e^{(\lambda+\mu i)\ln x} = x^{\lambda}\cos(\mu\ln x) + ix^{\lambda}\sin(\mu\ln x)$
 - So, in the case of complex roots, the general solution will be: $y(x) = x^{\lambda}(c_1 \cos(\mu \ln x) + c_2 \sin(\mu \ln x))$

Example:

 $x^2 y'' + 3xy' + 4y = 0$, the characteristic equation is : $r(r-1) + 3r + 4 = 0 \Rightarrow r_{1,2} = -1 \pm \sqrt{3}i$. $\Rightarrow y(x) = c_1 x^{-1} \cos(\sqrt{3} \ln x) + c_2 x^{-1} \sin(\sqrt{3} \ln x)$.

THANK YOU