Ordinary Differential Equations

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DATE : 12/10/2020

Introduction to Ordinary Differential Equations (ODE)

- Recall basic definitions of ODE,
  - order
  - linearity
  - initial conditions
  - solution
- Classify ODE based on(order, linearity, conditions)
- Classify the solution methods

Derivatives

Derivatives

Ordinary Derivatives

$$\frac{dy}{dx}$$

y is a function of one independent variable

Partial Derivatives

$$\frac{\partial u}{\partial y}$$

u is a function of more than one independent variable

Differential Equations

Differential Equations

Ordinary Differential Equations

$$\frac{d^2y}{dx^2} + 6xy = 1$$

involve one or more <u>Ordinary derivatives</u> of unknown functions

Partial Differential Equations

$$\frac{\partial^2 u}{\partial y^2} - \frac{\partial^2 u}{\partial x^2} = 0$$

involve one or more <u>partial derivatives</u> of unknown functions

Ordinary Differential Equations

Ordinary Differential Equations (ODE) involve one or more ordinary derivatives of unknown functions with respect to one independent variable

Examples:

$$\frac{dy}{dx} - y = e^x$$

$$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 2y = \cos(x)$$

y(x): unknown function

x: independent variable

Order of a differential equation

The **order** of an ordinary differential equations is the order of the highest order derivative

Examples:

$$\frac{dy}{dx} - y = e^x$$

First order ODE

$$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 2y = \cos(x)$$

Second order ODE

$$\left(\frac{d^2y}{dx^2}\right)^3 - \frac{dy}{dx} + 2y^4 = 1$$

Second order ODE

Solution of a differential equation

A **solution** to a differential equation is a function that satisfies the equation.

$$Example:$$

$$\frac{dx(t)}{dt} + x(t) = 0$$

$$Solution \quad x(t) = e^{-t}$$

Proof :

$$\frac{dx(t)}{dt} = -e^{-t}$$

$$\frac{dx(t)}{dt} + x(t) = -e^{-t} + e^{-t} = 0$$

Linear ODE

An ODE is linear if the unknown function and its derivatives appear to power one. No product of the unknown function and/or its derivatives

$$a_n(x)y^n(x)+a_{n-1}(x)y^{n-1}(x)+\cdots+a_1(x)y'(x)+a_0(x)y(x)=g(x)$$

$Examples:$

$$\frac{dy}{dx}-y=e^x$$ Linear ODE

$$\frac{d^2y}{dx^2}-5\frac{dy}{dx}+2x^2y=\cos(x)$$ Linear ODE

$$\left(\frac{d^2y}{dx^2}\right)^3-\frac{dy}{dx}+\sqrt{y}=1$$ Non-linear ODE

Boundary-Value and Initial value Problems

Initial-Value Problems

- The auxiliary conditions are at one point of the independent variable

$$y''+2y'+y=e^{-2x}$$

$$y(0)=1,\ \ y'(0)=2.5$$

same

Boundary-Value Problems

- The auxiliary conditions are not at one point of the independent variable

$$y''+2y'+y=e^{-2x}$$

$$y(0)=1,\ \ y(2)=1.5$$

- More difficult to solve than initial value problem

different

Classification of ODE

ODE can be classified in different ways

- Order
  - First order ODE
  - Second order ODE
  - N^{th} order ODE
- Linearity
  - Linear ODE
  - Nonlinear ODE
- Auxiliary conditions
  - Initial value problems
  - Boundary value problems

Solutions

- Analytical Solutions to ODE are available for linear ODE and special classes of nonlinear differential equations.
- Numerical method are used to obtain a graph or a table of the unknown function
- We focus on solving first order linear ODE and second order linear ODE and Euler equation

First Order Linear Differential Equations

- Def: A first order differential equation is said to be ***linear*** if it can be written

$$y' + p(x)y = g(x)$$

First Order Linear Differential Equations

- How to solve first-order linear ODE ?

Sol:

$$y' + p(x)y = g(x) \quad (1)$$

Multiplying both sides by $\mu(x)$, called an integrating factor, gives

$$\mu(x)\frac{dy}{dx} + \mu(x)p(x)y = \mu(x)g(x) \quad (2)$$

assuming $\mu(x)p(x) = \mu'(x), \quad (3)$ we get

$$\mu(x)\frac{dy}{dx} + \mu'(x)y = \mu(x)g(x) \quad (4)$$

First Order Linear Differential Equations

By product rule, (4) becomes

$$(\mu(x)y(x))' = \mu(x)g(x) \quad (5)$$

$$\Rightarrow \mu(x)y(x) = \int \mu(x)g(x)dx + c_1$$

$$\Rightarrow y(x) = \frac{\int \mu(x)g(x)dx + c_1}{\mu(x)} \quad (6)$$

Now, we need to solve $\mu(x)$ from (3)

$$\mu(x)p(x) = \mu'(x) \Rightarrow \frac{\mu'(x)}{\mu(x)} = p(x)$$

First Order Linear Differential Equations

$$\frac{\mu'(x)}{\mu(x)} = p(x) \Rightarrow (\ln \mu(x))' = p(x)$$

$$\Rightarrow \ln \mu(x) = \int p(x)dx + c_2$$

$$\Rightarrow \mu(x) = e^{\int p(x)dx + c_2} = c_3 e^{\int p(x)dx} \qquad (7)$$

to get rid of one constant, we can use

$$\mu(x) = e^{\int p(x)dx} \qquad (8)$$

The solution to a linear first order differential equation is then

$$y(x) = \frac{\int \mu(x)g(x)dx + c}{e^{\int p(x)dx}} \qquad (9)$$

Summary of the Solution Process

- Put the differential equation in the form (1)
- Find the integrating factor, $\mu(x)$ using (8)
- Multiply both sides of (1) by $\mu(x)$ and write the left side of (1) as $(\mu(x)y(x))'$
- Integrate both sides
- Solve for the solution $y(x)$

Example 1

$$y' - y = e^{2x}$$

Sol:

$$y(x) = e^{-\int p(x)dx}\left[\int e^{\int p(x)dx} g(x)dx + c\right]$$

$$= e^{-\int (-1)dx}\left[\int e^{\int (-1)dx} e^{2x}dx + c\right]$$

$$= e^{x}\left[\int e^{-x}e^{2x}dx + c\right]$$

$$= e^{x}\left[e^{x} + c\right]$$

$$= ce^{x} + e^{2x}$$

Example 2

$$xy' + 2y = x^2 - x,\ y(1) = \frac{1}{2}$$

Sol:

$$\Rightarrow y' + \frac{2}{x}y = x - 1$$

$$\Rightarrow y(x) = e^{-\int p(x)dx}\left[\int e^{\int p(x)dx} g(x)dx + c\right]$$

$$= e^{-\int \frac{2}{x}dx}\left[\int e^{\int \frac{2}{x}dx}(x-1)dx + c\right] = x^{-2}\left[\int x^2(x-1)dx + c\right]$$

$$= x^{-2}\left[\frac{1}{4}x^4 - \frac{1}{3}x^3 + c\right] = \frac{1}{4}x^2 - \frac{1}{3}x + cx^{-2}$$

Apply the initial condition to get c,

$$\Rightarrow \frac{1}{2} = \frac{1}{4} - \frac{1}{3} + c \Rightarrow c = \frac{7}{12}.$$

Second Order Linear Differential Equations

- Homogeneous Second Order Linear Differential Equations
  - real roots, complex roots and repeated roots

- Non-homogeneous Second Order Linear Differential Equations
  - Undetermined Coefficients Method

- Euler Equations

Second Order Linear Differential Equations

The general equation can be expressed in the form

$$ay''+by'+cy = g(x)$$

where a, b and c are constant coefficients

Let the dependent variable y be replaced by the sum of the two new variables: $y = u + v$

Therefore

$$\left[au''+bu'+cu\right]+\left[av''+bv'+cv\right]= g(x)$$

If v is a particular solution of the original differential equation

$$\left[au''+bu'+cu\right]=0$$

purpose

The general solution of the linear differential equation will be the sum of a "*complementary function*" and a "*particular solution*".

The Complementary Function (solution of the homogeneous equation)

$$ay''+by'+cy=0$$

Let the solution assumed to be: $y=e^{rx}$

$$\frac{dy}{dx}=re^{rx} \qquad \frac{d^2y}{dx^2}=r^2e^{rx}$$

$$e^{rx}(ar^2+br+c)=0$$

characteristic equation

→
- Real, distinct roots
- Double roots
- Complex roots

Real, Distinct Roots to Characteristic Equation

- Let the roots of the characteristic equation be real, distinct and of values r_1 and r_2. Therefore, the solutions of the characteristic equation are:

$$y = e^{r_1 x} \qquad\qquad y = e^{r_2 x}$$

- The general solution will be

$$y = c_1 e^{r_1 x} + c_2 e^{r_2 x}$$

- **Example**

$$y'' - 5y' + 6y = 0 \longrightarrow r^2 - 5r + 6 = 0$$

$$r_1 = 2, \quad r_2 = 3 \Rightarrow y = c_1 e^{2x} + c_2 e^{3x}$$

Equal Roots to Characteristic Equation

- Let the roots of the characteristic equation equal and of value $r_1 = r_2 = r$. Therefore, the solution of the characteristic equation is: $y = e^{rx}$

Let $y = Ve^{rx} \quad \Rightarrow y' = e^{rx}V' + rVe^{rx}$ and $y'' = e^{rx}V'' + 2re^{rx}V' + r^2Ve^{rx}$

where V is a function of x

$$ay'' + by' + cy = 0$$

$$\mathrm{ar}^2 + \mathrm{br} + \mathrm{c} = 0 \qquad 2\mathrm{ar} + \mathrm{b} = 0$$

$$V''(x) = 0 \rightarrow V = cx + d$$

$$y = be^{rx} + (cx + d)e^{rx} = c_1e^{rx} + c_2xe^{rx}$$

Complex Roots to Characteristic Equation

Let the roots of the characteristic equation be complex in the form $r_{1,2}$ =λ±μi. Therefore, the solution of the characteristic equation is:

$$y_1 = e^{(\lambda+\mu i)x} = e^{\lambda x}(\cos(\mu x) + i\sin(\mu x)),$$
$$y_2 = e^{(\lambda-\mu i)x} = e^{\lambda x}(\cos(\mu x) - i\sin(\mu x)).$$

$$u(x) = \frac{1}{2}(y_1 + y_2) = e^{\lambda x}\cos(\mu x),\ v(x) = \frac{1}{2i}(y_1 - y_2) = e^{\lambda x}\sin(\mu x)$$

It is easy to see that u and v are two solutions to the differential equation. Therefore, the geneal solution to the d.e. is :

$$y(x) = c_1 e^{\lambda x}\cos(\mu x) + c_2 e^{\lambda x}\sin(\mu x).$$

Examples

(I) Solve $y''+6y'+9y=0$

↓ characteristic equation

$$r^2+6r+9=0$$

↓

$$r_1=r_2=-3$$

↓

$$y=(c_1+c_2x)e^{-3x}$$

(II) Solve $y''-4y'+5y=0$

↓ characteristic equation

$$r^2-4r+5=0$$

↓

$$r_{1,2}=2\pm i$$

↓

$$y=e^{2x}(c_1\cos x+c_2\sin x)$$

Non-homogeneous Differential Equations (Method of Undetermined Coefficients)

$$ay''+by'+cy = g(x)$$

When $g(x)$ is constant, say k, a particular solution of equation is

$$y = k/c$$

When $g(x)$ is a polynomial of the form $a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n$ where all the coefficients are constants. The form of a particular solution is

$$y = \alpha_0 + \alpha_1 x + \alpha_2 x^2 + \ldots + \alpha_n x^n$$

Example

Solve $y''-4y'+4y=4x+8x^3$

complementary function

$y''-4y'+4y=0$

$r^2-4r+4=0$

characteristic equation

$r=2$

$y_c=(c_1+c_2x)e^{2x}$

$y=p+qx+rx^2+sx^3$

$y'=q+2rx+3sx^2$

$y''=2r+6sx$

$(2r+6sx)-4(q+2rx+3sx^2)+4(p+qx+rx^2+sx^3)=4x+8x^3$

equating coefficients of equal powers of x

$$\begin{cases} 2r-4q+4p=0 \\ 6s-8r+4q=4 \\ 4r-12s=0 \\ 4s=8 \end{cases}$$

$y_p=7+10x+6x^2+2x^3$

$$\begin{aligned} y &= y_c+y_p \\ &= (c_1+c_2x)e^{2x}+7+10x+6x^2+2x^3 \end{aligned}$$

Non-homogeneous Differential Equations (Method of Undetermined Coefficients)

- When $g(x)$ is of the form Te^{rx}, where T and r are constants. The form of a particular solution is

$$y = Ae^{rx}$$

$$A = \frac{T}{ar^2 + br + c}$$

- When $g(x)$ is of the form $C\sin nx + D\cos nx$, where C and D are constants, the form of a particular solution is

$$y = E\sin nx + F\cos nx$$

$$E = \frac{(c - n^2 a)C + nbD}{(c - n^2 a)^2 + n^2 b^2}$$

$$F = \frac{(c - n^2 a)C - nbD}{(c - n^2 a)^2 + n^2 b^2}$$

Example

Solve

$$3y''-6y'=18$$

$y=Cx$, $y'=C$, $y''=0$

$$3(0)-6(C)=18$$

$$C=-3$$

$$y_p=-3x$$

complementary function

$$3y''-6y'=0$$

$$3r^2-6r=0$$

characteristic equation

$$r_1=0,\ r_2=2$$

$$y_c=c_1+c_2e^{2x}$$

$$y=y_c+y_p$$
$$=-3x+c_1+c_2e^{2x}$$

Example

Solve

$$3y''+10y'-8y=7e^{-4x}$$

complementary function

$$3y''+10y'-8y=0$$

$$y=Cxe^{-4x}$$

$$y'=(1-4x)Ce^{-4x}$$

$$y''=(16x-8)Ce^{-4x}$$

$$3r^2+10r-8=(3r-2)(r+4)=0$$

characteristic equation

$$-24C+10C=7$$

$$r_1=2/3,\ r_2=-4$$

$$y_c=Ae^{2x/3}+Be^{-4x}$$

$$C=-\frac{1}{2}$$

$$y_p=-\frac{1}{2}xe^{-4x}$$

$$y=y_c+y_p$$

$$=-\frac{1}{2}xe^{-4x}+Ae^{2x/3}+Be^{-4x}$$

Example

Solve $y''+y'-6y=52\cos 2x$

complementary function

$$y''+y'-6y=0$$

$$r^2+r-6=(r-2)(r+3)=0$$

characteristic equation

$$r_1=2,\ r_2=-3$$

$$y_c=Ae^{2x}+Be^{-3x}$$

$$y=C\cos 2x+D\sin 2x$$

$$y'=-2(C\sin 2x-D\cos 2x)$$

$$y''=-4(C\cos 2x+D\sin 2x)$$

$$-10C+2D=52$$

$$-2C-10D=0$$

$$C=-5$$

$$D=1$$

$$y_p=-5\cos 2x+\sin 2x$$

$$y=y_c+y_p$$

$$=Ae^{2x}+Be^{-3x}-5\cos 2x+\sin 2x$$

Euler Equations

- Def: ***Euler*** equations

$$ax^2y''+bxy'+cy=0$$

  - Assuming x>0 and all solutions are of the form $y(x) = x^r$
  - Plug into the differential equation to get the characteristic equation

$$ar(r-1)+b(r)+c=0.$$

Solving Euler Equations: (Case I)

- The characteristic equation has two different real solutions r_1 and r_2.
- In this case the functions $y = x^{r_1}$ and $y = x^{r_2}$ are both solutions to the original equation. The general solution is:

$$y(x) = c_1 x^{r_1} + c_2 x^{r_2}$$

Example:

$2x^2 y'' + 3xy' - 15y = 0$, the characteristic equation is :

$$2r(r\text{-}1) + 3r\text{-}15 = 0 \Rightarrow r_1 = \frac{5}{2}, r_2 = -3.$$

$$\Rightarrow y(x) = c_1 x^{\frac{5}{2}} + c_2 x^{-3}.$$

Solving Euler Equations: (Case II)

- The characteristic equation has two equal roots $r_1 = r_2$=r.

- In this case the functions y = x^r and y = x^r lnx are both solutions to the original equation. The general solution is:

$$y(x) = x^r (c_1 + c_2 \ln x)$$

Example:

$$x^2 y'' - 7xy' + 16y = 0, \text{ the characteristic equation is :}$$

$$r(r\text{-}1) - 7r + 16 = 0 \Rightarrow r = 4.$$

$$\Rightarrow y(x) = c_1 x^4 + c_2 x^4 \ln x.$$

Solving Euler Equations: (Case III)

- The characteristic equation has two complex roots $r_{1,2}$ = λ±μi.

$$x^{\lambda+\mu i} = e^{(\lambda+\mu i)\ln x} = x^{\lambda}\cos(\mu \ln x) + i x^{\lambda}\sin(\mu \ln x)$$

So, in the case of complex roots, the general solution will be :

$$y(x) = x^{\lambda}(c_1 \cos(\mu \ln x) + c_2 \sin(\mu \ln x))$$

Example:

$x^2 y'' + 3xy' + 4y = 0$, the characteristic equation is :

$$r(r-1) + 3r + 4 = 0 \Rightarrow r_{1,2} = -1 \pm \sqrt{3}i.$$

$$\Rightarrow y(x) = c_1 x^{-1}\cos(\sqrt{3}\ln x) + c_2 x^{-1}\sin(\sqrt{3}\ln x).$$

THANK YOU