NAAC-Re Accredited" A"

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A

Project Work

Entitled

"The Pigeonhole Principle"

Submitted to

DEPARMENT OF MATHEMATICS

Vivekanand College, (Autonomous) Kolhapur

For the partial fulfillment of practical course for

The Award of B.Sc. Degree in Mathematics

By

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2018-2019

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CGPA=3.25

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CERTIFICATE

This is to certify that **Miss. Sarita Sureshlal Manchudiya** of the class B.Sc. III has satisfactorily completed the project work on the title *"The Pigeonhole Principle"* as a partial fulfillment of the practical course for the award of the B.Sc. Degree in Mathematics by Shivaji University, Kolhapur.

Place: Kolhapur

Date: 20/ 2/ 2019

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DECLARATION

It is hereby declare that the work reported in the project entitled *"The Pigeonhole Principle"* is completed and written by me and has not been copied from anywhere.

Place: Kolhapur

Date: 20/2/2019

Miss. Sarita Sureshlal Manchudiya

ACKNOWLEDGEMENT

I wish to express my deep sense of gratitude towards Mr. S.P. Patankar , Mr. S.T.Sutar, Department Of Mathematics, Vivekanand College, and Kolhapur for their valuable guidance to complete this project within time.

It is my proud privilege to express my sense of gratitude and sincere thanks to Principal Dr.S.Y.Honagekar & Prof. S.P. Patankar , Head of Department of Mathematics, Vivekanand College, and Kolhapur for providing all the available facilities of the college for completion of this project.

My sincere thanks to all those who have directly or indirectly involved in this project work.

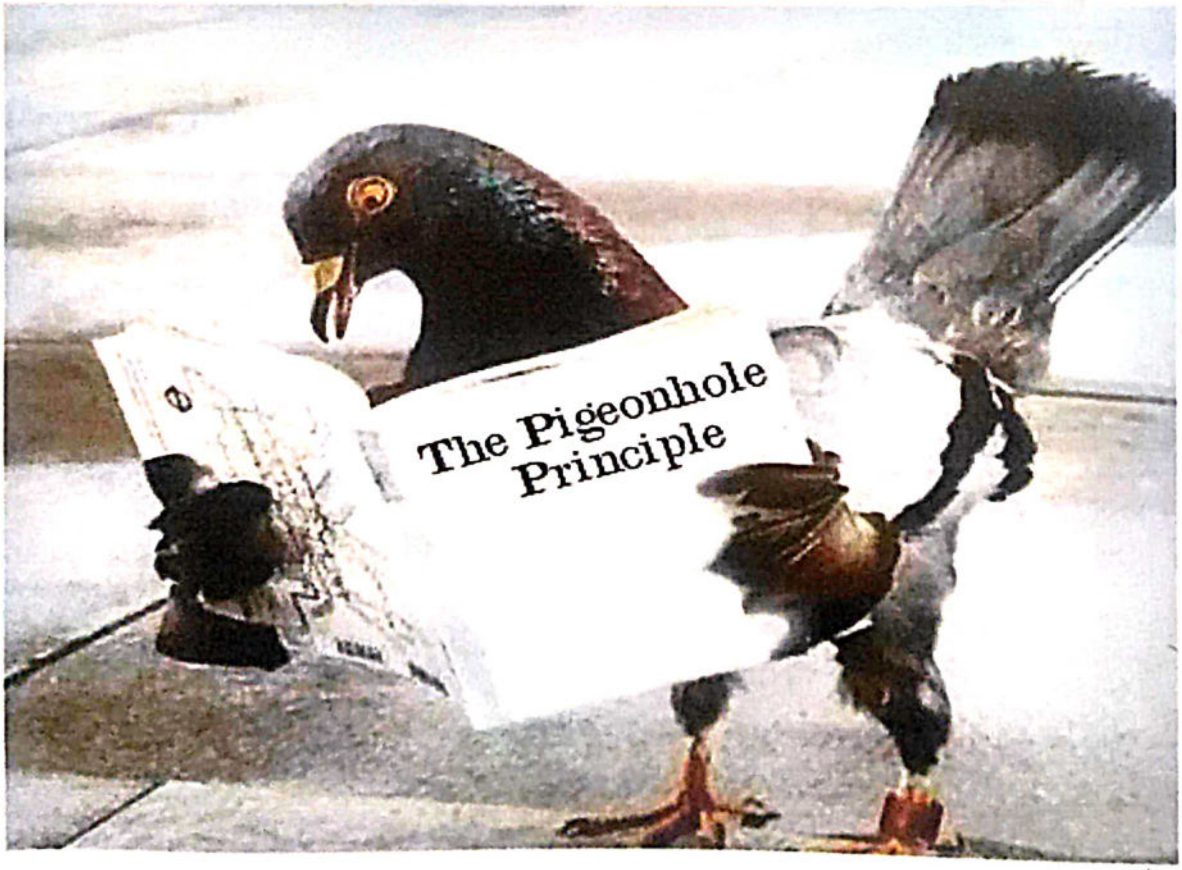
Miss. Sarita Sureshlal Manchudiya

INDEX

Introduction

In 1834, German mathematician Peter Gustav Lejeune Dirichlet (1805-1859) started a simple – but extremely powerful – mathematical principle which he called the Schubfachprinzip (drawer principle). Today it is known either as the Pigeonhole Principle, as Dirichlet Principle, or as The Cubby-hole Principle.

Pigeonhole principle is a fundamental but powerful tool in combinatorics. Unlike many other strong theorems, the principle itself is exceptionally simple. Unless you have looked into it thoroughly, it is hard to have a glimpse of its elegance and useful applications in mathematics.



The Basic Pigeonhole Principle:

Proposition PHP1:

(The Pigeonhole Principle, simple version.)

If k+1 or more pigeons are distributed among k pigeonholes, then at least one pigeonhole contains two or more pigeons.

Proof

The contrapositive of the statement is: If each pigeonhole contains at most one pigeon, then there are at most k pigeons. This is easily seen to be true.

The same argument can be used to prove a variety of different statements. We prove the general version of the Pigeonhole Principle and leave the others as exercises.

Proposition PHP2:

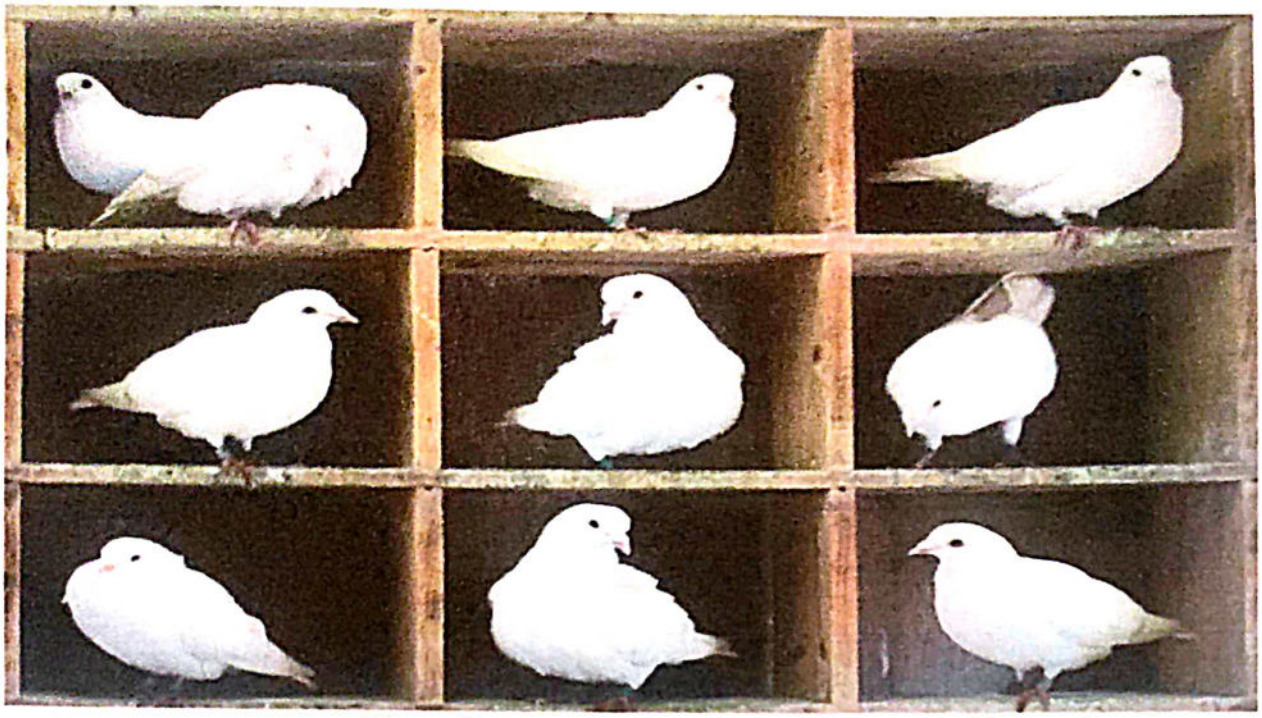
(The Pigeonhole Principle.)

If n or more pigeons are distributed among $k>0$ pigeonholes, then at least one pigeonhole contains at least $\left\lceil \frac{n}{k} \right\rceil$ pigeons.

Proof:

Suppose each pigeonhole contains at most $\left\lceil \frac{n}{k} \right\rceil - 1$ pigeons. Then, the total number of pigeons is at most

$$k(\left\lceil \frac{n}{k} \right\rceil - 1 < k\left(\frac{n}{k}\right) = n \text{ pigeons (because } \left\lceil \frac{n}{k} \right\rceil - 1 < \frac{n}{k} \leq \left\lceil \frac{n}{k} \right\rceil)$$



Other principles related to the pigeonhole principle:

- If n objects are put into n boxes and no box is empty, then each box contains exactly one object.
- If n objects are put into n boxes and no box gets more than one object, then each box has an object.

Example:

1. 51 numbers are chosen from the integers between 1 and 100 inclusively. Prove that 2 of the chosen integers are consecutive.

Solution.

Since the existence of consecutive integers is the main theme of the problem, it is natural to form pigeonholes using consecutive integers. We partition the 100 integers into
50 pairs of consecutive integers as pigeonholes: {1, 2}, {3, 4}, {5, 6}, ... , {99, 100}.

Let the 51 chosen integers be pigeons.

By Pigeonhole Principle, when we choose 51 integers, there is at least 1 pigeonhole (a pair of integers) containing $\left\lceil \frac{51}{50} \right\rceil = 2$ pigeons (chosen integers). Therefore, there are 2 consecutive integers among the 51 chosen integers.

2. 10 integers are chosen from 1 to 100 inclusively. Prove that we can find 2 disjoint non-empty subsets of the chosen integers such that the 2 subsets give the same sum of elements.

Solution.

Note that we are asked to find 2 subsets giving the same sum of elements. Therefore, we may let 'the possible sum of elements' be pigeonholes. Although we do not know what the 10 chosen numbers are, we know the range of the sums. Clearly, the sum must be an integer.

Moreover, the sum must be at least 1 (since all chosen integers are positive) and at most 91 +92+…. +100 =955.Therefore, we may set 955 pigeonholes as 1, 2, 3, ..., 955 and the subsets of the chosen numbers as pigeons. Obviously, there are $2^{10} - 1 = 1023$ pigeons (non-empty subsets).Therefore, there is at least 1 pigeonhole with
$\left\lceil \frac{1023}{955} \right\rceil = 2$ pigeons i.e. 2 subsets giving the same sum of elements.

Note that any one of them cannot be a subset of another. Otherwise, they must not have the same sum of elements. If the 2 subsets are disjoint, we are done. If they have common elements, we may remove the common elements from the 2 subsets. It reduces the sums by the same amount (namely the sum of common elements), so the 2 new subsets give the same sum again.

The pigeonhole principle can be phrased in terms of labels.

- If more than N objects are to be assigned labels from a set of N labels, then there is sure to be two objects with the same label.

This simple principle allows us to make some mighty surprising conclusions about the world.

EXAMPLE:

3. Twenty people in a room take part in handshakes. Each person shakes hands at least once and no one shakes the same person's hand more than once. Prove that two people took part in the same number of handshakes.

ANSWER:

Label each person by the number of handshakes she took part in. There are twenty people but only nineteen labels:1,2,3,...,18,19.By the pigeonhole principle, at least two people have the same label.

4. Eight positive numbers are chosen at random.

Explain why two of them are sure to differ by a multiple of seven.

ANSWER:

Label each number by the remainder it leaves when divided by seven. There are eight numbers and only seven labels: 0, 1, 2, 3, 4, 5, or 6. At least two numbers have the same label. The difference of these numbers is a multiple of seven.

5. Let $x_1, x_2, \ldots\ldots, x_{20}$ be 20 consecutive integers. Choose any 11 of them at random. Then at least two chosen integers differ by 10.

Answer:

Label each of the chosen numbers by its remainder upon division by 10. As there are 11 numbers and 10 labels, two must have the same label and hence differ by a multiple of 10.

Since the numbers $x_1, x_2, x_3, \ldots\ldots, x_{20}$ are consecutive, two cannot differ by 20 or more. Thus the two numbers that differ by a multiple of 10 can only differ by exactly 10.

THE GENERALISED PIGEONHOLE PRINCIPLE

If n pigeons are sitting in k pigeonholes, where n > k, then there is at least one pigeonhole with at least n/k pigeons.

Proof:

Assume there were not any pigeonhole with at least n/k pigeons.

Then every hole has < n/k pigeons, so the total number of pigeons is

< (n/k)(# holes) = (n/k)(k) = n.

But this says the number of pigeons is strictly less than n, and in fact there are exactly n pigeons.

So our assumption that there were no pigeonhole with at least n/k pigeons must have been incorrect, and this means the Generalized Pigeonhole Principle is true.

For example:-

6. If you have 5 pigeons sitting in 2 pigeonholes, then one of the pigeonholes must have at least 5/2 = 2:5 pigeons , but since (hopefully) the boxes can't have half-pigeons, then one of them must in fact contain 3 pigeons.

7. If 55 objects are to be put in 6 boxes, at least one box will possess at least 10 objects. (If not, six boxes with 9 or less objects will constitute only a total of 54 objects). Notice that 55/6 ≈9.17

8. Fifty & one integers are chosen at random from the numbers 1 through 100. Prove that at least two of the chosen integers will differ by 10.

ANSWER:

Label each of the chosen numbers by its remainder upon division by 10. As there are 51 numbers, some $\frac{51}{50} = 5.1$ of them, that is, at least 6 of them, must have the same label. Thus six numbers of the chosen numbers are spaced from each other by multiples of 10.

Since the six numbers are chosen from the range 1 to 100, it is not possible for all six to be 20 or more counts from each other. (Think about this.) Thus at least two of the six numbers differ by exactly 10.

How to construct pigeonholes?

Pigeons and pigeonholes are sometimes abstract. Therefore, drawing out some cases is the best way to find clues for the construction of the pigeonholes. From the trials, you will see how the pigeonholes should be constructed.

Example

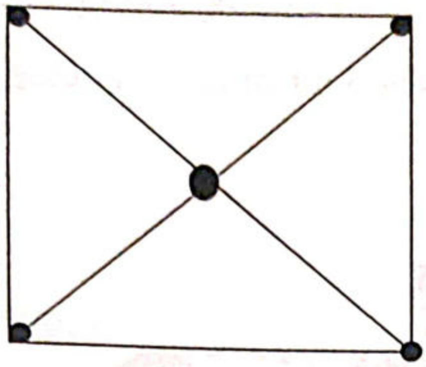
9. There are 5 points in a square of side length 2. Prove that there exist 2 of them having a distance not more than 2.

Solution.

After reading the question, we should try to think of the reason for the desired result.

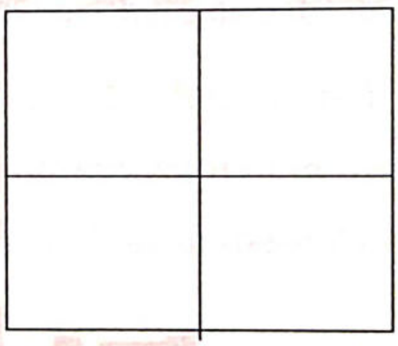
Since the unit square is bounded, in order to maximize the minimal distance between the points, we should put the points as 'far' as we can. The most intuitive way to do so is putting them at corners. However, there are only 4 corners, so one point should be placed so that it is 'quite' far from the 4 corners.

Clearly, the centre of the square is such a point. It is quite 'obvious' that the minimal distance between the points is the greatest now. (See Figure 9.1)



{fig 9.1:The 5 points at the corners and center.}

However, how can we claim the minimum distance cannot be greater than 2 in all circumstances? Actually, the figure tells us the answer. If we divide the square into 4 unit squares, then any two points in the same unit square are at most 2 apart. (See Figure 9.2) By Pigeonhole Principle, when 5 points are put into the squares, two of them must be in the same unit square. These two points are what we want to find.



{fig 9.2:The square is divided into 4 unit squares. There are always 2 points in 1 unit square.}

- The aid of the figure of the 'just satisfaction of the requirement' is important. It inspires us to think how we should divide the square. For example, you will know that dividing the square into 4 big triangles (using the 4 sides as the bases) is useless from Figure 9.1.

- If you check the Principle carefully, you will find that it is assumed implicitly that each pigeon are put in one pigeonhole only. Therefore, strictly speaking, some of the 4 unit squares should have lost a part of their common boundaries with other unit squares so as to be consistent with this implicit requirement of the theorem.

10. 27 points are aligned so that each row has 9 points and each column has 3 points. (A column is perpendicular to a row.) Each point is painted in red or blue. Prove that there exists a monochromatic rectangle (i.e. 4 vertices are of the same colour) with its sides parallel to the rows and columns.

Solution.

Try to draw the points to see how they 'behave'. If you try to draw and paint them randomly, you will probably find that some columns are exactly the same. Actually, this is the way to the solution. Every point may have 2 colours. Therefore, every column containing 3 points has $2^3 = 8$ colouring schemes. By Pigeonhole Principle, there are at least $\left\lceil \frac{9}{8} \right\rceil =$ 2 columns painted exactly the same. In each of these 2 columns, by Pigeonhole Principle again, we have at least 2 points in the column painted in the same colour. Obviously, these 4 points having the same colour in these 2 columns form a rectangle with the desired properties

- Drawing the points to see how they behave is important. It helps us find the way to construct pigeonholes.
- Actually, 21 points aligned in 7 rows and 3 columns are enough to find such a rectangle.

Finding the bound

In the sessions above, all problems only require a proof for the statement. For example, in Example 1, it tells us to prove choosing 51 numbers from 1 to 100 must give 2 consecutive integers. However, can this condition hold if 51 is replaced by a smaller number? Since questions are sometimes open, we have to seek for the best lower bound some time. How can we find the best lower bound? Pigeonhole Principle sometimes helps a lot.

To prove a number is the best lower bound, we have to check that any smaller integers lead to a counter example. Let's take Example 1 as an example:

Example:

11. Prove that 51 cannot be replaced by any smaller integer in Example 1

Solution:

From the solution in Example 1, we see that if any one pigeonhole contains 2 chosen integers, the condition is satisfied. Therefore, if the condition fails, each pigeonhole must contain at most 1 chosen number. We can use this clue to find a set of 50 numbers where each pigeonhole contains 1 chosen number only. It is not difficult to find out

{1, 3, 5, 7, ..., 99} and {2, 4, 6, ..., 100} are sets of 50 integers without consecutive integers. Therefore, 51 is the best lower bound.

12. Prove that 6 cannot be replaced by 5 .

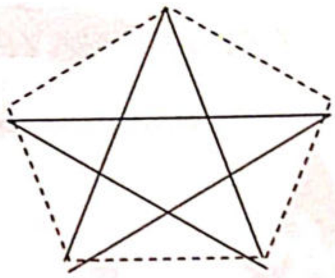
Solution.

We see that if any point is incident to 3 blue edges or 3 red edges, then there is a monochromatic triangle. Therefore, no triangle occurs only if all 5 points have no 3 blue edges and no 3 red edges incident to it. Since there are 4 edges incident to each point, we can conclude for the edges incident to each point, exactly 2 of them are red and exactly 2 of them are blue. Having known this information, we can draw a counterexample easily. (Figure 1) Hence, 6 cannot be replaced by 5.

{fig 1: A counter example.

(Solid lines are and dashed lines

are blue)}



- It proves that R(3,3)=6.

- Figure 1 is the only graph with 5 vertices in 2 colours having no monochromatic triangles. (To be precise in graph theory, we say it is unique up to homomorphism.) It consists of 2 monochromatic pentagons, one in red and one in blue.

13. If 19 is replaced by 18 , find a colouring scheme for the 72 points such that no monochromatic rectangles exist.

Solution.

If a pigeonhole has more than 1 pigeon, then there is a monochromatic rectangle. Therefore, the only way to make the condition fail is to let each pigeonhole contain no more than one pigeon. Since there are 18 columns (pigeons) and 18 pigeonholes, it forces each pigeonhole to accept one element. Moreover, if some of the columns have more than 1 choice of its pigeonholes, then clearly it can lead to more than 1 element in a pigeonhole.

Therefore, each column must have only exactly 1 pair of points in the same colour. Using this clue, we can find a colouring scheme for 18 columns to avoid rectangles:

RRRBBB BBBRRR YYYBBB

RBBRRY BRRBBY YBBYYR

BRYRYR RBYBYB BYRYRY

YYRYRR YYBYBB RRYRYY

{A counter example}

Pigeonhole Principle: Strong Form

Theorem 1.

Let $q_1, q_2, \dots, q_n$ n be positive integers. If

$$q_1 + q_2 + \cdots + q_n - n + 1$$

objects are put into n boxes, then either the 1st box contains at least q_1 objects, or the 2nd box contains at least q_2 objects,…, the nth box contains at least q_n objects.

Proof.

Suppose it is not true, that is, the ith box contains at most $q_i - 1$ objects, i:1,2,….,n. Then the total number of objects contained in the n boxes can be at most

$$(q_1 - 1) + (q_2 - 1) + \cdots + (q_n - 1) - q_1 + q_2 + \cdots + q_n - n,$$

which is one less than the number of objects distributed. This is a contradiction.

The simple form of the pigeonhole principle is obtained from the strong form by taking $q_1 - q_2 - \cdots - q_n - 2$.

Then $q_1 + q_2 + \cdots + q_n - n + 1 - 2n - n + 1 - n + 1.$

In elementary mathematics the strong form of the pigeonhole principle is most often applied in the special case when $q_1 - q_2 - \cdots - q_n - r$. In this case the principle becomes:

- If n(r-1)+ 1 objects are put into n boxes, then at least one of the boxes contains r or more of the objects. Equivalently.
- If the average of n nonnegative integers $a_1, a_2, \ldots, a_n$ is greater than r-1, i.e.,

$$\frac{a_1 + a_2 + \cdots + a_n}{n} > r - 1$$

then atleast one of the integers is greater than or equal to r.

example:

14. A basket of fruit is being arranged out of apples, bananas, and oranges. What is the smallest number of pieces of fruit that should be put in the basket in order to guarantee that either there are at least 8 apples or atleast 6 bananas or at least 9 oranges?

Answer: 8 + 6 + 9-3 + 1 = 21.

15. Given two disks, one smaller than the other. Each disk is divided into 200 congruent sectors. In the larger disk 100 sectors are chosen arbitrarily and painted red; the other 100 sectors are painted blue. In the smaller disk each sector is painted either red or blue with no stipulation on the number of red and blue sectors. The smaller disk is placed on the larger disk so that the centers and sectors coincide. Show that it is possible to align the two disks so that the number

of sectors of the smaller disk whose color matches the corresponding sector of the larger disk is at least 100.

Proof.

We fix the larger disk first, then place the smaller disk on the top of the larger disk so that the centers and sectors coincide. There 200 ways to place the smaller disk in such a manner. For each such alignment, some sectors of the two disks may have the same color. Since each sector of the smaller disk will match the same color sector of the larger disk 100 times among all the 200 ways and there are 200 sectors in the smaller disk, the total number of matched color sectors among the 200 ways is

100x200 = 20,000. Note that there are only 200 ways. Then there is at least one way that the number of matched color sectors is $\frac{20,000}{200} = 100$ or more

16. Show that every sequence $a_1, a_2, \dots, a_{n^2+1}$ of $n^2 + 1$ real numbers contains either an increasing subsequence of length n+ 1 or a decreasing subsequence of length n+ 1.

Proof.

Suppose there is no increasing subsequence of length n+1. We suffices to show that there must be a decreasing subsequence of length n+ 1.

Let l_k be the length of the longest increasing subsequence which begins with a_k, $1 \le k \le n^2 + 1$. Since it is assumed that there is no increasing subsequence of length n+ 1, we have $1 \le l_k \le n$ for all k. By the strong form of the pigeonhole principle, n+ 1 of the $n^2 + 1$ integers $l_1, l_2, \dots, l_{n^2+1}$ must be equal, say,

$$l_{k_1} - l_{k_2} - \cdots l_{k_{n+1}},$$

where, $1 \le k_1 < k_2 < \cdots < k_{n+1} \le n^2 + 1$.

If there is one k_i ($1 \leq i \leq n$) such that $a_{k_i} < a_{k_{i+1}}$, then any increasing subsequence of length $l_{k_{i+1}}$ beginning with $a_{k_{i+1}}$ will result a subsequence of length $l_{k_{i+1}} + 1$ beginning with a_{k_i} by adding a_{k_i} in the front; so $l_{k_i} > l_{k_{i+1}}$, which is contradictory to $l_{k_i} - l_{k_{i+1}}$. Thus we must have

$$a_{k_1} \geq a_{k_2} \geq \cdots \geq a_{k_{n+1}},$$

which is a decreasing subsequence of length n+ 1.

Ramsey Theory

Theorem 1.

Let S be a finite set with n elements. Let $P_r(S)$ be the collection of all r-subsets of S with r≥1, i.e,

$$P_r(S) - \{X \subseteq S : |X| - r\}.$$

Then for any integers p, q ≥ r there exists a smallest integer R(p, q; r)such that, if n ≥ R(p, q; r) and $P_r(S)$ is
2-colored with two color classes C_1 and C_2, then there is either a p-subset $S_1 \subseteq S$ such that $P_r(S_1) \subseteq C_1$, or a q-subset $S_2 \subseteq S$ such that $P_r(S_2) \subseteq C_2$.

Proof.

We proceed by induction on p, q, and r. For r-1, we have R (p, q; 1)– p+q-1. Note that every element of $P_1(S)$ is a singleton set and $|P_1(S)| - |S|$. For an n-set S with n≥p+q-1, if $|C_1| \geq p$, we take any p-subset
$S_1 \subseteq \cup_{X \in C_1} X$, then obviously $P_1(S_1) \subseteq C_1$. If $|C_1| < p$, then $|C_2| \geq q$; we take any q-subset
$S_2 \subseteq \cup_{X \in C_2} X$ and obviously have $P_1(S_2) \subseteq \boldsymbol{C_2}$. Thus R(p, q; 1)≤p+q-1 . For n-p+q-2, let $\boldsymbol{C_1}$ be the set of p-1 singleton sets and $\boldsymbol{C_2}$ the set of the other q-1 singleton sets. Then it is impossible to have a p-subset $\boldsymbol{S_1} \subseteq \boldsymbol{S}$ such that

$P_1(S_1) \subseteq C_1$ or a q-subset S_2 such that $P_1(S_2) \subseteq C_2$. Thus R(p, q; 1)≥p+q-1.

Moreover, for any integer r≥1 it can be easily verified that

R(r, q; r)-q, R(p, r; r)-p.

In fact, for p-r, let S be a q-set. For a 2-coloring $\{ C_1, C_2 \}$ of $P_r(S)$, if $C_1 - \emptyset$, then $P_r(S) - C_2$ and obviously $P_r(S_2) \subseteq C_2$ for $S_2 - S$. If $C_1 \neq \emptyset$, take an r-subset A∈ C_1 ; obviously, $P_r(A) - \{A\} \subseteq C_1$. Thus

R(k, q; k)≤q . Let $|S| \leq$ q−1 . If $C_1 - \emptyset$, then

$C_2 - P_r(S)$. It is clear that there is neither an r-subset

A ⊆ S such that $P_r(A) \subseteq C_1$ nor a q-subset B ⊆ S such that $P_r(B) \subseteq C_2$. Thus R(r, q; r)≥q. It is similar for the case R(p, r; r)-p.

Next we establish a recurrence relation about R(p, q; r) for r ≥ 2 as follows:

$$R(p,q;r) \leq R\big(p_1, q_{1;} r-1\big) + 1, p_1 - R(p-1, q; r),$$

$$q_1 - R(p, q-1; r)$$

Let n ≥ $R\big(p_1, q_{1;} r-1\big) + 1$ and $|S| - n$. Take an element x∈ S and let $S_1 - S - \{x\}$. Then $|S| - n - 1$ and $|S_1| \geq R(p_1, q_1; r-1)$. Let {C; D} be a 2-coloring of $P_r(S)$ and let,

$$C_1 - \{A \in C: x \notin A\}, D_1 - \{A \in D: x \notin A\}.$$

Obviously, $\{C_1, D_1\}$ is a 2-coloring of $P_r(S_1)$. Let

$$C_x - \{A \in P_{r-1}(S_1): A \cup \{x\} \in C\},$$

$$D_x - \{A \in P_{r-1}(S_1): A \cup \{x\} \in D\}.$$

For any $A \in P_{r-1}(S_1)$, it is obvious that either

$A \cup \{x\} \in C$ or $A \cup \{x\} \in D$; then either $A \in C_x$ or

$A \in D_x$. Thus $\{C_x, D_x\}$ is a 2-coloring of $P_{r-1}(S_1)$. Since $|S_1| \geq R(p_1, q_1; r-1)$. and by the induction hypothesis on k, we have (I) there exists a p_1-subset $X \subseteq S_1$ such that $P_{r-1}(X) \subseteq C_x$, or (II) there exists a q_1-subset

$Y \subseteq S_1$ such that $P_{r-1}(Y) \subseteq D_x$.

Case (I): Since $p_1 - R(p-1, q; r)$and $\{ C_1, C_2 \}$ is a 2-coloring of $P_r(S_1)$, by induction hypothesis on p (when r is fixed) there exists either a (p-1)-subset $X_1 \subseteq X$ such that $P_r(X_1) \subseteq C_1 \subset C$ or a q –subset $Y_1 \subseteq X$ such that $P_r(Y_1) \subseteq D_1 \subset D$. In the former case, consider the p-subset $X' - X_1 \cup \{x\} \subseteq S$. For any r-subset $A \subset X'$, if $x \notin A$, obviously $A \subset X_1$, so $A \in C$; if $x \in A$, obviously $A - \{x\}$ is an (r-1)-subset of X, so $A - \{x\} \in C_x$, then $A - (A - \{x\} \cup \{x\} \in C$. This means that X' is an r-subset of S and $P_r(X') \subseteq C$. In the latter case, we already have a q-subset $Y_1 \subseteq S$ such that $P_r(Y_1) \subseteq D$.

Case (II): Since $q_1 - R(p, q-1; r)$ and $\{ C_1, C_2 \}$ is a partition of $P_r(S_1)$, then by induction hypothesis on q (when r is fixed) there exists either a p-subset $X_1 \subseteq X$ such that $P_r(X_1) \subseteq C_1 \subset C$ or a (q-1)-subset $Y_1 \subseteq X$ such that $P_r(Y_1) \subseteq D_1 \subset D$. In the former case, we already have a p-subset $X_1 \subseteq S$ such that $P_r(X_1) \subseteq C$. In the latter case, we have a q-subset $Y' - Y_1 \cup \{x\} \subset S$ and $P_r(Y') \subseteq D$

Now we have obtained a recurrence relation:

$R(p, q; r) \leq R(R(p-1, q; r), R(p, q-1; r); r-1)+1$.

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Thank you.