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SHRI SWAMI VIVEKANAND SHIKSHAN SANSTHA

VIVEKANAND COLLEGE KOLHPUR.

Department of Mathematics

PROJECT REPORT ON

“Pascal Triangle”

For Mathematical studies 2019-2020

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CERTIFICATE

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VIVEKANAND COLLEGE, KOLHAPUR.

DEPARTMENT OF MATHEMATICS

This is to certify that, the project report entitled "Pascal Triangle" by Akankasha Ajit Ingale. As a partial fulfilment of B.Sc. III in Mathematics to Vivekanand College, Kolhapur.

This project has been completed under my guidance and supervision. To best of my knowledge and belief, the matter presented in project report is original and has not been submitted elsewhere for any other purpose.

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ACKNOWLEDGEMENT

It gives me a great Pleasure while presenting this report on "Pascal Triangle"

A since of prevailing satisfaction and achievement envelops the whole felling of completion of the project in the last year 2019-2020 under the guidance of Prof. Patankar S. P. I wish to express my deep sense of gratitude to for his valuable guidance and co-operations without which it would have been impossible to accomplished this success.

I am also thankful to all teaching and non-teaching staff that helped me during completion of this project.

Submitted by,

Akanksha Ajit Ingale .

DECLEARATION

I undersigned, here by declare that this seminar on titled "Pascal Triangle"

Is original work prepared by as under the guidance of Prof. Patankar S. P. the empirical findings in this seminar based on data collected by our team. The matter presented in this project is obtained from various sources on internet.

We give surety that any illegal matter is liable for punishment in any way the University deem to fit this work has not been submitted to the award of any degree of dither to Shivaji University, Kolhapur or any other university.

This work is humbly submitted to SHIVAJI UNIVERSITY as project under the curriculum.

Place:

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INTRODUCTION

Patterns are a big part of the study of mathematics and one of the most interesting aspects to incorporate into the math classroom. The discrete methods of combinational analysis, and their applications to the construction of mathematical models and solutions of applied problems in technology and the natural sciences, have brought about a great deal of interest in the study of the arithmetic and geometric properties of the so-called “Arithmetic triangles.” The classical example of the arithmetic triangle is, of course, the Pascal Triangle.

Pascal’s triangle is one of the most famous and interesting patterns in mathematics. In fact, while studying this triangle that initially appears to consist of a simple pattern of numbers, one learns that it contains many complex patterns. This paper briefly looks at the history of Pascal’s triangle and how it is defined and then explores not only its connection with algebra and probability, but also some of the intriguing patterns and topics contained within Pascal’s triangle.

HISTORY

Blaise Pascal(1623-1662)

The history of Pascal's triangle begins at least 500 years before its name sake, Blaise Pascal, was even born. In the 10th and 11th centuries, Indian and Persian Mathematicians first started working on this pattern of numbers. Also during the 10th century various Arab mathematicians developed a mathematical series for calculating the coefficients for $(1 + x)^n$ when n is a positive number. In addition, around 1070 Omar Khayyam, a Persian Mathematician, astronomer and philosopher worked on the binomial expansion and the numerical coefficients, which are the values of a row in Pascal's triangle.

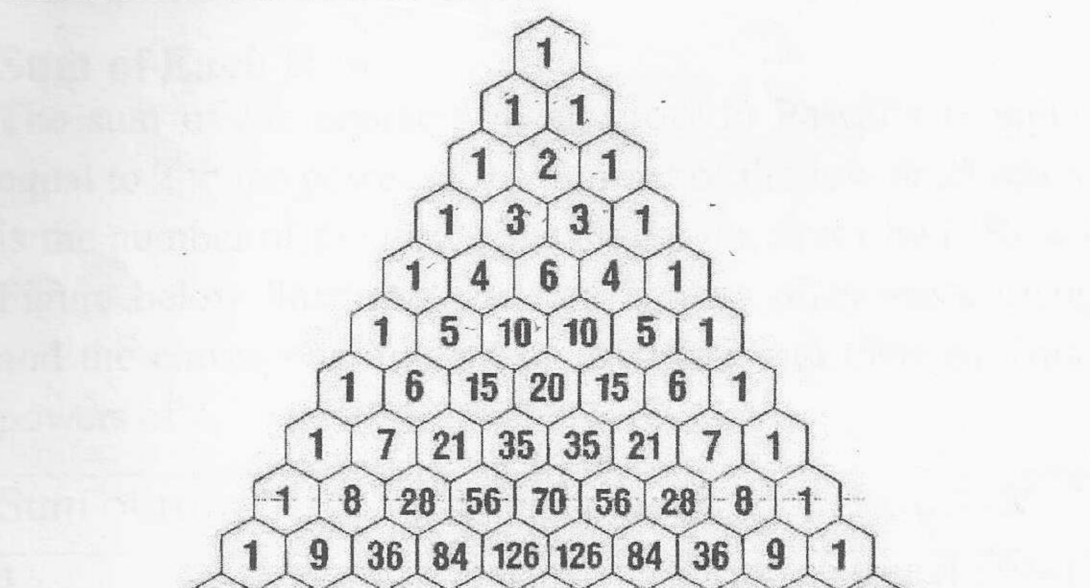
In China in the 13th century, hundreds of years before Pascal, Yang Hui worked on the exact same pattern as we know it today. Consequently, the Chinese will not refer the pattern as Pascal's triangle, and even to date it is known as Yang Hui's Triangle in China.

It wasn't until around 1654, that the French mathematician and philosopher, Blaise Pascal, began to investigate the triangle. His discussions with Peirre de Fermat on the chance of getting different values for rolls of dice led him to the triangle. These discussions later laid the foundation for the theory of probability. The two major areas where Pascal's triangle is used today are in algebra and probability specifically in regards to combinatorics. Pascal is credited because he investigated and took the information on this system of numbers, compiled and organized it so it made sense and was more useful. Pascal died in 1662 at the age of 39 before his work was published. In 1665, his work *Traite du triangle arithmetique* was published. In the early 1700s, two mathematicians, Pierre Raymond de Montmort and Abraham de Moivre, published articles each naming the triangle after Pascal and so it became known as Pascal's Arithmetic Triangle.

PASCAL'S TRIANGLE

CONSTRUCTION OF THE TRIANGLE: -

The easiest way to construct the triangle is to start at row zero and write only the number one. From there in order to obtain the numbers in the following rows, add the number directly above and left to the number directly above and right to acquire the new value. If there are no numbers on the left or the right just replace a zero for that missing number and proceed with the addition. Here is an illustration of rows zero to 17.



Some Patterns in Pascal's Triangle

- Each number is the sum of the two numbers above it.
- The outside numbers are all 1.
- The triangle is symmetric.
- The first diagonal shows the counting numbers.
- The sums of the rows give the powers of 2.
- Each row gives the digits of the powers of 11.
- Each entry is an appropriate "choose number."
- And those are the "binomial coefficients."
- The Fibonacci numbers are in there along diagonals.

Sum of Each Row

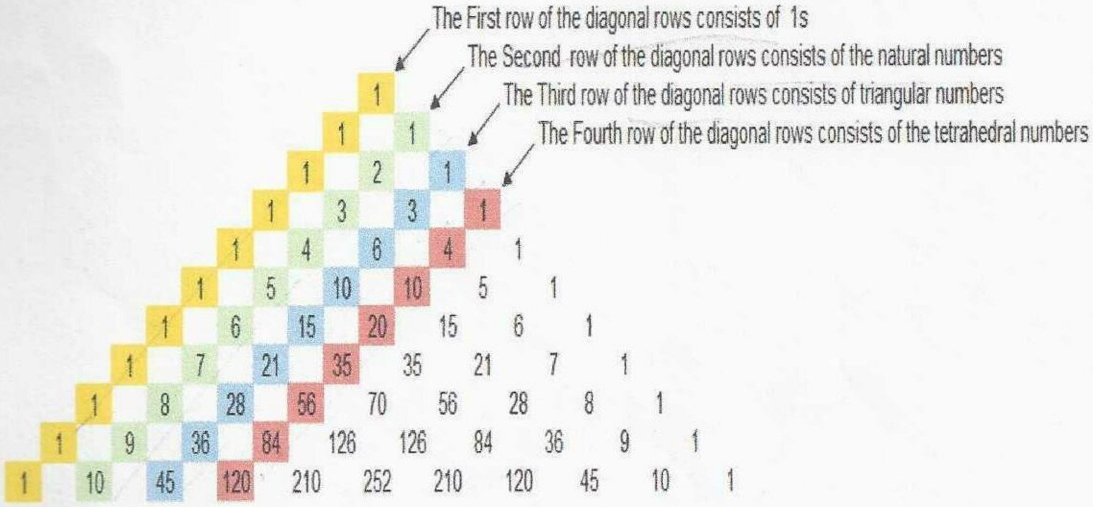
The sum of the numbers in any row in Pascal's triangle is equal to 2 to the power of the number of the row or 2^n where n is the number of the row. (Recall that the first row is Row 0.) Figure below illustrates the first 5 rows of Pascal's triangle and the corresponding sum of each row and their equivalent powers of 2.

Sum of rows	Pascal Triangle	Row#	2^n
1	1	Row0	$2^0=1$
1+1=2	1 1	Row1	$2^1=2$
1+2+1=4	1 2 1	Row2	$2^2=4$
1+3+3+1=8	1 3 3 1	Row3	$2^3=8$
1+4+6+4+1=16	1 4 6 4 1	Row4	$2^4=16$

The sum of the numbers on the nth row in Pascal's triangle is 2^n. For example, the sum of 10th row is 210.

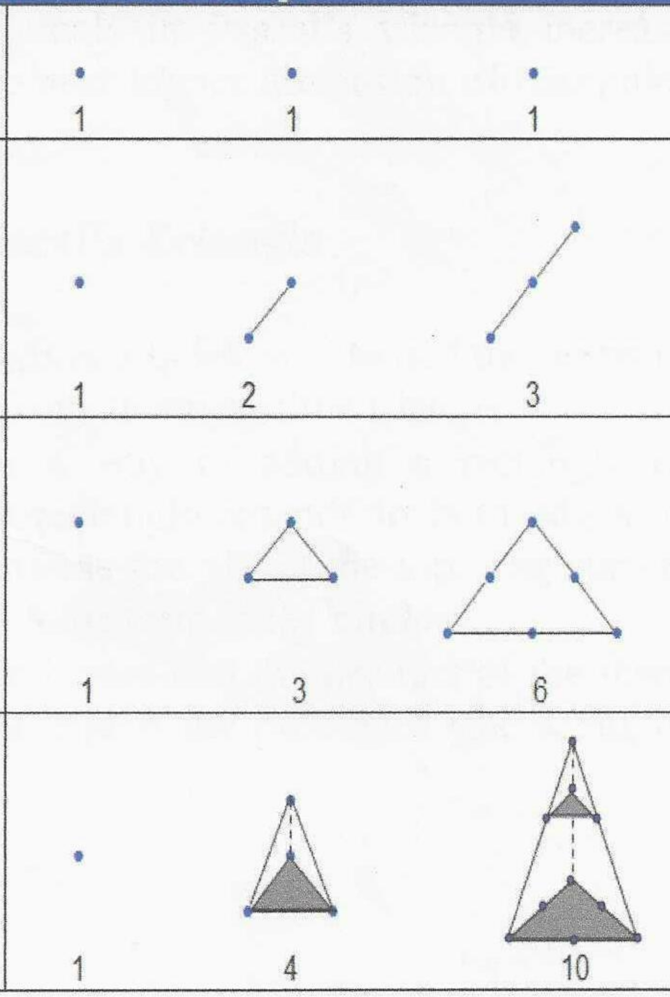
Diagonals of Pascal's Triangle

The diagonals of Pascal's triangle also have interesting patterns as is seen in Figure 14. All entries on the first diagonal are 1s. The second diagonal is made up of the counting numbers starting 1, 2, and 3 …. The third diagonal is made up of the triangular numbers 1, 3, and 6... and the fourth diagonal is made up of the tetrahedral numbers 1, 4, and 10…. Each of these sets of numbers is referred to as figurative numbers because they can be represented by regularly evenly spaced geometric figures.



As show in the table in Figure below, the first diagonal which contains a series of 1s can be geometrically represented by a series of individual points. The second diagonal which is the natural counting numbers can be represented as a series of points that form individual points. The second diagonal which is the natural counting numbers can be represented as a series of points that form individual lines. The third diagonal which contains the triangular numbers can be represented geometrically as a series of points that form triangular shapes. For example, the triangular number 3 can be visualized as a grouping of 3 points equally spaced that make a triangle. The same is true for the triangular numbers 6 which is shown in Figure 15 and the number 10 which can be visualized like a set of bowling pens.

Diagonals in Pascal's Triangle			
Diagonal 1 → all 1s a point - 0 dimensional	1	1	1
Diagonal 2 → counting numbers a line - 1 dimensional	1	2	3
Diagonal 3 → trianglular numbers a triangle - 2 dimension shape	1	3	6
Diagonal 4 → tetrahedral numbers a tetrahedron - 3 dimensional shape	1	4	10

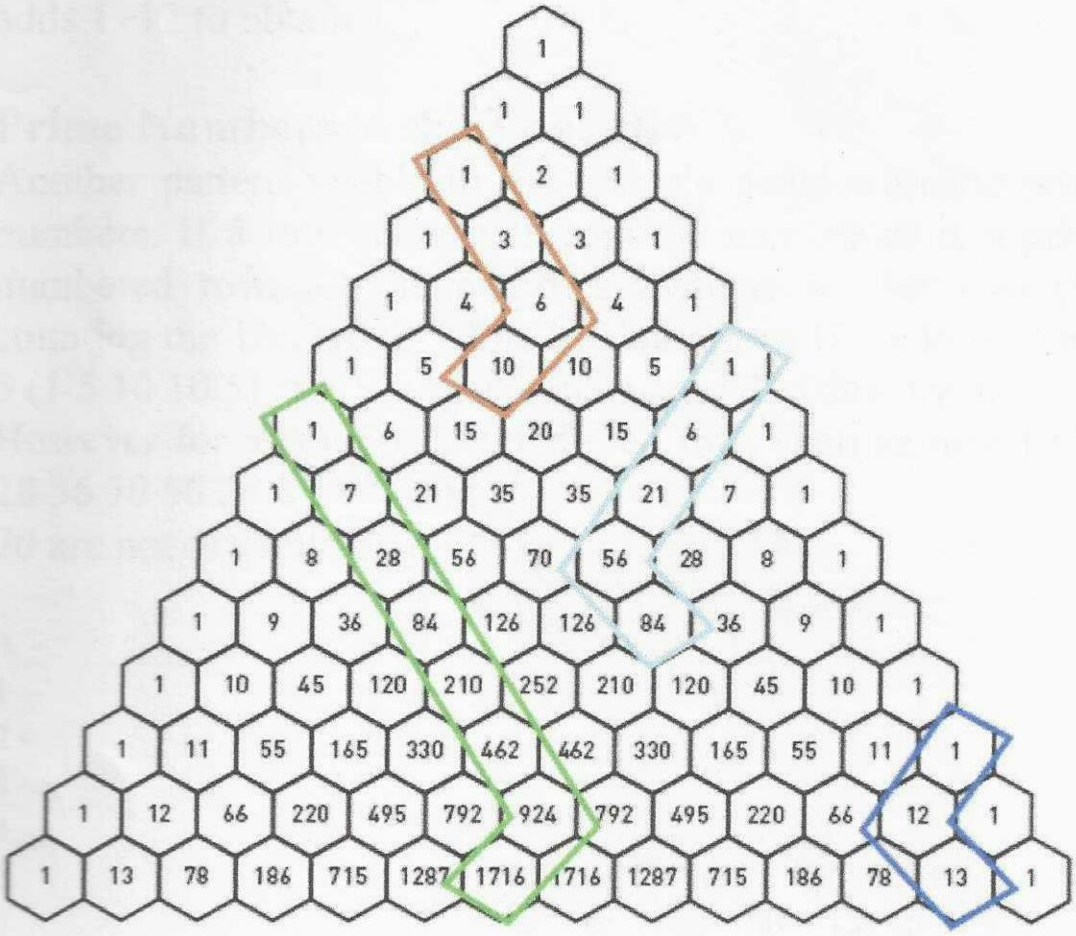


The fourth diagonal which contains the tetrahedral numbers can be represented geometrically as a series of points that form three dimensional tetrahedral shapes. The next diagonal is the pent tope numbers which is a geometric structure that is hard to visualize, but it is like a 4-dimensional tetrahedron. So the diagonals in Pascal's triangle increase with each diagonal to the next the fourth diagonal which contains the tetrahedral numbers can be represented geometrically as a series of points that form three dimensional tetrahedral shapes. The next diagonal is the Pentatope numbers which is a geometric structure that is hard to visualize, but it is like a 4-dimensional tetrahedron. So the diagonals in Pascal's triangle increase with each diagonal to the next higher dimension of triangular numbers.

The Arithmetic of Pascal's Triangle

- The "hockey stick" gives a quick way to add the terms in a diagonal, starting from an edge of the triangle.
- The "funnel" gives a way of adding a rectangle of numbers, where the rectangle extends to both edges of the triangle, and contains the "1" at the top. The sum is given by one less than the term in the circle.
- The Star of David indicates that the product of the three terms in one triangle equals the product of the terms in the other triangle.

HOCKEY STICKS IN PASCAL'S TRIANGLE



The "hockey-stick rule": Begin from any 1 on the right Edge of the triangle and follow the numbers left and down for any number of steps. As you go, add the numbers you encounter. When you stop, you can find the sum by taking A 90-degree turns on your path to the right and stepping down one. It is called the hockey-stick rule since the numbers involved form a long straight line like the handle of a hockey stick, and the quick turn at the end where the sum appears is like the part that contacts the puck. Figure illustrates four of them. The upper one adds 1+3+6 to obtain10, the second adds 1+6+21+56 to obtain84, the third adds

1+7+28+84+210+462+924 to obtain 1716, and the fourth adds 1+12 to obtain 13.

Prime Numbers in the Triangle:-

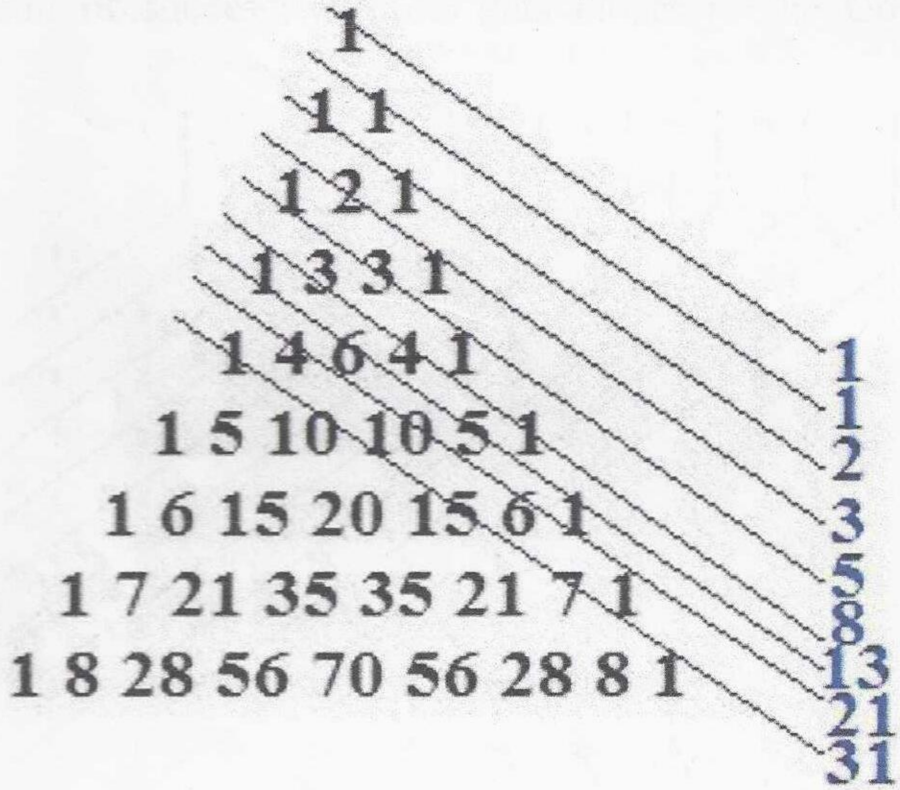
Another pattern visible in the triangle deals with the prime numbers. If a row starts with a prime number or is a prime numbered row, all the numbers that are in that row (not counting the 1's) are divisible by that prime. If we look at row 5 (1 5 10 10 51) we see that 5 and 10 are divisible by 5.
However for a composite numbered row, such as row 8 (1 8 28 56 70 56 28 8 1) 28 and
70 are not divisible by 8.

0	1
1	1 1
2	1 2 1
3	1 3 3 1
4	1 4 6 4 1
5	1 5 10 10 5 1
6	1 6 15 20 15 6 1
7	1 7 21 35 35 21 7 1
8	1 8 28 56 70 56 28 8 1

Suppose we number each row, starting from 0. Do you see a difference between the prime-numbered and the composite-numbered rows? Found it yet? If N is prime, then the N^{th} row contains only multiples of N, ignoring the 1's on the left and right. But if N is composite, then the N^{th} row has numbers other than 1 that are not multiples of N—for instance, 15 and 20 in the 6th row.

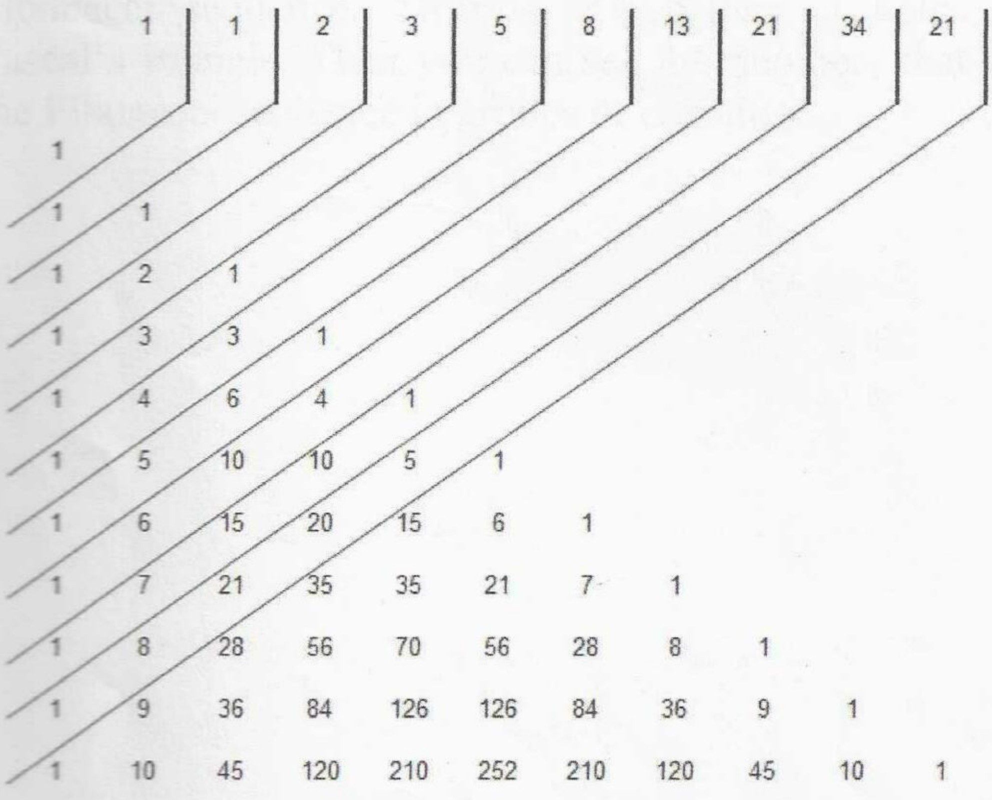
Fibonacci sequence in the triangle:-

By adding the numbers in the diagonals of the Pascal triangle the Fibonacci sequence can be obtained. There are various ways to show the Fibonacci numbers on the Pascal triangle



Leonardo Pisano Bigollo was an Italian mathematician who became better known by the nickname Fibonacci. In 1202 he wrote *Liber Abaci* which when translated means *The Book of the Abacus*. This book is famous for introducing the Hindu-Arabic numerals and the decimal system to the Western world, but it also contained a famous number sequence that he is remembered for today. This number sequence was actually known to Indian mathematicians as early as the 6th century, but it was Fibonacci who introduced this to the West.

The first two number of the Fibonacci series are 1 and 1, and then each number after that in the series is the sum of the previous two so the Fibonacci sequence is a recursive sequence of numbers. Also, another interesting note about the Fibonacci sequence is as the sequence goes farther along the ratio of successive terms gets closer to the Golden Ratio.



The recursive formula for the Fibonacci sequence is: Fn = Fn-1 + Fn-2where: F1 = 1 and F1 = 1
1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987 . . .
Even the Fibonacci sequence can be found in Pascal's triangle although finding it is a little less obvious than the figurative numbers. The diagram in above Figure illustrates how to identify pathways of numbers that sum to the terms of the Fibonacci sequence. To find this pattern it helps to tilt Pascal's triangle. Then you can see the numbers that sum to the Fibonacci sequence in groups or corridors.

How Pascal's Triangle Relates to Combinations and Binomials

The study of combinatories is a branch of mathematics that studies finite or discrete structures and includes permutations and combinations. If we start with a set of n objects and ask how many ways can we select a subset of r objects, we are asking how many **combinations** are possible. Note that the order in which the objects are selected does not matter. Take for example, making a fruit salad in which there are 5 fruits to choose from that include: strawberries, oranges, blackberries, apples, and grapes and you can choose to use or not use each fruit in your salad. (We are not discussing how much of each fruit is used.)

Let:

S = strawberries

O = oranges

B = blackberries

A = apples

G = grapes

1. When choosing zero fruit for the salad there is only one combination

No combination → 1 selection

2. If you choose only one fruit for the salad, since there are five fruits, there are five different ways to make a salad.

S, O, B, A, G → 5 selections

3. In choosing two different kinds of fruits the list of combinations for the fruit salad will look like:

SO, SB, SA, SG, OB, OA, OG, BA, BG, AG → 10 selections

4. In choosing three different fruits the list of combinations will look like
SOB, SOA, SOG, SBA, SBG, SAG, OBA, OBG, BAG, OAG $\rightarrow$ 10 selections
5. In choosing four different kinds of fruits the list of combinations will look like:

SOBA, SOBG, OBGA, SBAG, SOAG $\rightarrow$ 5 selections
6. And finally, when choosing all five fruits for the salad there is only one combination.

SOBAG $\rightarrow$ 1 selection
Listing all the possible combinations is tedious and it becomes very easy to make a mistake and miss options. Thus, it is almost impossible for larger combinations. Note that when choosing more than one different type of fruit for the salad, the order in which the fruit is chosen doesn't matter. For instance adding blackberries and then the apples is the same as adding the apples and then the blackberries. Thus, this is a combination instead of a permutation.
At this point, we want to establish a general fact behind what we see to be true for this example. We claim $C(n,r)$ is the number of r-element subsets of a set of S that contains n elements for any r, $0 \leq r \leq n$. We prove this by induction on n. If S = { } then S has 1 subset that contains no elements. And if S = $\{x\}$ then S has 1 subset that contains no elements. This corresponds to C (1,0) = 1. S also has 1 subset that contains 1 element which corresponds to the fact that C (1,1) = 1.
We assume that for $n > 1$, $C(n\text{-}1, r)$ is the number of r element subsets of a set with n-1 elements and for any r, $0 \leq r \leq n-1$.

Consider C (n,r) and a set $S_n = \{1, 2, 3, \ldots, n\text{-}1, n\}$. Organize the r-element subsets of S_n into two disjoint sets; A, which is the set of subsets of S that do contain n, and B, the subsets that do not contain n.

If we remove n from each of the subsets of A, we have the collection of all subsets of $[1, 2, 3,\ldots,(n\text{-}1)]$ that contain $(r\text{-}1)$ elements. Therefore, by induction, the size of set A is C $(n\text{-}1, r\text{-}1)$. Moreover, none of the sets in B contain n, so each is also a subset of $\{1, 2, 3 \ldots (n\text{-}1)\}$ that contains r elements. Our inductive analysis of this shows that the size of set B is C $(n\text{-}1, r)$.

Since $C(n,r) = C(n\text{-}1, r\text{-}1) + C(n\text{-}1, r)$ we have established that the number of r elements subsets of a set with n elements is $C(n, r)$. Now we would like to establish the formula, $C(n, r) = {}^nC_r$, where ${}^nC_r$ is defined as $n!/r!(n\text{-}r)!$

This too is an induction proof on n, with the Case $n = 0$ ($C(0, 0) = 1 = 0!/0!0!$) being obvious.

Now assume that if $n > 0$, then for all r, $0 \leq r \leq n\text{-}1$ we have $C(n\text{-}1, r) = (n\text{-}1)!/r!(n\text{-}1\text{-}r)!$

Note first that $C(n,0) = 1 = n!/0!(n\text{-}0)! = {}^nC_0$ which is the left end of the nth row of Pascal's triangle.

And $C(n,n) = 1 = n!/n!(n\text{-}n)! = {}^nC_n$ which is the right end of the nth row of Pascal's triangle.

We consider the situation where $1 \leq r \leq n\text{-}1$ and

$$C(n,r) = C(n\text{-}1, r\text{-}1) + C(n\text{-}1, r)$$
$$= [(n\text{-}1)!\,/(r\text{-}1)!\,((n\text{-}1)\text{-}(r\text{-}1))!] + [(n\text{-}1)!\,/r!\,(n\text{-}1\text{-}r)!]$$
$$= [(n\text{-}1)!\,/(r\text{-}1)!(n\text{-}r)!] + [(n\text{-}1)!/r!(n\text{-}1\text{-}r)!]$$
$$= n!/r!(n\text{-}r)!$$

This general formula is consistent with our example, e.g. $C(5,2) = 10$ and$5!/(2!(5\text{-}2)!) = 10$.

So how can Pascal's triangle are used in combinatorics? Think of Pascal's triangle as a simple way to do calculations. For example, look at the last column in the table in Figure. And compare that to Row 5 in Pascal's triangle. Now look at the last two columns in the table together.

The third entry is C(5,2) corresponds to the fifth row second position. Thus, if one wants to find the number of two element subsets in a five element set, rather than perform the calculations required, one can simply look at Pascal's triangle. In the combination format C(*n*, *r*) or *n* choose *r*, the row number in Pascal's triangle corresponds to the *n* or the total number of objects to choose from and the position within the row in Pascal's triangle corresponds to the *r* or the number of objects to be chosen. However, it must be remembered that the row and position in Pascal's triangle begins with zero.

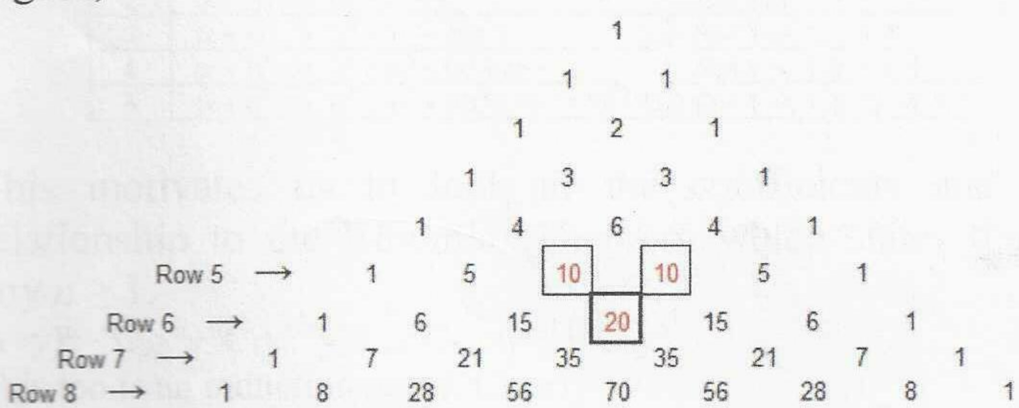
Ways to choose from 5 total fruits	C(n,r)	Combinations
There is 1 way to select no fruit from the 5 total fruits	C(5,0)	= 1
There are 5 ways to select 1 fruit from the 5 total fruits	C(5,1)	= 5
There are 10 ways to select 2 fruits from the 5 total fruits	C(5,2)	= 10
There are 10 ways to select 3 fruits from the 5 total fruits	C(5,3)	= 10
There are 5 ways to select 4 fruits from the 5 total fruits	C(5,4)	= 5
There is 1 ways to select 5 fruits from the 5 total fruits	C(5,5)	= 1

Pascal states the combinatorial relation in the formula for finding *fij* in the following way: $C(n+1, r+1) = C(n, r) + C(n, r+1)$

Using the example C (6,3) we let $n = 5$ and $r = 2$.

Then $C(5 + 1, 2 + 1) = C(5, 2) + C(5, 2 + 1)$ or

C (6,3) = C(5,2) + C(5,3), which is shown in the following Figure,



Recall that as early as the 10th century, various Arab mathematicians developed a mathematical series for calculating the coefficients for $(1 + x)^n$ when n was a positive number. As we will learn, the entries in Pascal's triangle correspond to the coefficients in a binomial expansion, and thus they are commonly referred to as binomial coefficients.

A binomial expression is an expression with two terms such as $x + y$ and binomial expansion refers to the formula to expand out binomials raised to a certain power. Given the binomial expression $(x + 1)^2$ we know that it is equal to $(x + 1)(x + 1)$. When this binomial expression is expanded (or multiplied through) it is equal to $1x^2 + 2x + 1$. On the following page figure 10 which contains a table showing the binomial expression $(x + 1)$ raised to the powers of zero through 5. Notice that for each expansion, the coefficients correspond to a row in Pascal's triangle.

Power	Binomial Expansion	Pascal's Triangle
0	$(x+1)^0 = 1$	Row 0 → 1
1	$(x+1)^1 = 1x+1$	Row 1 → 1, 1
2	$(x+1)^2 = 1x^2+2x+1$	Row 2 → 1, 2, 1
3	$(x+1)^3 = 1x^3+3x^2+3x+1$	Row 3 → 1, 3, 3, 1
4	$(x+1)^4 = 1x^4+4x^3+6x^2+4x+1$	Row 4 → 1, 4, 6, 4, 1
5	$(x+1)^5 = 1x^5+5x^4+10x^3+10x^2+5x+1$	Row 5 → 1, 5, 10, 10, 5, 1

This motivates us to look at the coefficients and their relationship to the Binomial Theorem which states that for any $n \geq 1$.

$(x+y)^n={}^nC_0x^n+{}^nC_1x^{n-1}y^r+\ldots{}^nC_{n-1}xy^{n-1}+{}^nC_ny^n$

This too is an induction proof. Clearly

$(x+y)^1={}^1C_0x+{}^1C_1y$ so the theorem is true for $n = 1$. Assume $n > 1$ and the theorem is true for $n-1$.

$(x+y)^n=(x+y)(x+y)^{n-1}=x(x+y)^{n-1}+y(x+y)^{n-1}$

$=x\{{}^{n-1}C_0x^{n-1}+{}^{n-1}C_1x^{n-2}y+\ldots+{}^{n-1}C_{n-1}y^{n-1}\}$

$+ y\{{}^{n-1}C_0x^{n-1}+{}^{n-1}C_1x^{n-2}y+\ldots+{}^{n-1}C_{n-1}y^{n-1}\}$

Collecting coefficients, the coefficient of x^n is and the coefficient of y^n is ${}^{n-1}C_0=1={}^nC_0$ and the coefficient of y^n is ${}^{n-1}C_{n-1}=1={}^nC_n$

Consider now a term $x^{n-r}y^r$ where $1 \leq r \leq n\text{-}1$, from the first expression we get the term ${}^{n-1}C_r*x*x^{n-1-r}*y^r$ and from the second we get ${}^{n-1}C^{r-1}*y*x^{n-r}*y^{r-1}$

When we add them we get the coefficient of term $x^{n-r}y^r$ to be

${}^{n-1}C_r+{}^{n-1}C_{r-1}=C(n-1,r)+C(n-1,r-1)$

$=C\ (n-1,r-1)+C\ (n-1,\ r)$

$=C\ (n,r)$

$=nC_r$

Coloring Mod 3 — Divisibility Only

An Easier Way to Color Modularly To find the remainder when you make a sum, you can just find the remainders of the things you're adding...and add *them*!

+ mod 3	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

For example: If you add the entries "5" and "10," you get the sum "15." The remainder of 15, mod 3, is 0.

Alternately, you could look at the remainders of "5" and "10," namely "2" and "1" respectively, and add them.
2 + 1 = 3, and the remainder of "3" mod 3 is 0.

So we get the same answer two different ways!

CONCLUSION

The complexity of something as simple as a sequence of number in a triangular shape is amazing. Each topic branches off into a web of interconnected, increasingly complicated mathematical topics and information. This is experienced even in the history of the various mathematicians who have discovered new patterns over the years. I found it interesting to note how many of the early mathematicians were also philosophers as philosophical parallels can be made to the complex interconnectedness of individuals in life as well. The ramifications of the simple yet increasingly complex study of Pascal's triangle in education are enormous encouraged to differentiate instruction as to engage students at different levels of learning style and ability. In the classroom, students could be working on a variety of activities as it relates to Pascal's triangle with some as simple as finding patterns while others finding the connections within the patterns. Most importantly, thought, is allowing students to experience the interconnected of math topics that Pascal's triangle naturally allows.

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