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In

Mathematics

By

SAMIKSHA SUBHASH BHOSALE

Roll No :- 7875

Exam Seat No :-

PRN No :- 2019071048

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Under the Guidance of

Mr. Patankar S.P (H.O.D) Department of Mathematics

Mr. Thorat S.P (Department of Mathematics

DECLERATION

I undersigned hereby declare that project entitled **" Differential Equations "**. Completed under the guidance of Mr. S.P. Patankar sir and Mr. S. P. Thorat sir (Department of Mathematics, Vivekanand College Kolhapur). Based on the experiment results and cited data . I declare that this is my original work which is submitted to , Vivekanand College Kolhapur in this academic year.

Ms. SAMIKSHA SUBHASH BHOSALE

Samiksha

28-05-2022

"Education for Knowledge, Culture and Science"
... Shikshanmaharshi Dr. Bapuji Salunkhe.

Shri. Swami Vivekanand Shikshan Sanstha, kolhapur's
VIVEKANAND COLLEGE (AUTONOMOUS), KOLHAPUR
DEPARTMENT OF MATHEMATICS

CERTIFICATE

This is to certify that Ms. SAMIKSHA SUBHASH BHOSALE has successfully compieted the project work on topic " " Differential Equations "" towards the partial fulfilment for the course of Bachelor of Science (Mathematics) work of Vivekanand College (Autonomous) Kolhapur affilited to Shivaji University, Kolhapur during the academic year 2021-2022. This report represents the bonafide work of student.

Place :- Kolhapur

Date :- 28-05-2022

Mr. S.P. Patankar
Head Dept. of
Mathematics

HEAD
Department of Mathematics
Vivekanand College, Kolhapur

Teacher Incharge

Examiner

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Place :- Kolhapur

Date :-

Ms. SAMIKSHA SUBHASH BHOSALE

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INTRODUCTION

A differential equation is any equation which contains derivatives, either ordinary derivatives or partial derivatives.

There is one differential equation that everybody probably knows, that is Newton's Second Law of Motion. If an object of mass m is moving with acceleration a and being acted on with force Fthen Newton's Second Law tells us.

A differential equation is a mathematical equation that relates some function with its derivatives. In applications, the functions usually represent physical quantities, the derivatives represent their rates of change, and the equation defines a relationship between the two. Because such relations are extremely common, differential equations play a prominent role in many disciplines including engineering, physics, economics, and biology.

In pure mathematics, differential equations are studied from several different perspectives, mostly concerned with their solutions—the set of functions that satisfy the equation. Only the simplest differential equations are solvable by explicit formulas; however, some properties of solutions of a given differential equation may be determined without finding their exact form.

If a self-contained formula for the solution is not available, the solution may be numerically approximated using computers. The theory of dynamical systems puts emphasis on qualitative analysis of systems described by differential equations, while many numerical methods have been developed to determine solutions with a given degree of accuracy.

Types of Differential Equation:

- **Ordinary differential equations:**

An ordinary differential equation (*ODE*) is an equation containing a function of one independent variable and its derivatives. The term "*ordinary*" is used in contrast with the term partial differential equation which may be with respect to *more than* one independent variable.

- **Partial differential equations:**

A partial differential equation (*PDE*) is a differential equation that contains unknown multivariable functions and their partial derivatives. (This is in contrast to ordinary differential equations, which deal with functions of a single variable and their derivatives.) PDEs are used to formulate problems involving functions of several variables, and are either solved in closed form, or used to create a relevant computer model.

- **Linear differential equations:**

A differential equation is *linear* if the unknown function and its derivatives have *degree* 1 (products of the unknown function and its derivatives are not allowed) and *nonlinear* otherwise. The characteristic property of linear equations is that their solutions form an affine subspace of an appropriate function space, which results in much more developed theory of linear differential equations.

- **Non-linear differential equations:**

Non-linear differential equations are formed by the *products of the unknown function and its derivatives* are allowed and its degree is > 1.There are very few methods of solving nonlinear differential equations exactly; those that are known typically depend on the equation having particular symmetries

Differential Equations :-

Differential equations are of great important in engineering because many physical laws and relations appear mathematically in the form of differential equations.

THE TECHNIQUE OF MATHEMATICAL MODELING :-

Mathematical modeling essentially consists of translating real world problems into mathematical problems solving the mathematical problems and interpreting these solutions in the language of real world i.e. Differential equations arise in many engineering and other applications as mathematical models of various physical and other systems.

For example if we drop a stone then its acceleration $y'' = \frac{d^2y}{dt^2}$ is equal to the acceleration of gravity g(a constant), Hence the model of this problem of "Free Fall" is $y'' = g$ (neglecting air resistance).

We have velocity $y' = \frac{dy}{dt} = gt + v_0$ where v_0 is the initial velocity with which the motion started (e.g. $v_0 = 0$)

We get the distance traveled

$$y = \frac{g}{2}t^2 + v_0 t + y_0$$

Where y_0 is the distance from 0 at the beginning.

We shall consider physical problems which lead to a differential equation of first order and first degree and that of second order which reduces to first order.

ORTHOGONAL TRAJECTORIES :-

1. Trajectory : A curve which cuts every member of a given family of curves according to some definite law is called a trajectory of the family
2. Orthogonal Trajectory : A curve which cuts every member of given family of curves at right angles is called as orthogonal trajectory of the family.
3. Orthogonal Trajectories : Two families of curves are said to be orthogonal if every member of either family cuts each member of the other family at right angles.

Thus if the given family consist of straight lines $y = mx$ (m = constant) represnting family of straight lines all passing through the origin then the family of circles $x^2 + y^2 = a^2$ (a is parameter) with centre at $(0, 0)$ represents a family of orthogonal trajectories to the family $y = mx$.

WORKING RULE TO FIND THE EQUATION OF ORTHOGONAL TRAJECTORIES :-

Step 1:-

Given $F(x,y,a) = 0$ where a is parameter.

Step 2:-

Differentiate F(x,y,a)=0 w.r.t. x and eliminate 'a' we thus forqm a differential equation of the family of the form

$\Phi\left(x,y,\frac{dy}{dx}\right)=0$

Step 3:-

Replace $\frac{dy}{dx}$ by $-\frac{dx}{dy}$. Then the differential equation of the family of orthogonal trajectories will be $\Phi\left(x,y,-\frac{dx}{dy}\right)=0$.

Step 4:-

The solution of step 3 is the family of orthogonal trajectories.

Example 1:

Find the orthogonal trajectories of the family of straight lines y = mx.

Solution:

Given, y = mx ------------------(1)

Differentiate (1) w.r.t. x

$\therefore \frac{dy}{dx} = m$

$m = \frac{y}{x}$ .. (from 1)

$\therefore \frac{dy}{dx} = \frac{y}{x}$ ---------------(2)

Which is the differential equation of the given family (1)

Replace $\frac{dy}{dx}$ by $-\frac{dx}{dy}$ in (2)

We get,

$$-\frac{dx}{dy} = \frac{y}{x} \quad \text{-----------------(3)}$$

Or $xdx + ydy = 0$

Which is the differential equation of the orthogonal trajectories

Integrating (3),

$$\int xdx + ydy = b$$

$$\frac{x^2}{2} + \frac{y^2}{2} = b$$

$$x^2 + y^2 = c^2$$

Which is the equation of the required orthogonal trajectories of (1).

ATMOSPHERIC PRESSURE:-

Example 1:

Find the atmospheric pressure p kg/m^2 at a height z meters above the sea – level

Solution:-

Consider a vertical column of air of unit cross-section. Let an element of this column be bounded by two horizontal planes at height z and $z + \delta z$ above the sea level. Let p and δp be the pressure at height z and $z + \delta z$ respectively

Let ρ be the average density of the element.

The thin column δz of air is in equilibrium under the action of forces.

(1) p kg upwards

(2) $(p + \delta p)$ kg downwards

(3) The weight $\rho g \, \delta z$ kg downwards

$$\therefore p = p + \delta p + \rho g \, \delta z$$

$$\Rightarrow \delta p + \rho g \, \delta z = 0 \text{ or } \frac{\delta p}{\delta z} = -\rho g$$

Taking the limit as $\delta z \to 0$

we have

$$\frac{\delta p}{\delta z} = -\rho g \quad \text{-----------------(1)}$$

Which is the differential equation giving atmospheric pressure at any height z

Now, we discuss equation (1) under two assumptions

(1) When the temperature is constant.

(2) When the temperature varies.

Case 1:- When the temperature is constant.

By Boyle's law

$p = k\rho$ or $\rho = p/k$

Substituting this value of ρ in (1) we have,

$$\frac{\delta p}{\delta z} = -\frac{\rho g}{k} \text{ or } \frac{\delta p}{p} = -\frac{g}{k}\delta z$$

Integrating,

$$\int \frac{\delta p}{p} = -\frac{g}{k}\int \delta z + c \text{ or } \log p = -\frac{g}{k} z + c \text{ ---(2)}$$

At the sea level, $z = 0$ $p = p_0$(say)

Then $c = \log p_0$

$\therefore$ From (2),

$$\log p = -\frac{g}{k}z + \log p_0$$

$$\text{or } \log \frac{p}{p_0} = -\frac{g}{k}z$$

$$\therefore p = p_0 e^{-\frac{gz}{k}}$$

Case 2:- when the temperature varies

$$\therefore \text{ Let } p = k\rho^n, n \neq 1 \text{ or } \rho = \left(\frac{p}{k}\right)^{1/n}$$

substituting this value of ρ in (1) we have,

$$\frac{dp}{dz} = -\left(\frac{p}{k}\right)^{1/n} .g \text{ or } p^{-\frac{1}{2}}\, dp = -g\, k^{-\frac{1}{n}}\, dz$$

Integrating

$$\int p^{-\frac{1}{2}}\, dp = -g\, k^{-\frac{1}{n}} \int dz + c$$

$$\text{or } \frac{n}{n-1} p^{1-\frac{1}{n}} = -g\, k^{-\frac{1}{n}}\, z + c$$

At the sea level , $z = 0$, $p = p_0$ (say)

Then,

$$c = \frac{n}{n-1} p_0^{1-\frac{1}{n}}$$

$\therefore$ from(3),

$$\frac{n}{n-1} p^{1-\frac{1}{n}} = -g\, k^{-\frac{1}{n}}\, z + \frac{n}{n-1} p_0^{1-\frac{1}{n}}$$

$$\text{or } \frac{n}{n-1}\left(p_0^{1-\frac{1}{n}} - p^{1-\frac{1}{n}}\right) = g\, k^{-\frac{1}{n}} z$$

SIMPLE ELECTRIC CIRCUITS :-

We shall consider circuits made up of

(1) Three passive elements – resistance, inductance and capacitance

(2) An active element – voltage source which may be a battery or generator.

1. Table of elements symbols and units

No	Element	Symbol	Unit
1	Time	t	Second
2	Quantity of electricity	q	Coulomb
3	Current	i	Ampere(A)
4	Resistance R	R	Ohm(Ω)
5	Inductance, L	L	Henry(H)
6	Capacitance, C	C	Farad(F)
7	Electromotive Force or voltage, F	F.	volt
8	Variable voltage Generator	V	volt

2. Basics Relations :

(i) $i = \frac{dq}{dt}$ or $q = \int i\, dt$

(ii)Voltage drop across resistance $R = R_i$

(iii) Voltage drop across inductance $L = L\frac{di}{dt}$

(iv) Voltage drop across capacitance $C = \frac{q}{c}$

3. Differential Equations

(i) Circuit involving L and R along with a voltage source (battery) E, all in series:

Consider a ckt. containing resistance R and inductance L in series with a voltage source (battery) E

Let i be the current flowing in the ckt. at any time t. Then by Kirchoff's 1st law

We have,

Sum of voltage drops across R and L = E

$$Ri + L\frac{di}{dt} = E$$

i.e.

$$\frac{di}{dt} + \frac{R}{L} = \frac{E}{L} \quad \text{----------(1)}$$

which is the Linear differential equation

$$\text{I.F.} = e^{\int \frac{R}{L}dt} = e^{\frac{Rt}{L}}$$

$$\text{i.e. } e^{\frac{Rt}{L}} = \int \frac{E}{L} e^{\frac{Rt}{L}}\,dt + c = \frac{E}{L}.\frac{L}{R}.e^{\frac{Rt}{L}} + c$$

$$\text{Hence, } i = \frac{E}{R} + c\,e^{\frac{-Rt}{L}} \quad \text{----------(2)}$$

If initially there is no current in the circuit i.e. i =0 when t = 0 we have $c = -\frac{E}{R}$

Thus (2) becomes

$$i = \frac{E}{R}\left(1 - e^{-\frac{Rt}{L}}\right). \text{ As } t \to \infty, i = \frac{E}{R}$$

$$R\frac{di}{dt} + \frac{1'}{C} = wE_0 \cos wt$$

$$\text{or } \frac{di}{dt} + \frac{1'}{C} = \frac{wE_0}{R} \cos wt \quad \text{-------------(1)}$$

which is linear differential equation

$$\text{I.F.} = e^{\int \frac{1}{RC} dt} = e^{\frac{t}{RC}}$$

The solution of equation (1) is

$$i.e^{\frac{t}{RC}} = \int \frac{wE_0}{R} \cos wt.e^{\frac{t}{RC}}.dt + K$$

$$= \frac{wE_0}{R}.\frac{e^{\frac{t}{RC}}}{\sqrt{\left(\frac{1}{RC}\right)^2 + w^2}}.\cos\left(wt - \tan^{-1}\frac{w}{1/RC}\right) + K$$

$$= \frac{wCE_0}{\sqrt{1+R^2C^2w^2}}.e^{\frac{t}{RC}}\cos(wt - \Phi) + K \text{ where } \tan\Phi = RCw$$

$$i = \frac{wCE_0}{\sqrt{1+R^2C^2w^2}}.\cos(wt - \Phi) + Ke^{-\frac{t}{RC}}$$

which gives current at any time t.

i.e. $R_i + \frac{q}{c} = E$

since $i = \frac{dq}{dt}$

this equation in terms of q can be written as

$$R\frac{dq}{dt} + \frac{q}{c} = E \text{ or } \frac{dq}{dt} + \frac{q}{Rc} = \frac{E}{R}$$

which is linear differential equation

$$\text{I.F.} = e^{\int \frac{1}{RC}dt} = e^{\frac{t}{RC}}$$

and therefore it's solution is

$$q\,e^{\frac{t}{RC}} = \int \frac{E}{R}.e^{\frac{t}{RC}}dt + B$$

$$= \frac{E}{R}\left(RCe^{\frac{t}{RC}}\right) + B$$

$$q = EC + B\,e^{-\frac{t}{RC}}$$

$$q = q_0 \text{ when } t = 0$$

$$q_0 = EC + B \text{ giving } B = q_0 - EC$$

$$q = EC + (q_0 - EC)e^{-\frac{t}{RC}}$$

$$q = EC\left(1 - e^{-\frac{t}{RC}}\right) + q_0 e^{-\frac{t}{RC}}$$

$$i = \frac{dq}{dt} = EC\frac{1}{RC}e^{-\frac{t}{RC}} - q_0 e^{-\frac{t}{RC}}$$

$$\therefore\ i = \left(\frac{E}{R} - \frac{q_0}{Rc}\right)e^{-\frac{t}{RC}}$$

Example 1:

A resistance of 100 ohms, an inductance of 0.5 henry are connected in series with a battery of 20 volts. Find the current in a circuit as a function of t.

Solution :

By Kirchoff's law, we have

$$\frac{dI}{dt} + RI = E \quad i = I \text{ read}$$

$$\frac{dI}{dt} + \frac{R}{L} I = \frac{E}{L} \text{ which is linear } \left(p = \frac{R}{L}, Q = \frac{E}{L} \right)$$

$$\text{I.F.} = e^{\int \frac{R}{L} dt} = e^{\frac{Rt}{L}} \text{ hence, the G.S. is}$$

$$Ie^{\frac{Rt}{L}} = \int \frac{E}{L}.e^{\frac{Rt}{L}}.dt + a = \frac{E}{R}.e^{\frac{Rt}{L}} + a$$

but at $t = 0, I = 0$

$$o = \frac{E}{R} + a \text{ giving } a = -\frac{E}{R}$$

$\therefore$ G.S. becomes,

$$Ie^{\frac{Rt}{L}} = \frac{E}{R}\left(-1 + e^{\frac{Rt}{L}} \right) \text{ or } I = \frac{E}{R}\left(1 - e^{\frac{-Rt}{L}} \right)$$

given R = 100 ohms L = 0.5 henry E = 20 volts

$$I = \frac{20}{100}\left(1 - e^{\frac{100t}{0.5}} \right) = \frac{1}{5}\left(1 - e^{200t} \right)$$

Example 2:

The equation of electromotive force in terms of current I for an electrical circuit having resistance R and a condenser of capacity C, in series, is $E = R_i + \int \frac{1'}{C} dt$ Find the circuit I at any time t, when $E = E_0 \sin wt$

Solution :

The given equation can be written as,

$$R_i + \int \frac{1'}{C} dt = E_0 \sin wt$$

Differentiating both sides w.r.t t,

we have,

Applications Of Differential Equations:

We present examples where differential equations are widely applied to model natural phenomena, engineering systems and many other situations.

The study of differential equations is a wide field in pure and applied mathematics, physics, and engineering. All of these disciplines are concerned with the properties of differential equations of various types. Pure mathematics focuses on the existence and uniqueness of solutions, while applied mathematics emphasizes the rigorous justification of the methods for approximating solutions. Differential equations play an important role in modelling virtually every physical, technical, or biological process, from celestial motion, to bridge design, to interactions between neurons. Differential equations such as those used to solve real-life problems may not necessarily be directly solvable, i.e. do not have closed form solutions. Instead, solutions can be approximated using numerical methods.

- **Application 1 : Exponential Growth – Population:**

Let P(t) be a quantity that increases with time t and the rate of increase is proportional to the same quantity P as follows

$dp/dt = kP$

where dp/dt is the first derivative of P, $k > 0$ and t is the time. The solution to the above first order differential equation is given by

$P(t) = A e^{kt}$

where A is a constant not equal to 0.

If P=P then $P_0 = A\,e^0$ which gives $A = P_0$

The final form of the solution is given by

$P(t) = P_0\, e^{kt}$

Assuming P_0 is positive and since k is positive, P(t) is an increasing exponential growth model.

- **Application 2: Newton's Law of Cooling**

It is a model that describes, mathematically, the change in temperature of an object in a given environment. The law states that the rate of change (in time) of the temperature is proportional to the difference between the temperature T of the object and the temperature Te of the environment surrounding the object.

$dT / dt = -k\,(T - Te)$

Let $x = T - Te$ so that $dx / dt = dT / dt$

Using the above change of variable, the above differential equation becomes

$dx / dt = -kx$

The solution to the above differential equation is given by

$x = A\,e^{-kt}$

substitute x by T - Te

$T - Te = A\,e^{-kt}$

Assume that at t = 0 the temperature T = To

$To - Te = A\,e^{0}$

which gives A = To – Te

The final expression for T(t) i given by

$T(t) = Te + (To - Te)e^{-kt}$

This last expression shows how the temperature T of the object changes with time.