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By

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Project name:- "Arithmetic Functions"

Under the Guidance of
Mr. Thorat S.P.
Head Of The Department Of Mathematics

"Education For Knowledge ,Science & culture"
-Shikshanmaharshi Dr. Bapuji Salunkhe.

Shri Swami Vivekanand Shikshan Sanstha's,

Vivekanand College, Kolhapur (Autonomous)

DEPARTMENT OF MATHEMATICS

CERTIFICATE

This is to certify that Mr./Ms./Mrs. "Prajakta Prabhakar More" has successfully completed the project work on topic "Arithmetic Functions" towards the partial fulfilment for the course of Bachelor of Science (Mathematics) work of Vivekanand College , Kolhapur(Autonomous) during the academic year 2022-2023. This report represents the bonafide work of student.

Place : Kolhapur

Date :

Mr. S. P. Thorat

Head Of Dept. Of Mathematics

HEAD
Department of Mathematics
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DECLERATION

I undersigned hereby declare that project entitled "Arithmetic Functions". Completed under the guidance of Mr.S.P. Thorat sir. (Department of Mathematics Vivekanand College (Autonomous), Kolhapur). Based on the experiment results and cited data. I declare that this is my original work which is submitted to Vivekanand College, Kolhapur in this academic year.

Mr./Ms. :- Prajakta Prabhakar More

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Place :- Kolhapur

Date :-

Mr./Ms. :- “Prajakta Prabhakar More”

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INTRODUCTION

Arithmetic Functions

An arithmetic function is an important function with many interesting properties frequently occurred in number theoretic investigations.

Defintion :

A function f : N→N , N is the set of natural numbers is called an Arithmatic function.

eg. 1] $f(n) = n$,$n \epsilon N$

2] $g(n) = n^2$,$n \epsilon N$

are Arithmatic functions.

3] $f(n) = \log n$,$n \epsilon N$ is not an Arithmatic function.

Multiplcative Arithmatic Function

Definition :

If f(n) is an arithmetic function such that f(mn) = f(m)f(n) where, gcd(m,n) = 1, then f(n) is said to be Multiplcative Arithmatic Function.

Eg. 1] If f(n) = n, n∈N then

F(mn) = mn = f(m)f(n)

'f' is multiplicative function.

2] If g(n) = 2n, n∈N then

g(mn) = 2mn

g(m)g(n) =(2m)(2n)

g(mn) ≠g(m)g(n)

'g' is not multiplicative function

Totally Multiplicative Arithmetic Function

Defintion :

If f(n) is an arithmetic function such that f(mn) = f(m)f(n) for all m,nϵN, then f(n) is said to be Totally Multiplicative Arithmatic Function.

eg.

1] If $f(n) = n^2$, nϵN then

$f(mn) = (mn)^2 = m^2 n^2 = f(m)f(n) \quad \forall\ m, n\epsilon N$

'f' is totally multiplicative arithmatic function.

Defintion :

If f(n) and g(n) are two arithmetic functions then their product and quotient are defined as follows,

$(fg)(n) = f(n)g(n)$

$\left(\frac{f}{g}\right)(n) = \frac{f(n)}{g(n)}$

Theorem : If f and g are multiplicative arimetic functions then (fg)(n) and $\left(\frac{f}{g}\right)(n)$ are also multiplicative.

Proof :

Since f and g are multiplicative

$\therefore f(mn) = f(m)f(n)\ ,\ g(mn) = g(m)g(n) \quad \forall(m,n) = 1$

Consider, $(fg)(mn) = f(mn)g(mn)$ ... by definition

$= f(m)f(n)g(m)g(n)$

$= [f(m)g(m)][f(n)g(n)]$

$= (fg)(m).\ (fg)(n) \quad \forall(m,n) = 1$

$\left(\frac{f}{g}\right)(mn) = \frac{f(mn)}{g(mn)}$

$= \frac{f(m)}{g(m)}\frac{f(n)}{g(n)}$

$= \left(\frac{f(m)}{g(m)}\right)\left(\frac{f(n)}{g(n)}\right)$

$= \left(\frac{f}{g}\right)(m)\left(\frac{f}{g}\right)(n) \qquad \forall (m,n) = 1$

Note :

If f and g are totally multiplicative arimetic functions then (fg)(n) and $\left(\frac{f}{g}\right)(n)$ are also totally multiplicative.

Theorem : If f is an arithmetic function such that f is multiplicative then f(1)= 1

Proof : Given , f is a multiplicative arithmetic function

$\therefore f(mn) = f(m)f(n) \quad ,\forall (m,n) = 1$

Consider,

$f(m) = f(m.1)$

$\therefore f(m).1=f(m).f(1)$

$\Rightarrow f(1) = 1$

Lemma : Let f and g be two arithmetic functions and m,n∊ N then

$$\sum_{(d|m)(D|n)} f(d)g(D) = \sum_{(d|m)} f(d) \times \sum_{(D|n)} g(D)$$

Proof :

$$\sum_{(d|m)(D|n)} f(d)g(D) = \sum_{j=1,2,\dots,s\ k=1,2,\dots,t} f(d_j)g(D_k)$$

$$= f(d_1)g(D_1) + f(d_1)g(D_2) + \dots + f(d_1)g(D_t) +$$
$$f(d_2)g(D_1) + f(d_2)g(D_2) + \dots + f(d_2)g(D_t) + \dots +$$
$$f(d_s)g(D_1) + f(d_s)g(D_2) + \dots + f(d_s)g(D_t)$$

$$= [f(d_1) + \dots + f(d_s)][g(D_1) + \dots + g(D_t)]$$

$$\sum_{(d|m)(D|n)} f(d)g(D) = \sum_{(d|m)} f(d) \times \sum_{(D|n)} g(D)$$

Lemma : Let $S_1 = \{d_1, d_2, \ldots, d_k\}$ is the set of positive divisors of m and

$S_2 = \{e_1, e_2, \ldots, e_l\}$ is the set if positive divisors of n. If (m,n) = 1 then the set

$S = \{d_i e_j | d_i \in S_1, \in S_2\}$ is the set of positive divisors of mn without repetition.

Proof: To prove this we have to show

(i) $d_i e_j | mn$ ∀i,j

If $d_i | m$, ∀ i ; $e_j | n$, ∀j then $d_i e_j | mn$ ∀i,j

(ii) **If $d_i e_j = d_r e_s$ then $d_i | d_r e_s$**

$d_i | d_r$ [$\because (d_i, e_s) = 1$ for (m,n) =1]

Similarly we can show that $d_r | d_i$

$\therefore d_i = d_r$

⟹ i=r

Similarly,we can show j=s

(iii) **If f|mn then f=$d_i e_j$ for some i,j**

Let f|mn

Let(f,m) = d

Then,d|m ⟹ d =d_i for some i

Again d|f gives f= d× (some integer)= d_ie (say)

If we can show that this e∈S_2

Ie. e|n then we are through.

Now (f,m) =d

$\left(\frac{f}{d}, \frac{m}{d}\right) = 1$

$(e, d_k) = 1$

And f|mn gives d_i e|mn

ie. $e | \frac{m}{d} n \Rightarrow e | d_k n$

but $(e, d_k) = 1$

$\therefore e | n$

$\therefore e = e_j$ for some $j \in \{1,2,\ldots,s\}$

$\therefore f = d_i e_j$

Theorem : If f(n) is a multiplicative arithmetic function then the function

$$\sum_{d|n} f(d) \text{ is also multiplicative.}$$

Proof : Let ,

$$F(n) = \sum_{d|n} f(d)$$

We prove that F(n) is multiplicative.

Let, $S_1 = \{d_1, d_2, \ldots, d_k\}$ is the set of positive divisors of m and

$S_2 = \{e_1, e_2, \ldots, e_l\}$ is the set if positive divisors of n. If (m,n) = 1 then the set

$S = \{d_i e_j | d_i \in S_1, \in S_2\}$ is the set of positive divisors of mn without repetition.

We are to prove F(m)F(n) = F(mn) if (m,n) = 1 if f(n) is multiplicative.

$$F(m)F(n) = \sum_{d|m} f(d) \times \sum_{d|n} f(d)$$

$$= \sum_{1 \le i \le k} f(d_i) \times \sum_{1 \le j \le l} f(e_j)$$

$$= \sum_{1 \le i \le k, 1 \le j \le l} f(d_i) f(e_j)$$

$$= \sum_{d_i \in S_1, e_j \in S_2} f(d_i) f(e_j)$$

$$= \sum_{D|mn} f(D)$$

$$= F(mn)$$

$\therefore F(mn) = F(m)F(n)$

Definition :

Let A be the set of arithmetic functions then A≠Φ

Let us define an operation * called as **convolution** on A such that

$$(f * g)(n) = \sum_{d|n} f(d)g\left(\frac{n}{d}\right)$$

The product * defined above is known as **Dirichlet product.**

Theorem : If f and g are multiplicative arithmetic functions the f*g is also multiplicative.

Proof : Given, f and g are multiplicative arithmetic functions

Let ,(m,n) =1

Let, $S_1 = \{d_1, d_2, \dots, d_k\}$ is the set of positive divisors of m and

$S_2 = \{e_1, e_2, \dots, e_l\}$ is the set if positive divisors of n. If (m,n) = 1 then the set

$S = \{d_i e_j | d_i \in S_1, \in S_2\}$ is the set of positive divisors of mn without repetition.

$$(f * g)(m) \times (f * g)(n) = \sum_{d|m} f(d)g\left(\frac{m}{d}\right) \times \sum_{e|n} f(e)g\left(\frac{n}{e}\right)$$

$$= \sum_{1 \le i \le k} f(d_i)\, g\left(\frac{m}{d_i}\right) \times \sum_{1 \le j \le l} f(e_j)\, g\left(\frac{n}{e_j}\right)$$

$$= \sum_{1 \le i \le k, 1 \le j \le l} f(d_i)\, g\left(\frac{m}{d_i}\right) f(e_j) g\left(\frac{n}{e_j}\right)$$

$$= \sum_{1 \le i \le k, 1 \le j \le l} f(d_i)\, f(e_j) g\left(\frac{m}{d_i}\right) g\left(\frac{n}{e_j}\right)$$

$$= \sum_{1 \le i \le k, 1 \le j \le l} f(d_i\, e_j) g\left(\frac{mn}{d_i e_j}\right)$$

$$= \sum_{D|mn} f(D)\, g(D)$$

$$= (f * g)(mn)$$

∴f*g is multiplicative.

Convolution

If $f, g: N \to C$ are two arithmetic functions from the positive integers to the complex numbers, the *Dirichlet convolution* $f * g$ is a new arithmetic function defined by:

$$(f * g)(n) = \sum_{d|n} f(d)g(\frac{n}{d}) = \sum_{ab=n} f(a)g(b)$$

where the sum extends over all positive divisors *d* of *n*, or equivalently over all distinct pairs (*a*, *b*) of positive integers whose product is *n*.

This product occurs naturally in the study of Dirichlet series such as the Riemann zeta function. It describes the multiplication of two Dirichlet series in terms of their coefficients:

$$(\sum_{n\geq 1} \frac{f(n)}{n^8})(\sum_{n\geq 1} \frac{g(n)}{n^8}) = (\sum_{n\geq 1} \frac{(f * g)(n)}{n^8})$$

Properties of convolution(*)

1] * is commutative. ie f*g = g*f

$$f * g(n) = \sum_{d|n} f(d)g\left(\frac{n}{d}\right)$$

$$= \sum_{d|n} f\left(\frac{n}{d}\right) g(d)$$

$$= g * f(n)$$

∴f*g = g*f

2] * is associative ie. (f*g)*h = f*(g*h)

$$((f * g) * h)(n) = \sum_{d|n} (f * g)(d)h\left(\frac{n}{d}\right)$$

$$= \sum_{de=n} (f * g)(d)h(e)$$

$$= \sum_{de=n} h(e)(f * g)(d)$$

$$= \sum_{de=n} h(e) \sum_{k|d} f(k)g\left(\frac{d}{k}\right)$$

$$= \sum_{de=n} h(e) \sum_{d=kl} f(k)g(l)$$

$$= \sum_{de=n,kl=d} h(e)f(k)g(l)$$

$$= \sum_{n=kle} h(e)f(k)g(l)$$

$$= \sum_{km=n} f(k) \sum_{le=m} g(l)\, h(e)$$

$$= \sum_{km=n} f(k) \sum_{l|m} g(l)\, h\left(\frac{m}{l}\right)$$

$$= \sum_{km=n} f(k).\, g * h(m)$$

$$= \sum_{k|n} f(k).\left[g * h\left(\frac{n}{k}\right)\right]$$

$= (f * (g * h))(n)$

$\therefore (f*g)*h = f*(g*h)$

3] Existance of identity

If 'f' is any function and e is such that,

e(n) = 1 for n=1

= 0 for $n \neq 1$ then,

$$(f * e)(n) = \sum_{d|n} f(d)e\left(\frac{n}{d}\right)$$

$$= f(1)e(n) + f(d_1)e\left(\frac{n}{d_1}\right) +...+ f(n)e(n)$$

$$= f(1).0 + f(d_1).0 +...+ f(n).1$$

$$= f(n)$$

$\therefore f*e = f$

Similarly we can show that f= e*f

∴ 'e' is identity w.r.t *

From property [1],[2],[3],

∴ A is commutative semi group under*

Definition :

If f*g = f = g*f then g is said to be inverse of f and is written as $g = f^{-1}$

Remark : For every arithmetic function its inverse may not exist.

Note: If we define the operation addition ⊕ as (f⊕g)(n) = f(n)+g(n), then A will be an integral domain under ⊕ and*.

Some Important Arithmatic Functions :

Euler's Function

An arithmetic function Φ : N→ N , defined by

Φ(n) = 1, if n = 1

= the number of a's(<n) such that (a,n) =1, if n > 1

And is called as Euler's Totient function.

Theorem : For any prime 'p', $\Phi(p) = p-1$

Proof:

Since, 'p' is prime ,each of 1,2,3,...,p-1 is relatively prime to 'p'

ie. for each integer a, $1 \leq a \leq p-1$ is such that (a,p) = 1

$\Phi(p) = p-1$

Theorem : $\Phi(n)$ is multiplicative.

ie. if (m,n) = 1, then $\Phi(mn) = \Phi(m)\,\Phi(n)$s

Proof :

Given that, (m,n) = 1

We consider the product mn. Then mn numbers can be arranged in n lines, each containing m numbers.

Thus,

$$\begin{array}{cccccc}
1, & 2, & \ldots, & k, & \ldots, & m \\
m+1, & m+2, & \ldots, & m+k, & \ldots, & m+m \\
2m+1, & 2m+2, & \ldots, & 2m+k, & \ldots, & 2m+m \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
(n-1)m+1, & (n-1)m+2, & \ldots, & (n-1)m+k & \ldots, & (n-1)m+m
\end{array}$$

We consider the vertical column beginning with k.

If (k,m) = 1 , all the terms of this column will be prime to m.

But if k and m have a common divisor, no number in the column will be prime to m.

Now the first row contains $\Phi(m)$ numbers prime to m.

$\therefore$ $\Phi(m)$ vertical columns in each of which every term is prime to m.

Let us suppose that the vertical column which begins with k is one of these.

This column is in arithmetic progression, the terms of which when divided by n leaves remainders 0,1,2,3,4, ... ,n-2,n-1

Hence, the columns contains $\Phi(n)$ integers prime to n.

Thus in the table there are $\Phi(m).\Phi(n)$ integers, which are prime to m and n also and therefore to mn

ie. $$\Phi(mn) = \Phi(m)\,\Phi(n)$$

$\therefore \Phi(n)$ is multiplicative

Expression for $\Phi(p^\alpha)$

The numbers from 1 to p^α are as follows :

1,	2,	...,	p,	...,	2p,	...,	pp,
p^2+1,	p^2+2,	...,	$2p^2$,	...,	$3p^2$,	...,	$p^2.p = p^3$
p^3+1,	p^3+2,	...,	$2p^3$,	...,	$3p^3$,	...,	$p^3.p = p^4$
:	:	:	:	:	:	:	:
:	:	:	:	:	:	:	:
$p^{\alpha-1}+1$,	$p^{\alpha-1}+2$,	...,	$2p^{\alpha-1}$,	...,	$3p^{\alpha-1}$,	...,	$p^{\alpha-1}.p = p^\alpha$

In each row there are p number of a's such that $(a,p^\alpha)\neq 1$

$\therefore$there are in total $p.p^{\alpha-1}$ numbers such that $(a,p^\alpha)\neq 1$

$\therefore \Phi(p^\alpha) = p^\alpha -$ (the number of a's such that $(a,p^\alpha)\neq 1$)

$$= p^\alpha - p^{\alpha-1}$$

$$= p^\alpha\left(1-\frac{1}{p}\right)$$

Now if $m = p_1^{\alpha_1}p_2^{\alpha_2}\dots p_n^{\alpha_n}$ then,

$\Phi(m) = \Phi(p_1^{\alpha_1}p_2^{\alpha_2}\dots p_n^{\alpha_n})$

$= \Phi(p_1^{\alpha_1})\,\Phi(p_2^{\alpha_2})\dots\Phi(p_n^{\alpha_n})$ ...since Φ is multiplicative

$= p_1^{\alpha_1}\left(1-\frac{1}{p_1}\right)p_2^{\alpha_2}\left(1-\frac{1}{p_2}\right)\dots p_n^{\alpha_n}\left(1-\frac{1}{p_n}\right)$

$= n\prod_{i=1}^{n}\left(1-\frac{1}{p_i}\right)$ where p_i's are distinct prime factors of m

Theorem : Prove that $\Phi(ab) = \Phi(a)\Phi(b)\frac{(a,b)}{\emptyset((a,b))}$

Proof :

Suppose, $(a,b) = d$

If $a = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_n^{\alpha_n}$, $b = q_1^{\beta_1} q_2^{\beta_2} \dots q_n^{\beta_n}$ then

$\Phi(ab) = \Phi(p_1^{\alpha_1} p_2^{\alpha_2} \dots p_n^{\alpha_n} q_1^{\beta_1} q_2^{\beta_2} \dots q_n^{\beta_n})$

$$= ab \prod_{(p|ab)} \left(1 - \frac{1}{p}\right), \text{p's are distinct prime factors of ab}$$

$$\frac{\Phi(ab)}{ab} = \prod_{(p|ab)} \left(1 - \frac{1}{p}\right)$$

$$= \frac{\prod_{(p|a)}\left(1-\frac{1}{p}\right) \prod_{(p|b)}\left(1-\frac{1}{p}\right)}{\prod_{(p|d)}\left(1-\frac{1}{p}\right)}$$

$$= \frac{\frac{\Phi(a)}{a} \frac{\Phi(b)}{b}}{\frac{\Phi(d)}{d}}$$

$$= \frac{1}{ab} \Phi(a)\, \Phi(b) \frac{d}{\Phi(d)}$$

$$= \Phi(a)\Phi(b)\frac{(a,b)}{\emptyset((a,b))}$$

Examples :

1] What are the positive integers a,b that satisfy the expression

$\Phi(ab) = \Phi(a)+\Phi(b)$?

Solution :

We know,

$\Phi(ab) = \Phi(a)\Phi(b)\,\frac{d}{\emptyset(d)}$ where $d = (a,b)$

Given, $\Phi(ab) = \Phi(a)+\Phi(b)$

$\therefore \Phi(a)+\Phi(b) = \Phi(a)\Phi(b)\frac{d}{\emptyset(d)}$

$$\therefore \frac{\Phi(a)+\Phi(b)}{\Phi(a)\Phi(b)} = \frac{d}{\emptyset(d)}$$

$$\therefore \frac{1}{\Phi(a)} + \frac{1}{\Phi(b)} = \frac{d}{\emptyset(d)}$$

$$\therefore \frac{\Phi(d)}{\Phi(a)} + \frac{\Phi(d)}{\Phi(b)} = d$$

$\therefore \frac{1}{m} + \frac{1}{n} = d$, where $m = \frac{\Phi(a)}{\Phi(d)}$, $n = \frac{\Phi(b)}{\Phi(d)}$

m,n and d are positive integers.

Only possible values of m,n and d are :

d = 2,m = n = 1 ...(1)

d = 1, m = n = 2 ...(2)

for case (1),

Φ(a) = Φ(b) = 2 then , a = b = 2

for case (2),

Φ(a) = Φ(b) = 2 then , one of a,b is 3 and the other is 4.

Thus, the possible values are (2,2),(3,4) and (4,3) .

2] Find Φ(225)

Solution :

$\Phi(225) = \Phi(5^2.3^2)$

$= 225 \times \left(1 - \frac{1}{5}\right) \times \left(1 - \frac{1}{3}\right)$

$= 225 \times \frac{4}{5} \times \frac{2}{3}$

= 120

Divisor Function

An arithmetic function d : N ⟶ N defined by,

d(n) = numberof divisors of n , n ϵ N

is called as divisor function and is denoted by d(n) or τ(n).

We note that,

d(n) = 1 , if n = 1

= 2 ,if n is a prime number

>2, if n is composite

Expression for d(n)

Since, 1, p , p^2,p^3, … , p^α are the divisors of p^α,where p is prime, then,

d(p^α) = α + 1

d($p^\alpha q^\beta$) =(α + 1) (β + 1)

if n = $p^\alpha q^\beta$,then divisors of n are,

1	p	p^2	…	p^α
q	pq	p^2q	…	$p^\alpha q$
q^2	pq^2	p^2q^3	…	$p^\alpha q^2$

:	:	:	:	:
q^{β}	pq^{β}	p^2q^{β}	...	$p^{\alpha}q^{\beta}$

Therefore the number of divisors is $(\alpha + 1)(\beta + 1)$

Thus, $d(n) = (\alpha + 1)(\beta + 1)$

Similarly if, $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_n^{\alpha_n}$ then,

$d(n) = d(p_1^{\alpha_1} p_2^{\alpha_2} \dots p_n^{\alpha_n})$ ...$(p_i, p_j) = 1, i \neq j$

$= (\alpha_1 + 1)(\alpha_2 + 1) \dots (\alpha_n + 1)$

eg. 1] $d(4) = d(2^2) = 2 + 1 = 3$

2] $d(4320) = d(2^5 . 3^3 5^1)$

$=(5+1)(3+1)(1+1)$

$= (6)(4)(2) = 48$

Theorem : d is multiplicative. ie. $d(mn) = d(m)d(n)$ if $(m,n) = 1$

Proof : Let, $m = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_i^{\alpha_i}$, p_i's are distinct primes and α_i's are positive integers.

$\therefore d(m) = (\alpha_1 + 1)(\alpha_2 + 1) \dots (\alpha_i + 1)$ ...(1)

$n = q_1^{\beta_1} q_2^{\beta_2} \dots q_j^{\beta_j}$, q_j's are distinct primes and β_j's are positive integers

$\therefore d(n) = (\beta_1 + 1)(\beta_2 + 1) \dots (\beta_j + 1)$ ...(2)

$d(mn) = d(p_1^{\alpha_1} p_2^{\alpha_2} \dots p_i^{\alpha_i} q_1^{\beta_1} q_2^{\beta_2} \dots q_j^{\beta_j})$

$= (\alpha_1 + 1)(\alpha_2 + 1) \dots (\alpha_i + 1)\ (\beta_1 + 1)(\beta_2 + 1) \dots (\beta_j + 1)$

$= [(\alpha_1 + 1)(\alpha_2 + 1) \dots (\alpha_i + 1)][(\beta_1 + 1)(\beta_2 + 1) \dots (\beta_j + 1)]$

= d(m)d(n)   from eqn.(1) and (2)

∴d is multiplicative.

Note : d is not totally multiplicative.

$d(8) = d(2^3) = 4$

$(4,2) = 2 \neq 1$

d(4) = 3

d(2) = 2

$\therefore d(8) \neq d(4)d(2)$

Examples :

1] Show that $\Phi(m) \geq \frac{m}{d(m)}$

Solution : Suppose, $m = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_i^{\alpha_i}$, then ,

$$\Phi(m)d(m) = m\left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right) \dots \left(1 - \frac{1}{p_i}\right)(\alpha_1 + 1)(\alpha_2 + 1) \dots (\alpha_i + 1)$$

$$\geq m\left(\frac{1}{2}\right)^i 2^i$$

$$\geq m$$

This gives, $\Phi(m) \geq \frac{m}{d(m)}$

2] Prove that d(n) is odd if and only if n is a perfect sqare.

Solution :

Suppose n = $p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_i^{\alpha_i}$, then

d(n) = d($p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_i^{\alpha_i}$)

$= (\alpha_1 + 1)(\alpha_2 + 1) \ldots (\alpha_i + 1)$

Now, d(n) is odd if and only if $(\alpha_i + 1)$ is odd for each i.

ie. if and only if α_i is even for each i

ie. if and only if $\alpha_i = 2k_i$ for each i

Thus, n = $p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_i^{\alpha_i}$

$= p_1^{2k_1} p_2^{2k_2} \ldots p_i^{2k_i}$

$= \left(p_1^{k_1} p_2^{k_2} \ldots p_i^{k_i}\right)^2$

$= m^2$ where $m = p_1^{k_1} p_2^{k_2} \ldots p_i^{k_i}$

Hence d(n) is odd if and only if n is perfect square.

3] Find the smallest positive integer having only 10 positive integers.

Solution :

Suppose, $n = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_i^{\alpha_i}$

$d(n) = d(p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_i^{\alpha_i}) = 10$ ... (given)

$\therefore d(n) = (1)(10)$ or $d(n) = (2)(5)$

case(1) :

$(\alpha_1 + 1)(\alpha_2 + 1) = (1)(10)$

$\Rightarrow (\alpha_1 + 1) = 1$

$\Rightarrow \alpha_1 = 0$

$\Rightarrow (\alpha_2 + 1) = 10$

$\Rightarrow \alpha_2 = 9$

$n = q^9$ Since 2 is smallest prime number

$\therefore n = 2^9$

Case (2) :

$(\alpha_1 + 1)(\alpha_2 + 1) = (2)(5)$

$\Rightarrow (\alpha_1 + 1) = 2$

$\Rightarrow \alpha_1 = 1$

$\Rightarrow (\alpha_2 + 1) = 5$

$\Rightarrow \alpha_2 = 4$

$n = pq^4$ Since 2,3 are first two smaller prime numbers

$\therefore n = (3)(2^4)$

$= 489$

$\therefore$from, case (1) and case (2),

$n = 489$

REFERENCES

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