A Project Submitted to

Vivekanand College, Kolhapur (Empowered Autonomous)

KOLHAPUR

Affiliated to

Shivaji University, Kolhapur

For the Degree of Bachelor of Science

In

Mathematics

By

Name: Sandesh Daji Jadhav

Roll No: 8292

Exam Seat No:600392

B.Sc. III (Mathematics)

Year: 2023-24

Under the Guidance of

Prof. S.P. Thorat

(Head and Associate Professor , Department of Mathematics)

Vivekanand College (Empowered Autonomous),Kolhapur

Dissemination Of Education For Knowledge ,Science & culture"
-Shikshanmaharshi Dr. Bapuji Salunkhe

Shri. Swami Vivekanand Shikshan Sanstha, Kolhapur

Vivekanand College, Kolhapur (Empowered Autonomous)

DEPARTMENT OF MATHEMATICS

Certificate

This is to certify that Mr. Sandesh Daji Jadhav has successfully completed the project work on topic "Application of GRAPH THEORY" towards the partial fulfilment for the course of Bachelor of Science (Mathematics) work of Vivekanand College, Kolhapur (Empowered Autonomous) affiliated to Shivaji University, Kolhapur during the academic year 2023-2024.

Place: Kolhapur

Date: 23|03|24

Examiner

Mr. S.P. Thorat

Head Dept. of Mathematics

HEAD
DEPARTMENT OF MATHEMATICS
VIVEKANAND COLLEGE, KOLHAPUR
(EMPOWERED AUTONOMOUS)

DECLARATION

I undersigned hereby declare that project entitled " Application of GRAPH THEORY " completed under the guidance of(Name of teacher)based on the experiment results and cited data. I declare that this is my original work which is submitted to Vivekanand College, Kolhapur in 2023-2024 academic year.

Name: Sandesh Daji Jadhav.
Sign :

ACKNOWLEDGEMENT

On the day of completion of this project, the numerous memories agreeing rushed in my mind with full of gratitude to this encouraged and helped me a lot at various stages of this work.

I offer sincere gratitude to all of them. I have great pleasure to express my deep sense of indebtedness and heart of full gratitude to my project guide Mr. S.P. Thorat sir for his expert and valuable guidance and continuous encouragement given to me during the course of project work.

I am thankful to prin. Dr. R.R. Kumbhar sir (Principal, Vivekanand College) and Mr.S.P.Thorat sir (H.O.D Dept. of Mathematics) for allowing me to carry out our project work and extending me all the possible infra-structural facilities of department.

I would like to thank all my teachers Mr. G. B. Kolhe sir, Ms. P. P. Kulkarni mam, Ms. S.M. Malavi for co-operation, help and maintaining cheerful environment during my project.
I would also like to thanks non-teaching staff Mr. D.J. Birnale sir and Mr. S.S Suryavanshi Sir.

I would like to thanks my entire dear friends for their constant encouragement and co-operation .1 am indebted to my parents who shaped me to this status with their blunt less vision and selfless agenda.

Place: Kolhapur
Date: 23/03/24
Name: Sandesh Daji Jadhav

CONTENTS

INTRODUCTION:

In Mathematics, **graph theory** is the study of *Graphs,* which are mathematical structure used to model pairwise relations between objects. A graph in this context is made up of *vertices* (also called *nodes* or *points*) which are connected by *edges.* A distinction is made between **undirected graphs,** where edges link two vertices symmetrically, and **directed graphs**, where edges link two vertices asymmetrically. Graphs are one of the principal objects of study in discrete Mathematics

History

Leonhard Euler (1707-1783) is considered to be the most prolific mathematician in history. Euler discovered his talents in mathematics while attending the University of Basel. By 1726, the 19-year-old Euler had finished his work at Basel and published his first paper in mathematics. Euler worked in almost every branch of mathematics, both pure and applied, and his ability to perform complicated mathematical calculations in his head was legendary.

Euler decided to analyze the problem of the Königsberg bridges. The first step was to transform the actual diagram of the city and its bridges into a *graph*. In this case, a graph must have *vertices* and *edges*. Furthermore, a graph must have a rule that tells how the edges join the various vertices. In the Königsberg Bridge Problem, the vertices represent the landmasses connected by the bridges, and the bridges themselves are represented by the edges of the graph.

Finally, a *path* is a sequence of edges and vertices, just as the path taken by the people in Königsberg is a sequence of bridges and landmasses.

More than one century after Euler's paper on the bridges of Konigsberg and while Johaan /benedict listing was introducing the concept of topology.

Chapter 1: Terminology

1.1.Nodes or Vertex:

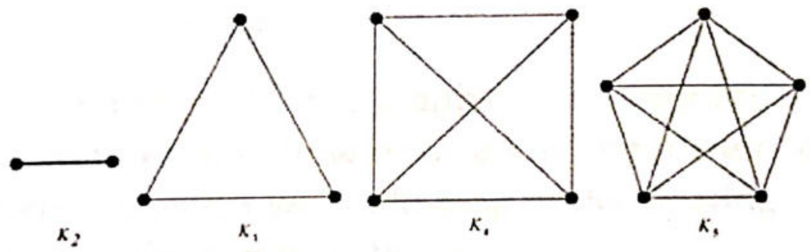
A point in a network or diagram at which lines or pathways intersect or branch. A graph in this context is made up of vertices (also called nodes or points) which are connected by edges (also called links or lines).

1.2.Edges:

The nodes are connected with each other via edges.

1.3.Simple Graph:

In graph theory, a simple graph is a type of undirected graph that has no self-loops or multiple edges between the same pair of vertices.



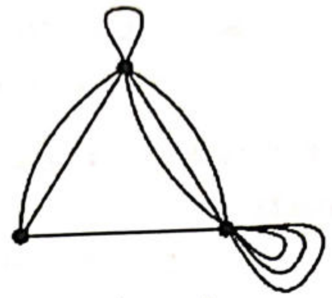
1.4. multigraph :

A multigraph is a generalization that allows multiple edges to have the same pair of endpoints.

Thus two vertices may be connected by more than one edge. In multigraph no loops are allowed.

1.5. Pseudograph:

A pseudo graph is a graph G with a self-loop and numerous edges. A pseudograph is a graph in which loops (edges that connect a vertex to itself) and multiple edges (more than one edge connecting two vertices) can exist.

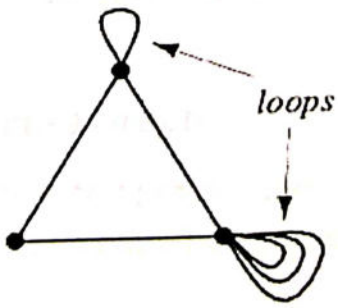


pseudograph

1.6. Loop:

In graph theory, a **loop** (also called a **self-loop** or a *buckle*) is an edge that connects a vertex to itself. A simple graph contains no loops.

Depending on the context, a graph or a multigraph may be defined so as to either allow or disallow the presence of loops (often in concert with allowing or disallowing multiple edges between the same vertices).



1.7. Directed graph:

A directed graph is defined as a type of graph where the edges have a direction associated with them.

Applications of Directed Graph

Directed graphs have many applications across a wide range of fields. Here are some examples:

- Social networks: Social networks are often modeled as directed graphs, where each person is a vertex and relationships such as friendships or following are represented as edges.
- Transportation networks: Transportation systems such as roads, airports, or subway systems can be modeled as directed graphs, with vertices representing locations and edges representing connections between them.
- Computer networks: Computer networks such as the internet can be represented as directed graphs, with vertices representing devices such as computers or routers and edges representing connections between them.
- Project management: Project management can be model as a directed graph, with vertices representing tasks and edges representing dependencies between them.

Disadvantages of Directed Graph

May be more complex: Directed graphs can be more complex than undirected graphs, since each edge has a direction associated with it.

May require more processing power: Analyzing directed graphs may require more processing power than analyzing undirected graphs, since the directionality of the edges must be taken into account.

1.8. Un-directed graph:

An undirected graph is a type of graph where the edges have no specified direction assigned to the them.

Applications of Undirected Graph:

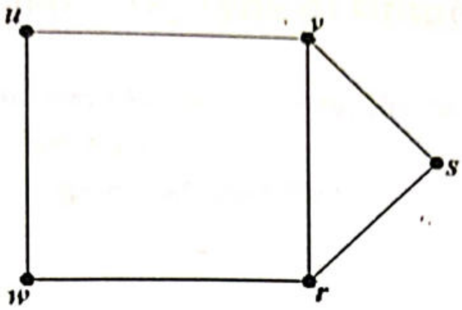
Social Networks: Undirected graphs are used to model social networks where people are represented by nodes and the connections between them are represented by edges.

Traffic flow optimization: Undirected graphs are used in traffic flow optimization to model the flow of vehicles on road networks. The vertices of the graph represent intersections or road segments, and the edges represent the connections between them. The graph can be used to optimize traffic flow and plan transportation infrastructure.

Website analysis: Undirected graphs can be used to analyze the links between web pages on the internet. Each web page is represented by a vertex, and each link between web pages is represented by an edge.

1.9. Degree of vertex:

In graph theory, the degree of a vertex is defined as the count of the number of connections of edges with that vertex



e.g. Degree of vertex u= 2, degree of vertex v=3.

1.10. Isolated and pendent vertices:

A vertex with degree zero is called an **isolated vertex.**

A **pendant vertex** can also be found to be described as an **end vertex**.

In the context of trees, a **pendant vertex** is usually known as a **terminal node**, a leaf node or just **leaf**.

Chapter2: Type of Graph:

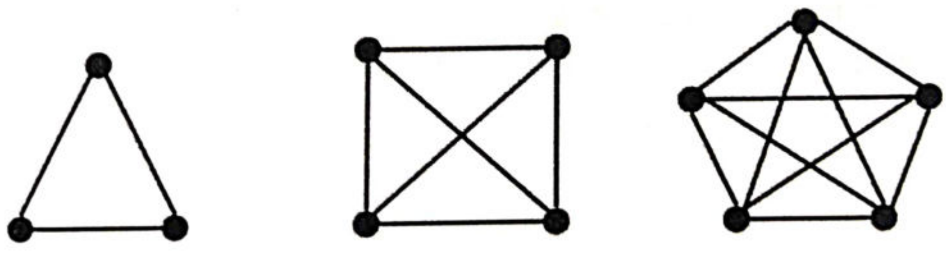
2.1. Null graph:

In the mathematical field of graph theory, the term "**null graph**" may refer either to the order zero graph.

Alternatively, to any edgeless graph (the latter is sometimes called an "empty graph").

2.2. Complete graph:

A complete graph is an undirected graph in which every pair of distinct vertices is connected by a unique edge. In other words, every vertex in a complete graph is adjacent to all other vertices. A complete graph is denoted by the symbol K_ n, where n is the number of vertices in the graph.



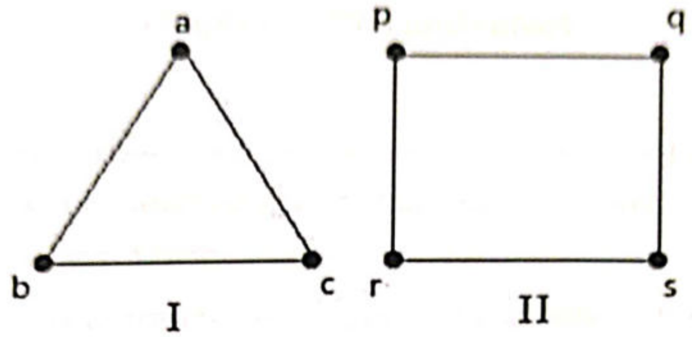
K_3 K_4 K_5

2.3. Regular graph:

A graph is called regular graph if degree of each vertex is equal. A graph is called K regular if degree of each vertex in the graph is K.

Example:

Consider the graph below:



2.4. Cycle:

In graph theory, a cycle in a graph is a non-empty trail in which only the first and last vertices are equal. A directed cycle in a directed graph is a non-empty directed trail in which only the first and last vertices are equal.

A graph without cycles is called an acyclic graph. A directed graph without directed cycles is called a directed acyclic graph. A connected graph without cycles is called a tree.

2.5. Platonic graph:

a Platonic graph is a graph that has one of the Platonic solids as its skeleton. There are 5 Platonic graphs, and all of them are regular, polyhedral:

Tetrahedral graph – 4 vertices, 6 edges

Octahedral graph – 6 vertices, 12 edges

Cubical graph – 8 vertices, 12 edges

Icosahedral graph – 12 vertices, 30 edges

Dodecahedral graph – 20 vertices, 30 edges

Chapter 3: Tree and Forest

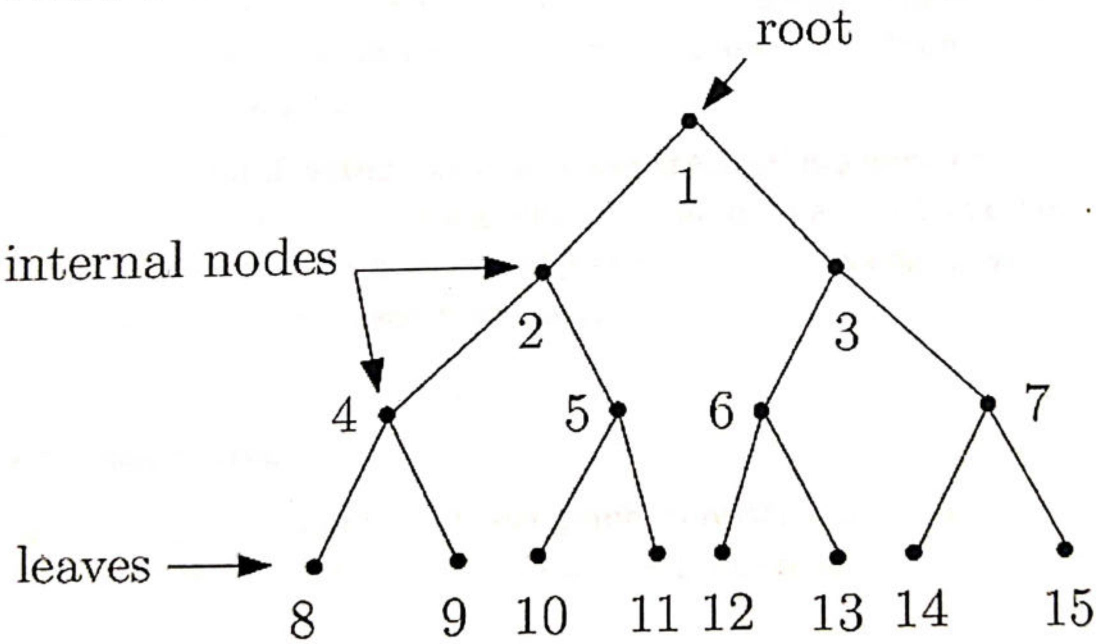
3.1. Tree:

In graph theory, a tree is an undirected, connected and acyclic graph. In other words, a connected graph that does not contain even a single cycle is called a tree.

A tree represents hierarchical structure in a graphical form. The elements of trees are called their nodes and the edges of the tree are called branches. A tree with n vertices has (n-1) edges.

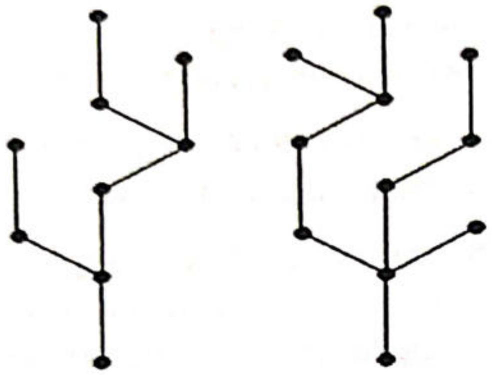
Trees provide many useful applications as simple as a family tree to as complex as trees in data structures of computer science.

A leaf in a tree is a vertex of degree 1 or any vertex having no children is called a leaf.



3.2. Forest:

In graph theory, a forest is an undirected, disconnected, acyclic graph. In other words, a disjoint collection of trees is known as forest. Each component of a forest is tree.



3.3. Poly tree and Poly forest:

Polytree (also called directed tree, oriented tree or singly connected network) is a directed acyclic graph whose underlying undirected graph is a tree. In other words, if we replace its directed edges with undirected edges, we obtain an undirected graph that is both connected and acyclic.

A Poly forest (or directed forest or oriented forest) is a directed acyclic graph whose underlying undirected graph is a forest. In other words, if we replace its directed edges with undirected edges, we obtain an undirected graph that is acyclic.

3.4. Types of Trees:

Path graph: A path graph (or linear graph) consists of n vertices arranged in a line, so that vertices i and i + 1 are connected by an edge for i = 1, ..., n – 1.

Starlike tree: starlike tree consists of a central vertex called root and several path graphs attached to it. More formally, a tree is starlike if it has exactly one vertex of degree greater than 2.

Star tree: A star tree is a tree which consists of a single internal vertex (and n – 1 leaves). In other words, a star tree of order n is a tree of order n with as many leaves as possible.

Caterpillar tree: A caterpillar tree is a tree in which all vertices are within distance 1 of a central path subgraph.

Lobster tree: A lobster tree is a tree in which all vertices are within distance 2 of a central path subgraph.

Regular tree: A regular tree of degree d is the infinite tree with d edges at each vertex. These arise as the Cayley graphs of free, and in the theory of Tits buildings.

Decision tree: A decision tree is a decision support hierarchical model that uses a tree-like model of decisions and their possible consequences, including chance event outcomes, resource costs, and utility. It is one way to display an algorithm that only contains conditional control statements.

Chapter 4: Graph isomorphism and Graph operations

4.1. Graph isomorphism:

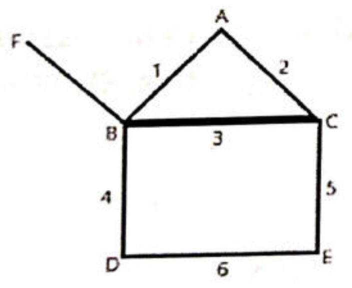
The isomorphism graph can be described as a graph in which a single graph can have more than one form. That means two different graphs can have the same number of edges, vertices, and same edges connectivity. These types of graphs are known as isomorphism graphs.

4.2. Complex graph operations:

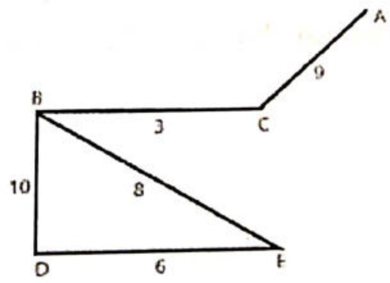
In the mathematical field of graph theory, graph operations are operations which produce new graphs from initial ones. They include both unary (one input) and binary (two input) operations.

4.2.1. Union:

Let G1(V1, E1) and G2(V2, E2) be two graphs as shown below in the diagram. The union of G1 and G2 is a graph G=G1∪G2, where vertex set V=V1 ∪ V2 and edge set E= E1 ∪ E2.



G1(V1,E1)

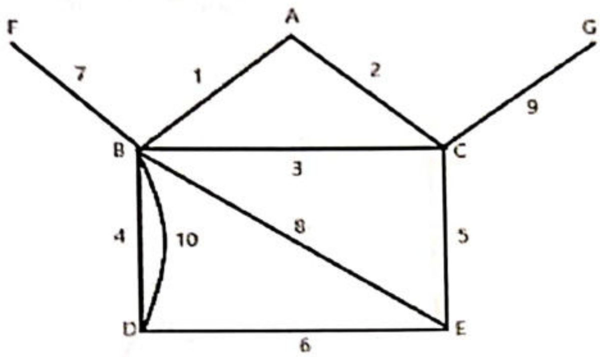


G2(V2,E2)

For the above two graphs G1 and G2, we have vertices and edges as V1= { A, B, C, D, E, F } and E1= { 1,2,3,4,5,6,7 } and V2= {B, C, D, E, G} and E2= {3,6,8,9,10} respectively.

So, in order to find the union of graphs G1 and G2, which can be denoted as G= G1∪G2. The vertex set of graph G will be V=V1∪V2= {A, B, C, D, E, F, G} and the edge set of graph G will be E=E1∪E2 = {1,2,3,4,5,6,7,8,9,10}.

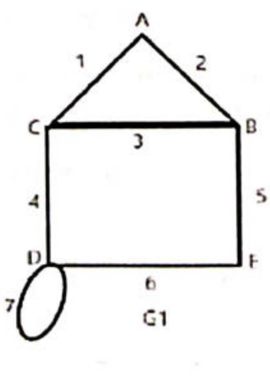
The resultant union graph G with all the vertices of set V and edges of set E will be as shown:



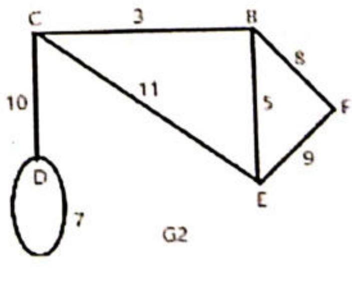
G(V,E)

4.2.2. Intersection:

Let G1(V1,E1) and G2(V2,E2) be two graphs. Then the intersection of G1 and G2 is a graph G=G1∩G2, whose vertex set V=V1∩V2 and edge set E=E1∩E2.



G1

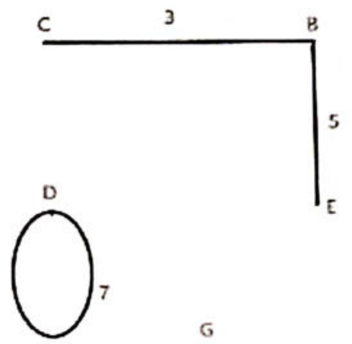


G2

For the above two graphs G1 and G2, we have vertices and edges as V1= { A, B, C, D, E } and E1= { 1,2,4,5,6,7 } and V2= {B, C, D, E, F} and E2= {3,5,7,8,9,10,11} respectively.

So, in order to find the intersection of graphs G1 and G2, which can be denoted as G= G1∩G2. The vertex set of graph G will be V=V1∩V2= {B, C, D, E} and the edge set of graph G will be E=E1∩E2 ={3,5,7}.

The resultant intersection graph G with all the vertices of set V and edges of set E will be as shown:



4.2.3. Sum of two Graphs :

The graph sum of graphs G and H is the graph with adjacency matrix given by the sum of adjacency matrices of G and H. A graph sum is defined when the orders of G and H are the same. The example illustrated above shows the graph sum K_5+C_5 of the pentatope graph K_5 and the cycle graph C_5, corresponding to adjacency matrices

4.2.4. Ring Sum of Two Graphs:

The ring sum of two graphs, denoted as G1 ⊕ G2, is a new graph formed by combining the vertices and edges of G1 and G2. Here's how it works:

The vertex set of the ring sum graph, G, is the union of the vertex sets of G1 and G2: V(G) = V(G1) ∪ V(G2).

The edge set of G consists of edges that are either in G1 or G2, but not both: E(G) = E(G1) ∪ E(G2) – E(G1) ∩ E(G2).

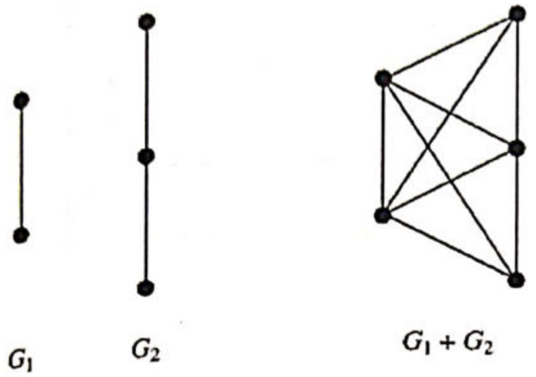
In simpler terms, an edge is included in G if and only if it is an edge of G1 or an edge of G2, but not both12. This operation allows us to create a new graph by combining the structures of the original graphs while avoiding duplicate edges.

Remember, the ring sum is a mathematical concept used in graph theory, and it provides a way to construct new graphs from existing ones.

4.2.4. Graph join:

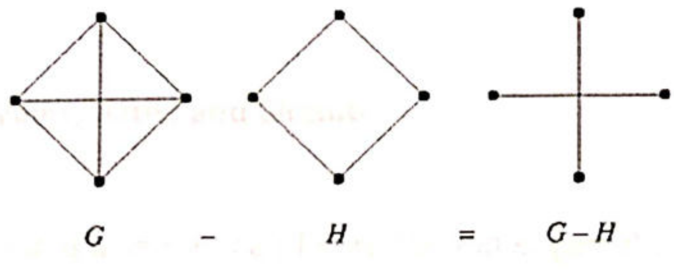
The join G=G_1+G_2 of graphs G_1 and G_2 with disjoint point sets V_1 and V_2 and edge sets X_1 and X_2 is the graph union G_1 union G_2 together with all the edges joining V_1 and V_2 (Harary 1994, p. 21). Graph joins are implemented in the Wolfram Language as Graph Join [G1, G2].

A complete k-partite graph K (i, j, ...) is the graph join of empty graphs on i, j, ... nodes. A wheel graph is the join of a cycle graph and the singleton graph. Finally, a star graph is the join of an empty graph and the singleton graph . The following table gives examples of some graph joins. Here K^__n denotes an empty graph (i.e., the graph complement of the complete graph K_n), C_n a cycle graph, and K_1 the singleton graph.



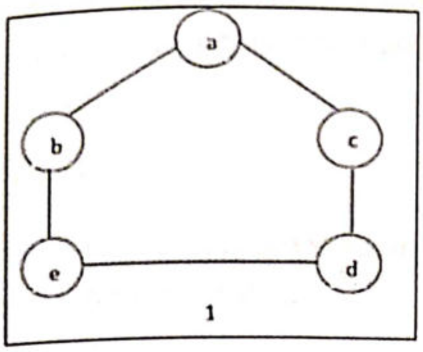
4.2.5. Difference of Graphs:

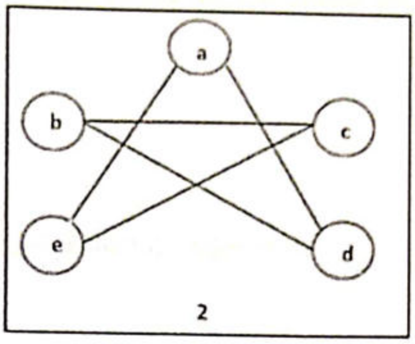
The graph difference of graphs G and H is the graph with adjacency matrix given by the difference of adjacency matrices of G and H. A graph difference is defined when the orders of G and H are the same, and can be computed in the Wolfram Language using Graph Difference.



4.2.6. Graph compliment:

The complement of a graph G is a graph G’ on the same set of vertices as of G such that there will be an edge between two vertices (v, e) in G’, if and only if there is no edge in between (v, e) in G. Complement of graph G(v, e) is denoted by G'(v, e’).





Chapter 5: Walks, paths and Circuits

5.1. Walk:

A walk is a sequence of vertices and edges of a graph i.e. if we traverse a graph then we get a walk. Edge and Vertices both can be repeated.

Open walk- A walk is said to be an open walk if the starting and ending vertices are different i.e. the origin vertex and terminal vertex are different.

5.2. Closed walk:

A walk is said to be a closed walk if the starting and ending vertices are identical i.e. if a walk starts and ends at the same vertex, then it is said to be a closed walk.

5.3. Trail:

Trail is an open walk in which no edge is repeated. Vertex can be repeated.

5.4. Circuit:

A circuit is a sequence of adjacent nodes starting and ending at the same node. Circuits never repeat edges. However, they allow repetitions of nodes in the sequence. Circuit is a closed trail.

There are two particular categories of circuits with specific characteristics:

Eulerian: this circuit consists of a closed path that visits every edge of a graph exactly once

Hamiltonian: this circuit is a closed path that visits every node of a graph exactly once.

5.5. Path:

It is a trail in which neither vertices nor edges are repeated i.e. if we traverse a graph such that we do not repeat a vertex and nor we repeat an edge. As path is also a trail, thus it is also an open walk. Another definition for path is a walk with no repeated vertex. This directly implies that no edges will ever be repeated and hence is redundant to write in the definition of path. Vertex not repeated
Edge not repeated

5.6. Connected graph:

A connected graph is one in which there is a path connecting any two points in the graph, or one that is connected in the sense of a topological space.

5.7. Component:

The components of any graph partition its vertices into disjoint sets, and are the induced subgraphs of those sets. A graph that is itself connected has exactly one component, consisting of the whole graph. Components are sometimes called connected components.

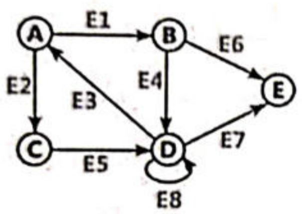
Chapter 6: Representation of Graphs as matrix

6.1. Adjacency matrix:

In graph theory, an adjacency matrix is nothing but a square matrix utilised to describe a finite graph. The components of the matrix express whether the pairs of a finite set of vertices (also called nodes) are adjacent in the graph or not. In graph representation, the networks are expressed with the help of nodes and edges, where nodes are the vertices and edges are the finite set of ordered pairs.

6.2. Incidence matrix:

Incidence matrix is a common graph representation in graph theory. It is different to an adjacency matrix, which encodes the relation of vertex-vertex pairs.



	E1	E2	E3	E4	E5	E6	E7	E8
A	1	1	-1	0	0	0	0	0
B	-1	0	0	1	0	1	0	0
C	0	-1	0	0	1	0	0	0
D	0	0	1	-1	-1	0	1	1
E	0	0	0	0	0	-1	-1	0

Chapter 7: Graph in circuits

7.1. Loop matrix:

A loop matrix or circuit matrix is represented by B_a. For a graph with n nodes and b branches, loop matrix B_a is a rectangular matrix with b columns (equal to number of branches) and as many as rows as there are loops.

Circuits	Branches					
	1	2	3	4	5	6
1	1	1	1	0	0	0
2	0	0	− 1	1	1	0
3	0	− 1	0	0	0	1
4	1	1	0	1	1	0
5	1	0	1	0	0	1

This circuit matrix B_a is given by,

$$B_a = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 1 & 0 \\ 1 & -1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

7.2. Cut set matrix:

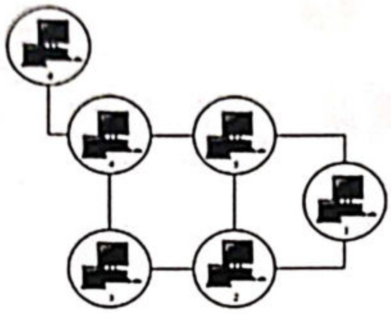
A connected graph can be separated into two parts by removing certain branches of the graph. This is equivalent to cutting a graph into two parts hence it is refered as Cut Set Matrix in Network Analysis.

Chapter 8: Application of in Computer Science

8.1. Networks:

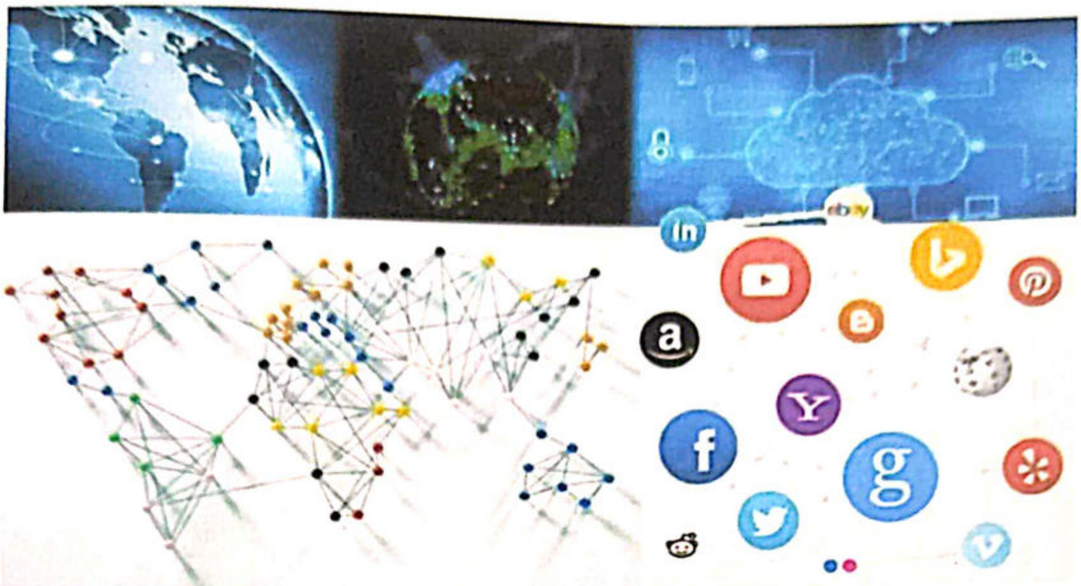
Graph theory can be used in computer networks, for security purpose or to schematize network topologies. Graphs are used to define the flow of computation.

Graphs are used to represent networks of communication. Graphs are used to represent data organization. Graph transformation systems work on rule-based in-memory manipulation of graphs. Graph databases ensure transaction-safe, persistent storing and querying of graph structured data. Graph theory is used to find shortest path in road or a network. In Google Maps, various locations are represented as vertices or nodes and the roads are represented as edges and graph theory is used to find the shortest path between two nodes.



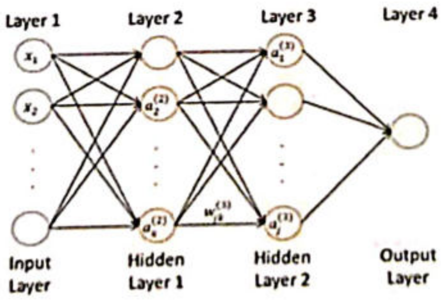
8.2. Webpage:

Webpage can represented by a direct graph. The vertices are the web pages available at website a directed edge from page A to page B exists if and only if A contains a link to B.



8.3. Neural Networks:

A graph neural network (GNN) belongs to a class of artificial neural networks for processing data that can be represented as graphs. The series of algorithms that attempt to identify underlying relationships in set of data by using process that mimic the way the human brain operates.

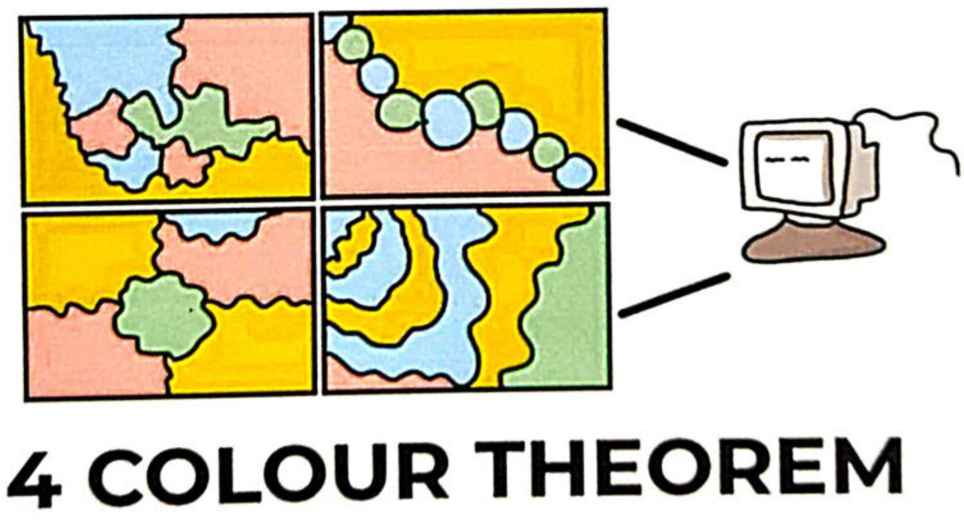


Chapter 9: Application of graph theory in Security

9.1. The Four colour theorem:

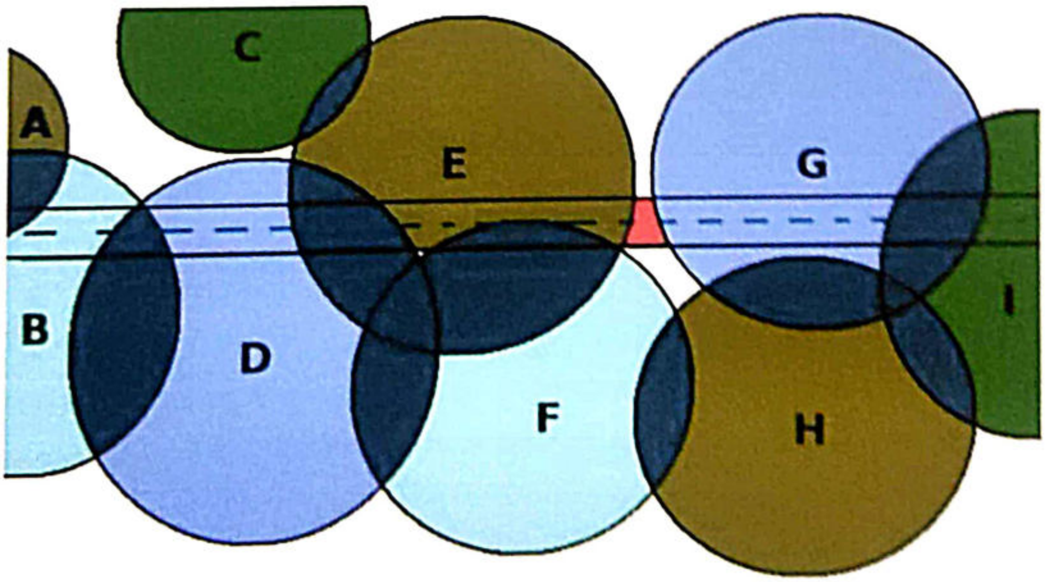
In mathematics, the four colour theorem, or the four colour map theorem, states that no more than four colours are required to colour the regions of any map so that no two adjacent regions have the same colour. Adjacent means that two regions share a common boundary of non-zero length (i.e., not merely a corner where three or more regions meet).

It was the first major theorem to be proved using a computer. Initially, this proof was not accepted by all mathematicians because the computer-assisted proof was infeasible for a human to check by hand. The proof has gained wide acceptance since then, although some doubts remain



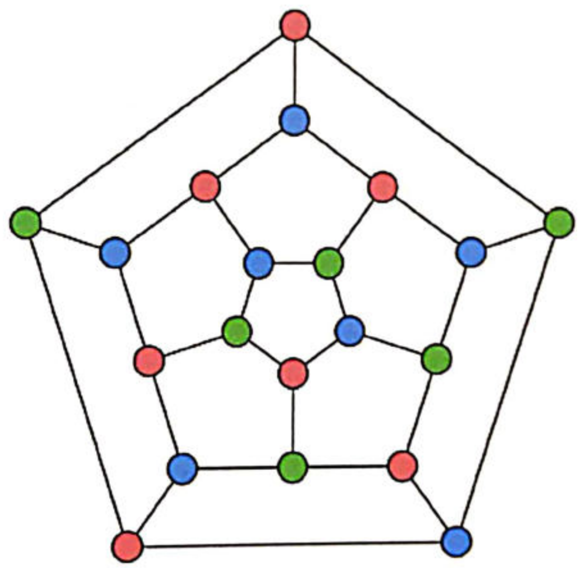
9.2: Applying of the four colour theorem in wireless a cell tower placement plan:

Consider the cell tower placement map shown above, where each cell tower broadcast channel is likened to a colour, and channel–colours are limited to four, the task of finding where to economically position broadcast towers for maximum coverage is equitable to the four-color map problem. The two challenges are: 1. Elimination of the no-coverage spots (marked red in the diagram above) 2. Allocation of a different channel in the spots where channel overlap occurs (marked in blue). In analogy, colours must be different, so that cell phone signals are handed off to a different channel. Each cell region therefore uses one control tower with a specific channel and the region or control tower adjacent to it will use another tower and another channel. It is not hard to see how by using 4 channels, a node colouring algorithm can be used to efficiently plan towers and channels in a mobile network, a very popular method in use by mobile service providers today.



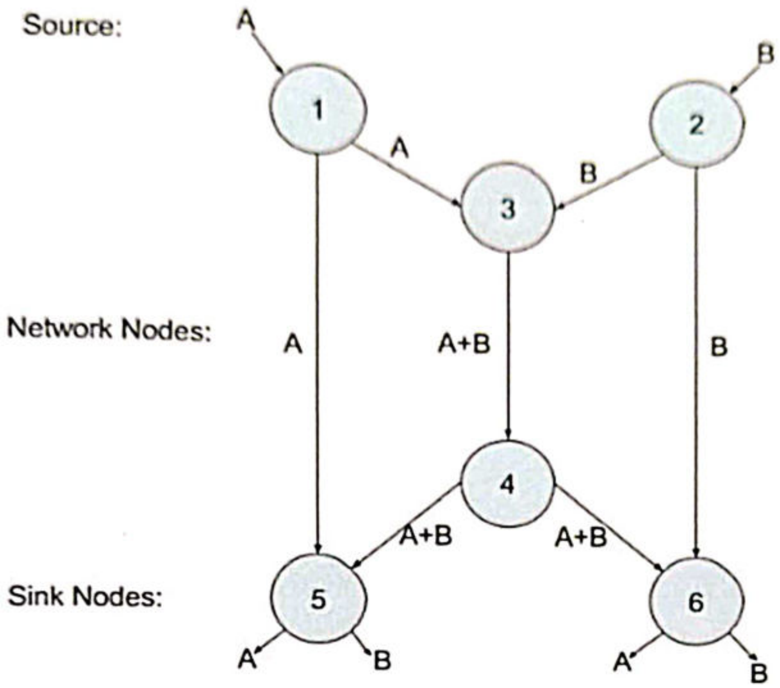
9.3. Node Colouring Theorem:

As can be seen in the map below, borders wander making it a difficult problem to analyze a map. Instead of using a sophisticated map with many wandering boundaries, it becomes a simpler problem if we use node colouring. If two nodes are connected by a line, then they can't be the same colour. Wireless Service providers employ node colouring to make an extremely complex network map much more manageable.



9.4. Network coding:

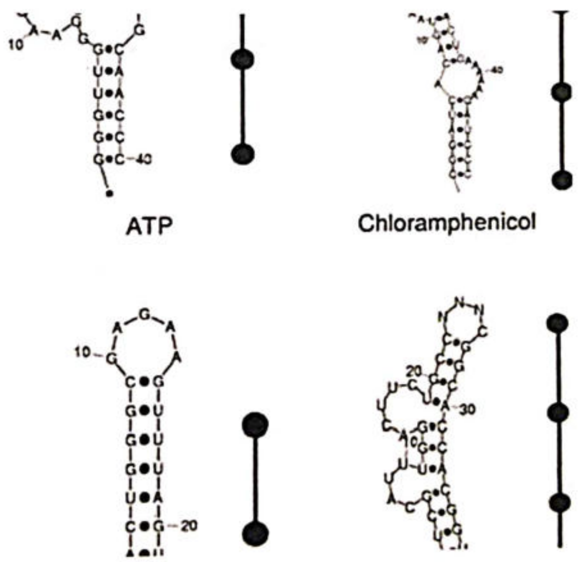
Network coding is a networking technique in which transmitted data is encoded and decoded to increase network throughput, reduce delays and make the network more robust. In network coding, algebraic algorithms are applied to the data to accumulate the various transmissions.



Chapter 10: Application of Graph theory in Molecular Biology

10.1. Application of Graph theory in Molecular Biology :

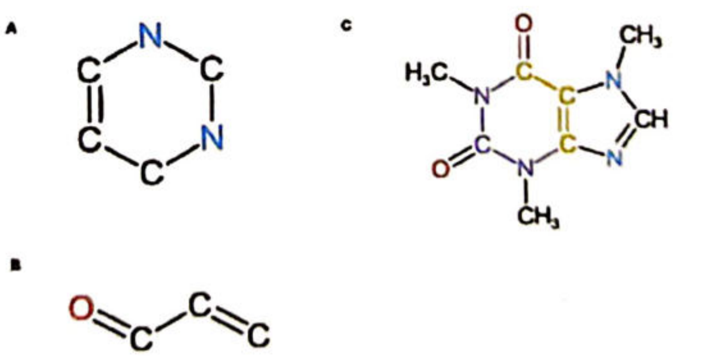
Graphs are also used for knowledge representation in the Gene Ontology (GO), and bipartite graphs between biological concepts and scientific papers that are written about them are another form of knowledge representation. A second application of graphs to molecular biology is to model measured



Chapter 11: Application of Graph theory in Molecular Chemistry

11.1. Application of Graph theory in Molecular Chemistry:

Chemical graph theory is the application of discrete mathematics to chemistry applied to model physical and biological properties of chemical compounds. Various topological indices which are derived from graph theory can model the geometric structure of chemical compounds. Several of these topological indices are used to construct boiling point models for alkanes with 1–12 carbon atoms. The models are used to predict the boiling points of a set of alkanes with 13–22 carbon atoms. Similarly, melting point models are considered for a family of alkanes having 10–20 carbon atoms and only one methyl group. Methyl-substituted alkanes such as these are especially important in the production of diesel and jet fuels since they enable a lower pour point for the resulting synthetic fuels. All models produced in this manner may be used to predict physical properties of compounds for which no experimental data exist.



Chapter 12: Fingerprint recognition using Graph Representation

12.1. Fingerprint recognition using Graph Representation

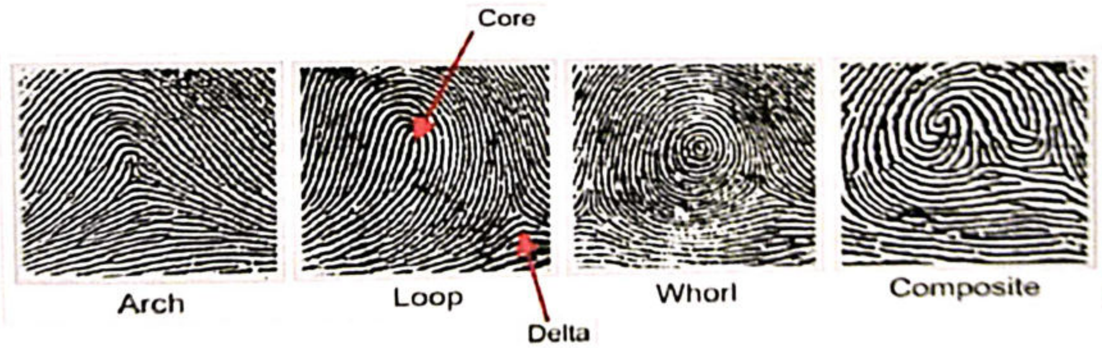
A fingerprint is represented in the form of a graph whose nodes correspond to ridges in the print. Edges of the graph connect nodes that represent neighbouring or intersecting ridges. Hence the graph structure captures the topological relationships within the fingerprint.

There are three characteristic of Fingerprints:

1. There are no similar fingerprint in the world.
2. Fingerprints are unchangeable.
3. Fingerprints are one of the unique feature for identification systems.

12.2. Type of Fingerprint:

All fingerprints can be classified into three basic patterns: loops, whorls, and arches.



References

https://en.wikipedia.org/wiki/Graph_theory

https://www.encyclopedia.com/science/encyclopedias-al

Balakrishnan, V. K.; *Graph Theory*, McGraw-Hill; 1 edition (February 1, 1997).

https://www.javatpoint.com/graph-theory-tree-and-forest

https://math.stackexchange.com/questions/1124172/difference-of-two-graphs

https://www.geeksforgeeks.org/complement-of-graph/

https://mathworld.wolfram.com/GraphJoin.html