Closure operator and α −ideals in 0-distributive lattices

S.S. Khopade[1], S. P. Thorat[2] and Laxmi Rathour[3]

[1] Karmaveer Hire Arts, Science, Commerce and Education College, Gargoti, Maharashtra, India – 416209

email: santajikhopade@gmail.com

[2] Vivekanand College, Kolhapur, Maharashtra, India – 416003

email: thoratsanjay15@gmail.com

[3] Department of Mathematics, National Institute of Technology, Chaltlang, Aizawl, Mizoram, India – 796 012

email: laxmirathour817@gmail.com

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Abstract

A closure operator on the lattice of the ideals of a bounded 0-distributive lattice is introduced. It is observed that the ideals which are closed with respect to this closure operator are α-ideals in it and conversely.

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1.Introduction

As a generalization of the concept of distributive lattices on one hand and pseudo-complemented lattices on the other, 0-distributive lattices are introduced by Varlet [6]. C. Jayaram [3] defined and studied α-ideals in, 0-distributive lattices. Additional properties of α-ideals in 0-distributive lattice are obtained by Pawar et. al. in [4] and [5]. Separation theorem for α-ideals in 0-distributive lattice is proved in [2]. In [4], the authors have obtained a characterization of an α-ideal using a closure operator on the lattice of all ideals of a 0-distributive lattice. In this paper we introduce a new closure operator on the lattice of all ideals of a 0-distributive lattice and characterize α-ideals in terms of the ideals which are closed with respect to this closure operator. Further it is observed that in a given 0-distributive lattice the ideals which are closed under this closure operator are the α-ideals in it and conversely.

2 Preliminaries

Following are some basic concepts and results needed in the sequel from references. For other non-explicitly stated elementary notions please refer to [1]. A lattice L with 0 is said to be 0-distributive if $a \wedge b = 0$ and $a \wedge c = 0$ imply $a \wedge (b \vee c) = 0$ for any $a, b, c \ in \ L$. Throughout this paper L will denote a bounded 0-distributive lattice unless otherwise specified. For a lattice L , $I(L)$ denotes the set of all ideals of L .Then $(I(L), \wedge, \vee)$ is a lattice where $I \wedge J = I \cap J$ and $I \vee J = (I \cup J]$, for any two ideals I and $J \ of \ L$. For any non- empty subset A of L, define $A^* = \{x \in L : x \wedge a = 0, for \ each \ a \in A\}$. By A^{**} we mean$(A^*)^*$. Note that when $A = \{a\}$ then $A^* = (a]^*$ and also denoted by $(a)^*$. An ideal I in L is called annihilator ideal if $I = A^*$, for a non-empty subset $A \ of \ L$. Let $L \ and \ L'$ denote bounded 0-distributive lattices and $f: L \longrightarrow L'$ be homomorphism, f is called annihilator preserving homomorphism if $f(A^*) = \{f(A)\}^*$ for any non-empty subset $A \ of \ L$. An ideal I of L is called α-ideal if $\{x\}^{**} \subseteq I \ for \ each \ x \in I$. Closure operator on L is a mapping $f: L \longrightarrow L$ satisfying the following conditions: (i) $x \leq f(x)$, $ii)$ $x \leq y \Longrightarrow f(x) \leq f(y) \ and \ f(f(x)) = f(x)$.

Result 2.1.(Varlet [6]). A lattice L with 0 is 0-distributive if and only if A^* is an ideal for any non-empty subset A of L.

Following result can be proved easily.

Result 2.2. In a 0-distributive lattice L, for all $a, b, c \in L$ we have

i) $\{a\}^{**} \cap \{b\}^{**} = \{a \wedge b\}^{**}$.
ii) $\{a\}^{*} \cap \{b\}^{*} = \{a \vee b\}^{*}$.
iii) $\{a\}^{**} = \{b\}^{**} \Longrightarrow \{a \wedge c\}^{**} = \{b \wedge c\}^{**}$.

Result 2.3 (Pawar and Mane [4]). In a bounded 0-distributive lattice L following statements are equivalent.

(i) $For\ x, y \in L,\ \{x\}^* = \{y\}^*,\ x \in I \Longrightarrow y \in I$.
(ii) $I = \cup \{\{x\}^{**} | x \in I\}$.
(iii) $For\ x, y \in L, h(x) = h(y), x \in I \Longrightarrow y \in I,$

$$where\ h(x) = \{M\ /\ is\ minimal\ prime\ ideal\ containing\ x\}.$$

(iv) I is an α-ideal.

Result 2.4 (Jayaram [2]). Let L be a 0-distributive lattice. Let I be anα-ideal and S be a meet sub semi lattice of L such that $I \cap S = \emptyset$. Then there exists a prime α-ideal P in L containing $I$and disjoint with S.

Result 2.5 (Pawar and Mane [4]). Every annihilator ideal in a 0-distributive lattice L is an α-ideal.

Result 2.6 (Pawar and Khopade [5]). Let $L\ and\ L'$ be any two bounded 0-distributive lattices and let $f: L \longrightarrow L'$be an annihilator preserving onto homomorphism, Then

(i) If I is an α-ideal of L, then $f(I)$ is an α-ideal of L'.
(ii) If $I'$is an α-ideal of L', then $f^{-1}(I')$ is an α-ideal of L.

3 Closure Operator

In this section we introduce a closure operator on $I(L)$.

Define $B(L) = \{\{a\}^{**} \,/\, a \in L\}$. L being 0-distributive lattice, $B(L) \subseteq I(L)$ (by result 2.1) but, $B(L)$ is not necessarily a sublattice of the lattice $I(L)$. For this consider the following example.

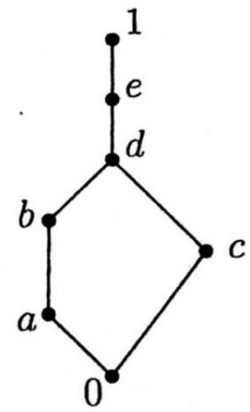

Figure 3.1

Example 3.1 Consider the bounded 0-distributive lattice $L = \{0, a, b, c, d, e, 1\}$ as shown by the Hasse Diagramme in Figure 3.1. Here $\{a\}^{**} = \{0, a, b\}$ and $\{c\}^{**} = \{0, c\}$. Hence $\{a\}^{**} \vee \{c\}^{**} = \{0, a, b, c, d\} \notin B(L)$. Hence the set $B(L)$ is a poset under set inclusion but need not be a sublattice of the lattice $I(L)$.

For $\{a\}^{**}, \{b\}^{**} \in B(L)$. Define $\{a\}^{**} \sqcap \{b\}^{**} = \{a \wedge b\}^{**}$ and

$\{a\}^{**} \sqcup \{b\}^{**} = \{a \vee b\}^{**}$. Then we have

Theorem 3.1 $(B(L), \sqcap, \sqcup)$ is a bounded lattice.

Proof. Obviously, $\{a \wedge b\}^{**}$ is the infimum of $\{a\}^{**}$ and $\{b\}^{**}$ in $(B(L), \subseteq)$. To prove $\{a \vee b\}^{**}$ is the supremum of $\{a\}^{**}$ and $\{b\}^{**}$ in $(B(L), \subseteq)$. $\{a \vee b\}^{**}$ is and upper bound of $\{a\}^{**}$ and $\{b\}^{**}$ in $(B(L), \subseteq)$. Let $\{c\}^{**}$ be any other upper bound of $\{a\}^{**}$ and $\{b\}^{**}$ in $(B(L), \subseteq)$. Let $t \in \{a \vee b\}^{**}$. Then $(t] \cap \{a \vee b\}^{*} = \{0\}$. By result 2.2 (ii) we get $(t] \cap [\{a\}^{*} \cap \{b\}^{*}] = \{0\}$, which implies $(t] \cap \{a\}^{*} \subseteq \{b\}^{**}$. But as $\{b\}^{**} \subseteq \{c\}^{**}$ we get $(t] \cap \{a\}^{*} \subseteq \{c\}^{**}$. Thus $(t] \cap \{a\}^{*} \cap \{c\}^{*} = \{0\}$, implies $(t] \cap \{c\}^{*} \subseteq \{a\}^{**}$. Again as $\{a\}^{**} \subseteq \{c\}^{**}$,we get $(t] \cap \{c\}^{*} \subseteq \{c\}^{**}$,that is $(t] \cap \{c\}^{*} = \{0\}$. Therefore $(t] \subseteq \{c\}^{**}$which yields $t \in \{c\}^{**}$. This shows that $\{a \vee b\}^{**} \subseteq \{c\}^{**}$ and hence $\{a \vee b\}^{**}$ is the supremum of $\{a\}^{**}$ and $\{b\}^{**}$ in $(B(L), \subseteq)$. As $\{0\}^{**} = \{0\}$ and $\{1\}^{**} = L$ belong to $B(L)$, $(B(L), \sqcap, \sqcup)$ is a bounded lattice.

Corollary 3.1. The lattice $(B(L), \sqcap, \sqcup)$ is a homomorphic image of the lattice L.

Proof. Define $\theta: L \rightarrow B(L)$ by $\theta(a) = \{a\}^{**}$ for each $a \in L$. Then $\theta(a \wedge b) = \{a \wedge b\}^{**} = \{a\}^{**} \sqcap \{b\}^{**} = \theta(a) \sqcap \theta(b)$ and $\theta(a \vee b) = \{a \vee b\}^{**} = \{a\}^{**} \sqcup \{b\}^{**} = \theta(a) \sqcup \theta(b)$ hold for all $a, b \in L$. Hence θ is a homomorphism. As θ is onto, the result follows.

Remark 3.1. Note that the homomorphism θ is not necessarily one-one. For this consider the 0-distributive lattice in Example 3.1. Here for $a \neq b$ in L we have $\{a\}^{**} = \{b\}^{**}$.

For any ideal I of L, define $\delta(I) = \{\{a\}^{**} / a \in I\}$ for any Ideal $\bar{I}$ of $B(L)$, define $\overleftarrow{\delta}(\bar{I}) = \{a \in L / \{a\}^{**} \in \bar{I}\}$. With these notations we prove

Theorem 3.2.

(i) $\delta(I)$ is an ideal of $B(L)$, for any ideal I of L.

(ii) $\overleftarrow{\delta}(\bar{I})$ is an ideal of L for any ideal of $\bar{I}$ of $B(L)$.

(iii) For any two ideals, I and J of $L, I \subseteq J \Longrightarrow \delta(I) \subseteq \delta(J)$.

(iv) For any two ideals $\bar{I}$ and $\bar{J}$ of $B(L), \bar{I} \subseteq \bar{J} \Longrightarrow \overleftarrow{\delta}(\bar{I}) \subseteq \overleftarrow{\delta}(\bar{J})$.

Proof. (i). Let I be any ideal of L. As $0 \in I, \{0\}^{**} = \{0\} \in \delta(I)$. Hence $\delta(I)$ is non-empty. Let $\{a\}^{**}, \{b\}^{**} \in B(L)$ such that $\{a\}^{**} \subseteq \{b\}^{**}$ and $\{b\}^{**} \in \delta(I)$. Then $\{b\}^{**} = \{x\}^{**}$ for $some$ $x \in I$. Thus $\{a\}^{**} = \{a\}^{**} \sqcap \{b\}^{**} = \{a\}^{**} \sqcap \{x\}^{**} = \{a \wedge x\}^{**}$. As $a \wedge x \in I$, we get $\{a\}^{**} \in \delta(I)$. Let $\{a\}^{**}, \{b\}^{**} \in \delta(I)$. Therefore $\{a\}^{**} = \{x\}^{**}$ and $\{b\}^{**} = \{y\}^{**}$ for some $x, y \in I$. Hence $\{a\}^{**} \sqcup \{b\}^{**} = \{x\}^{**} \sqcup \{y\}^{**} = \{x \vee y\}^{**}$. As $x \vee y \in I$, we get $\{x \vee y\}^{**} \in \delta(I)$. Hence $\{a\}^{**} \sqcup \{b\}^{**} \in \delta(I)$. Therefore $\delta(I)$ is an ideal of $B(L)$.

(ii) Let $\bar{I}$ be any ideal of $B(L)$. $\{0\}^{**} = \{0\} \in \bar{I}$ implies $0 \in \overleftarrow{\delta}(\bar{I})$. Hence $\overleftarrow{\delta}(\bar{I})$ is non-empty. Let $a, b \in L$ $such$ $that$ $a \leqq b$ and $b \in \overleftarrow{\delta}(\bar{I})$. Then $\{a\}^{**} \subseteq \{b\}^{**}$ and $\{b\}^{**} \in \bar{I}$. $\bar{I}$ being an ideal we get $\{a\}^{**} \in \bar{I}$. But then $a \in \overleftarrow{\delta}(\bar{I})$. Let $a, b \in \overleftarrow{\delta}(\bar{I})$. Then $\{a\}^{**}, \{b\}^{**} \in \bar{I}$ implies $\{a\}^{**} \sqcup \{b\}^{**} = \{a \vee b\}^{**} \in \bar{I}$. Therefore $a \vee b \in \overleftarrow{\delta}(\bar{I})$. This proves $\overleftarrow{\delta}(\bar{I})$ is an ideal of L.

(iii) Let I and J be two ideals of L such that $I \subseteq J$. Let $\{a\}^{**} \in \delta(I)$. Then $\{a\}^{**} = \{x\}^{**}$ for $some$ $x \in I$. But then, since $I \subseteq J$ we get $x \in J$. This is turns gives $\{a\}^{**} \in \delta(J)$. Hence $\delta(I) \subseteq \delta(J)$.

(iv) Let $\bar{I}$ and $\bar{J}$ be any two ideals of $B(L)$ such that $\bar{I} \subseteq \bar{J}$. Let $x \in \overleftarrow{\delta}(\bar{I})$. Then $\{x\}^{**} \in \bar{I}$ implies $\{x\}^{**} \in \bar{J}$. Hence $x \in \overleftarrow{\delta}(\bar{J})$ and the result follows.

As $\delta(I)$ is an ideal of $B(L)$, for any ideal I of L, we have the mapping $\delta: I(L) \to I(B(L))$ is well defined where $I(B(L))$ denotes the lattice of all ideals of the lattice $B(L)$. Further we have

Theorem 3.3 $\delta: I(L) \to I(B(L))\{0,1\}$ is a homomorphism.

Proof: Let I and J be any ideals in $I(L)$. $\delta(I \cap J) \subseteq \delta(I) \cap \delta(J)$ (by Theorem 3.2 (iii)). Let $\{a\}^{**} \in \delta(I) \cap \delta(J)$. Then $\{a\}^{**} \in \delta(I)$ implies $\{a\}^{**} = \{i\}^{**}$ for some $i \in I$ and $\{a\}^{**} \in \delta(J)$ gives $\{a\}^{**} = \{j\}^{**}$ for some $j \in J$. Thus $\{a\}^{**} = \{i\}^{**} \sqcap \{j\}^{**} = \{i \wedge j\}^{**}$. As $i \wedge j \in I \cap J$, we get $\{a\}^{**} \in \delta(I \cap J)$. This shows that $\delta(I) \cap \delta(J) \subseteq \delta(I \cap J)$. Combining both the inclusions we get $\delta(I \cap J) = \delta(I) \cap \delta(J)$.

Now, again by Theorem 3.2 – (iii), $\delta(I) \vee \delta(J) \subseteq \delta(I \vee J)$. Let $\{a\}^{**} \in \delta(I \vee J)$. Hence $\{a\}^{**} = \{y\}^{**}$ for some $y \in I \vee J$. Therefore $y \leq i \vee j$ for some $i \in I$ and $j \in J$. This yields $\{y\}^{**} \subseteq \{i \vee j\}^{**} = \{i\}^{**} \sqcup \{j\}^{**}$. Therefore $\{a\}^{**} = \{y\}^{**} \in \delta(I) \vee \delta(J)$. Hence $\delta(I \vee J) \subseteq \delta(I) \vee \delta(J)$. Combining both the inclusions we get $\delta(I \vee J) = \delta(I) \vee \delta(J)$.

This proves that $\delta: I(L) \to I(B(L))$ is a homomorphism. Again $\delta((0]) = \{\{0\}^{**}\} = \{\{0\}\}$ $and \delta((1]) = \{\{1\}^{**}\} = \{L\}$, shows δ is a $\{0,1\}$ homomorphism.

By theorem 3.2., we get two mappings $\delta: I(L) \to I(B(L))$ and $\overleftarrow{\delta} : I(B(L)) \to I(L)$. Hence $\delta \circ \overleftarrow{\delta} : I(B(L)) \to I(B(L))$ and $\overleftarrow{\delta} \circ \delta : I(L) \to I(L)$. About these two mappings we have

Theorem 3.4.

(i) $\delta \circ \overleftarrow{\delta}$ is a identity mapping on $I(B(L))$.

(ii) $\overleftarrow{\delta} \circ \delta$ is a closure operator on $I(L)$.

Proof. (i) Let $\bar{I}$ be any ideal of $B(L)$. Let $\{x\}^{**} \in \delta \circ \overleftarrow{\delta}(\bar{I}) = \delta(\overleftarrow{\delta}(\bar{I}))$. Hence $\{x\}^{**} = \{y\}^{**}$ for some $y \in \overleftarrow{\delta}(\bar{I})$. But then $\{y\}^{**} \in \bar{I}$, which implies $\{x\}^{**} \in \bar{I}$. This gives $\delta \circ \overleftarrow{\delta}(\bar{I}) \subseteq \bar{I}$. Conversely, let $\{x\}^{**} \in \bar{I}$. Then $x \in \overleftarrow{\delta}(\bar{I})$ and consequently

$\{x\}^{**} \in \delta(\overleftarrow{\delta}(\bar{I}))$.(since $\overleftarrow{\delta}(\bar{I})$is an ideal of L). Hence $\bar{I} \subseteq \delta \circ \overleftarrow{\delta}(\bar{I})$. From both the inclusions we get $\delta \circ \overleftarrow{\delta}(\bar{I}) = \bar{I}$. Hence $\delta \circ \overleftarrow{\delta}$ is an identity mapping on $I(B(L))$.

(ii)Let $I \in I(L)$ and $x \in I$. Then $\{x\}^{**} \in \delta(I)$ and by Theorem 3.2 –(i), $\delta(I)$is an ideal of $B(L)$, which yields $x \in \overleftarrow{\delta} \circ \delta(I)$. Hence $I \subseteq \overleftarrow{\delta} \circ \delta(I)$. Let $I, J \in I(L)$ and $I \subseteq J$. As δ and $\overleftarrow{\delta}$ are isotone mappings (by Theorem 3.2), we get $\overleftarrow{\delta} \circ \delta(I) \subseteq \overleftarrow{\delta} \circ \delta(J)$.

Finally, Let $I \in I(L)$. As $I \subseteq \overleftarrow{\delta} \circ \delta(I)$, applying (II) we get $\overleftarrow{\delta} \circ \delta(I) \subseteq \overleftarrow{\delta} \circ \delta\left(\overleftarrow{\delta} \circ \delta(I)\right)$. Conversely, let $x \in \overleftarrow{\delta} \circ \delta\left(\overleftarrow{\delta} \circ \delta(I)\right)$.Then $\{x\}^{**} \in \delta\left(\overleftarrow{\delta} \circ \delta(I)\right)$ implies $\{x\}^{**} = \{y\}^{**}$ for some $y \in \overleftarrow{\delta} \circ \delta(I)$. But then $\{y\}^{**} \in \delta(I)$, which implies $\{x\}^{**} \in \delta(I)$. This gives $x \in \overleftarrow{\delta} \circ \delta(I)$.This proves $\overleftarrow{\delta} \circ \delta\left(\overleftarrow{\delta} \circ \delta(I)\right) \subseteq \overleftarrow{\delta} \circ \delta(I)$. Combining both the inclusions we get $\overleftarrow{\delta} \circ \delta\left(\overleftarrow{\delta} \circ \delta(I)\right) = \overleftarrow{\delta} \circ \delta(I)$.

From (3.1), (3.2) and (3.3) we get $\overleftarrow{\delta} \circ \delta$ is a closure operator on $I(L)$.

Remark 3.2. The mapping $\delta: I(L) \to I(B(L))$ is a homomorphism follows from Theorem 3.3. Let $\bar{I}$ be any ideal of $B(L)$. As $\overleftarrow{\delta}(\bar{I})$ is an ideal of L and $\delta \circ \overleftarrow{\delta}(\bar{I}) = \bar{I}$, we get the mapping $\delta: I(L) \to I(B(L))$ is onto. Hence the lattice $I(B(L))$ is homomorphic image of lattice $I(L)$.

4 $\alpha -$ ideals

In this section we show that the ideals in L which are closed with respect to the closure operator $\overleftarrow{\delta} \circ \delta$ defined on $I(L)$ are $\alpha -$ ideals in L and conversely. Let $C(L)$ denote the set of all ideals in L which are closed with respect to the closure operator $\overleftarrow{\delta} \circ \delta$ defined on $I(L)$.

Thus $C(L) = \{ I \in I(L): \overleftarrow{\delta} \circ \delta(I) = I \}$. Obviously, (0] and (1] belong to $C(L)$. Hence $C(L)$ is a non-empty subset of $I(L)$ but not necessarily a sublattice of the lattice $I(L)$.This follows by the 0-distributive lattice given in example 3.1. Here $C(L) = \{(0], (b], (c]\}$ and $(b] \vee (c] = (d]$. As $(d] \notin C(L)$, the subset $C(L)$is not a sublattice of the lattice $I(L)$. Though $C(L)$does not form a sublattice of the lattice I(L), it forms a lattice on its own. This we prove in the following theorem.

Theorem 4.1. $(C(L), \overline{\wedge}, \underline{\vee})$ is a bounded lattice where $\overline{\wedge}$ and $\underline{\vee}$ are defined by $I \overline{\wedge} J = I \cap J$ and $I \underline{\vee} J = \overleftarrow{\delta} \circ \delta(I \vee J)$ for $I, J \in C(L)$

Proof: (i) First we prove that for $I, J \in C(L), I \cap J \in C(L)$. As $\overleftarrow{\delta}$ and δ are isotone mappings, we get $\overleftarrow{\delta} \circ \delta$ is also isotone. Hence $\overleftarrow{\delta} \circ \delta(I \cap I) \subseteq \overleftarrow{\delta} \circ \delta(I) \cap \overleftarrow{\delta} \circ \delta(J)$.

Let $x \in \overleftarrow{\delta} \circ \delta(I) \cap \overleftarrow{\delta} \circ \delta(J)$. Then $\{x\}^{**} \in \delta(I) \cap \delta(J) = \delta(I \cap J)$. This gives $x \in \overleftarrow{\delta} \circ \delta(I \cap J)$. Hence $\overleftarrow{\delta} \circ \delta(I) \cap \overleftarrow{\delta} \circ \delta(J) \subseteq \overleftarrow{\delta} \circ \delta(I \cap J)$. Combining both the inclusions we get $\overleftarrow{\delta} \circ \delta(I \cap J) = \overleftarrow{\delta} \circ \delta(I) \cap \overleftarrow{\delta} \circ \delta(J) = I \cap J$ (since $I, J \in C(L)$). This proves $I \cap J \in C(L)$. Thus the infimum of $I, J \in C(L)$ is $(I \cap \mathrm{J})$. Hence $I \overline{\wedge} J = I \cap J$.

(ii) First note that, by Theorem 3.4- (ii), $\overleftarrow{\delta} \circ \delta(I) \in C(L)$, for any ideal I of L. Let $I, J \in C(L)$. Then $I = \overleftarrow{\delta} \circ \delta(I) \subseteq \overleftarrow{\delta} \circ \delta(I \vee J)$ and $J = \overleftarrow{\delta} \circ \delta(J) \subseteq \overleftarrow{\delta} \circ \delta(I \vee J)$ (since $\overleftarrow{\delta} \circ \delta$ is isotone). Thus $\overleftarrow{\delta} \circ \delta(I \vee J)$ is an upper bound of $I \ and \ J \ in \ C(L)$. Let $K \in C(L)$, such that $I \subseteq K$ and $J \subseteq K$. Then $I \vee J \subseteq K$ implies $\overleftarrow{\delta} \circ \delta(I \vee J) \subseteq \overleftarrow{\delta} \circ \delta(K) = K$ (since $K \in C(L)$). This shows that $\overleftarrow{\delta} \circ \delta(I \vee J)$ is the supremum of $I \ and \ J \ in \ C(L)$ i.e. $I \underline{\vee} J = \overleftarrow{\delta} \circ \delta(I \vee J)$. As $(0] \in C(L)$ and $L \in C(L)$, $(C(L), \overline{\wedge}, \underline{\vee})$ is a bounded lattice.

We know that the lattice $I\big(B(L)\big)$ is a homomorphic image of the lattice $I(L)$ (see Remark 3.2). But interestingly we have

Theorem 4.2. The lattice $C(L)$is isomorphic with the lattice $I\big(B(L)\big)$.

Proof. Define the mapping $\psi: C(L) \to I\big(B(L)\big)$ by $\psi(I) = \delta(I)$ for each $I \in C(L)$, which is clearly a well defined mapping.

(i)Let $\psi(I) = \psi(J)$ for $I, J \in C(L)$. Then we have $\delta(I) = \delta(J)$. Therefore $\overleftarrow{\delta} \circ \delta(I) = \overleftarrow{\delta} \circ \delta(J)$ which implies $I = J$ (since $I, J \in C(L)$). This shows that ψ is one-one.

(ii) Let $\overline{I}$ be any ideal of $B(L)$. Then $\overleftarrow{\delta}(\overline{I})$ is an ideal of L (by theorem 3.2-(ii)) and $\delta \circ \overleftarrow{\delta}(\overline{I}) = \overline{I}$ (by theorem 3.4-(i)). Then $\overleftarrow{\delta} \circ \delta\left(\overleftarrow{\delta}(\overline{I})\right) = \overleftarrow{\delta}\left(\delta\left(\overleftarrow{\delta}(\overline{I})\right)\right) = \overleftarrow{\delta}\left(\delta \circ \overleftarrow{\delta}(\overline{I})\right) = \overleftarrow{\delta}(\overline{I})$. This shows that $\overleftarrow{\delta}(\overline{I}) \in C(L)$). As $\psi\left(\overleftarrow{\delta}(\overline{I})\right) = \delta\left(\overleftarrow{\delta}(\overline{I})\right) = \delta \circ \overleftarrow{\delta}(\overline{I}) = \overline{I}$, we get ψ is onto.

(iii)Let $I, J \in C(L)$). Then by definition of ψ and by theorem 3.3 we get, $\psi(I \barwedge J) = \psi(I \cap J) = \delta(I \cap J) = \delta(I) \cap \delta(J) = \psi(I) \cap \psi(J)$. And by definition of $\underline{\vee}$ in $C(L)$ we get $\psi\big(I \,\underline{\vee}\, J\big) = \delta\big(I \,\underline{\vee}\, J\big) = \delta(\overleftarrow{\delta} \circ \delta(I \vee J)) = \delta(I \vee J)$ (since $\overleftarrow{\delta} \circ \delta$ is an identity map). Thus $\psi\big(I \,\underline{\vee}\, J\big) = \delta(I \vee J) = \delta(I) \vee \delta(J) = \psi(I) \vee \psi(J)$. This proves that ψ is a homomorphism. From (i) – (iii) we get ψ is an isomorphism.

Following theorem gives a necessary and sufficient conditions for an ideal $I\ of\ L$ to be a member of $C(L)$.

Theorem 4.3. For any ideal $I\ of\ L$, following statements are equivalent.

(i). $I \in C(L)$.

(ii). For $x, y \in L$, $\{x\}^{**} = \{y\}^{**}, x \in I \Rightarrow y \in I$

(iii). For $x, y \in L$, $\{x\}^{*} = \{y\}^{*}, x \in I \Rightarrow y \in I$

(iv). $I = \cup \{\{x\}^{**} : x \in I\}$.

(v). For $x, y \in L, h(x) = h(y), x \in I \Rightarrow y \in I$,

$$where\ h(x) = \{M : M\ is\ a\ minimal\ prime\ ideal\ containing\ x\}.$$

(vi). I is an $\alpha -$ ideal.

Proof. The equivalence of the statements (iii) to (vi) follows by Result 2.3.

(ii) $\Leftrightarrow$ (iii): As $\{x\}^{**} = \{y\}^{**} \Leftrightarrow \{x\}^{*} = \{y\}^{*}$ $for\ any\ x, y \in L$, the equivalence follows. (i) $\Rightarrow$(ii): Let $I \in C(L)$. Let $x, y \in L$ such that $\{x\}^{**} = \{y\}^{**}$ and $x \in I$. As $x \in I$, we have $\{x\}^{**} \in \delta(I)$. But then, by assumption, we get $\{y\}^{**} \in \delta(I)$. This gives $y \in \overleftarrow{\delta} \circ \delta(I)$. Again by assumption that $I \in C(L)$, we get $y \in I$. Thus the implication follows. (ii) $\Rightarrow$ (i): Let $I \in I(L)$ satisfying condition in (ii). By Theorem 3.4, we have $I \subseteq \overleftarrow{\delta} \circ \delta(I)$.

To prove $\overleftarrow{\delta} \circ \delta(I) \subseteq I$. On contrary assume that $\overleftarrow{\delta} \circ \delta(I) \nsubseteq I$. Then there exists $x \in \overleftarrow{\delta} \circ \delta(I)$ such that $x \notin I$. Then $\{x\}^{**} \in \delta(I)$ which implies $\{x\}^{**} = \{y\}^{**}$ for some $y \in I$. But then, by assumption, $x \in I$; a contradiction. Hence $\overleftarrow{\delta} \circ \delta(I) \subseteq$

I. Combining both the inclusions, we get $\overleftarrow{\delta} \circ \delta(I) = I$. Hence $I \in C(L)$ and the implication follows. Hence all the statements are equivalent.

Using the property that $I \in C(L)$ if and only if I is an α-ideal, proved in theorem, we get

Corollary 4.1. $(a] \in C(L)$ if and only if $(a] = \{a\}^{**}$ for any $a \in L$.

Proof. Let $(a] \in C(L)$. Then by Theorem 4.3, $(a]$ is an α-ideal of L. This gives $\{a\}^{**} \subseteq (a]$ (by definition of α-ideal). As we obviously have $(a] \subseteq \{a\}^{**}$, the proof of if part follows. Conversely, suppose $(a] = \{a\}^{**}$. We know that every annihilator ideal is an α-ideal, therefore $\{a\}^{**} = (a]$ is an α-ideal. Thus again by Theorem 4.3, we get $(a] \in C(L)$.

$I^* \in C(L)$ For any ideal I in L, because I^* is an α-ideal of L (see result 2.5). Hence we have

Corollary 4.2. The lattice $(C(L), \overline{\wedge}, \underline{\vee})$ is a pseudo complemented lattice.

Define $A_0(L) = \{\{x\}^* : x \in L\}$. Then $(A_0(L), \widehat{\wedge}, \widetilde{\vee})$ is a lattice, where $\{x\}^* \mathbin{\widehat{\wedge}} \{y\}^* = \{x \vee y\}^*$ and $\{x\}^* \mathbin{\widetilde{\vee}} \{y\}^* = \{x \wedge y\}^*$. This lattice is called as a lattice of all annulets of L. For any ideal I in L, the set $\{\{x\}^* : x \in I\}$ is a filter in $A_0(L)$ and for any filter F in $A_0(L)$, the set $\{x \in L : \{x\}^* \in F\}$ is an ideal of L. Let $\mathcal{F}(A_0(L))$ denote the lattice of all filters in $A_0(L)$. Then the maps $\alpha : I(L) \to \mathcal{F}(A_0(L))$ defined by $\alpha(I) = \{\{x\}^* : x \in I\}$ and $\beta : \mathcal{F}(A_0(L)) \to I(L)$ defined by $\beta(F) = \{x \in L : \{x\}^* \in F\}$ are well defined isotone maps.

We need the following results from [4]:

Lemma 4.1 (Theorem 9 in [4]).

The map $\beta \circ \alpha : I(L) \to I(L)$ is a closure operator on $I(L)$.

Lemma 4.2 (Theorem 10 in [4]).

For any ideal of I in L, following statements are equivalent.

(i). I is an α-ideal.

(ii). $\beta \circ \alpha(I) = I$.

Using above two lemmas and Theorem 4.3 we get $C(L) = \{I \in I(L): \overleftarrow{\delta} \circ \delta(I) = I\} = \{I \in I(L): \beta \circ \alpha(I) = I\}$. Hence an ideal I in L is closed with respect to the closure operator $\overleftarrow{\delta} \circ \delta$ if and only if it is closed with respect to the closure operator $\beta \circ \alpha$ defined on $I(L)$. Thus we have

Corollary 4.3. For any ideal I of L, $\overleftarrow{\delta} \circ \delta(I) = I$ if and only if $\beta \circ \alpha(I) = I$.

Let I be an ideal of L. If there exists a prime ideal P of L such that $I \subseteq P$ *and* P is minimal in the class of all prime ideals containing I, then P is called a prime ideal belonging to I. We know that any prime ideal of L need not be an α-ideal. For this consider the lattice $L = \{0, a, b, c, d, e, 1\}$ whose Hasse Diagram is as in Figure 3.1. The ideal (e] is a prime ideal but not an α-ideal. For, $d \in$(e] but $(d]^{**} = L \nsubseteq$(e].

In the following theorem we show that a prime ideal belonging to an α-ideal is an α-ideal.

Theorem 4.4. Let I be an α-ideal of L. Let P be a prime ideal belonging to I, then P is an α-ideal.

Proof. Suppose P is not an α-ideal. Hence there exist x, y *in* L *such that* $\{x\}^{**} = \{y\}^{**}, x \in P$ *but* $y \notin P$ (see theorem 4.3). Consider the filter $F = (L \backslash P) \vee [x \wedge y)$. Claim that $F \cap I = \emptyset$. Let $F \cap I \neq \emptyset$. Select $a \in F \cap I$. Then $a \in F$ implies $a \geq r \wedge s$ for some $r \in (L \backslash P)$ and $s \geq x \wedge y$. But then $a \geq r \wedge x \wedge y$ and therefore $r \wedge x \wedge y \in I$ as ($a \in I$). Since $\{x\}^{**} = \{y\}^{**}$, using the Result 2.2, we get $\{r \wedge x\}^{**} = \{r \wedge y\}^{**}$ and hence $\{r \wedge x \wedge y\}^{**} = \{r \wedge y\}^{**}$. Since $r \wedge x \wedge y \in I$ and I is an α-ideal, by theorem 4.3, we get $r \wedge y \in I$. Hence $r \wedge y \in P$ (since $I \subseteq P$). Now $r \wedge y \in P$, P is a prime ideal and $r \notin P$ imply $y \in P$; which contradicts our assumption. Hence we must have $F \cap I = \emptyset$. Therefore, by result 2.4, there exists a prime ideal Q containing I and disjoint with F. Thus $Q \subseteq P$. Moreover $F \cap Q = \emptyset$ and $x \wedge y \in F$ implies $x \wedge y \notin Q$. Hence Q$\neq P$ (since $x \in P \Rightarrow x \wedge y \in P$) *i.e.* $Q \subset P$. But this contradicts to the fact that P is minimal in the class of all prime ideals containing I . Hence we must have P is an α-ideal.

Making an appeal to Theorem 4.1, Theorem 4.3 and Result 2.6, we establish

Corollary 4.4. Let L *and* L' be bounded 0-distributive lattices and let $f: L \to L'$ be an annihilator preserving onto homomorphism. Then we have (i). If $I \in C(L), then\ f(I) \in C(L')$.

(ii). If $I' \in C(L'), then\ f^{-1}(I') \in C(L)$.

5 Conclusion

The present investigation provides a new way to define closure operator on the lattice of all ideals of bounded 0-distributive lattice. Moreover the ideals closed with respect to this closure operator are α-ideals. Therefore this work will motivate and useful to study more properties of α-ideals.

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